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On the bounds of the expected nearest neighbor distance

ABSTRACT

The risk of nearest neighbor procedures may be bounded using the distance to the nearest neighbor. For bounded support, covering numbers may be used to estimate the expected distance. In this paper, we give some contributions for special distributions having unbounded support $S = (-\infty, \infty)$ for which we derive upper and lower bounds on the expected nearest neighbor distance of logistic and Laplace's distributions as typical, and evaluate the bounds of the risk for their distributions.

Keywords: Nearest neighbor classification, expected nearest neighbor distance, logistic distribution, Laplace's distribution.

1. INTRODUCTION

The nearest neighbor rule was first studied by Fix and Hodges [1]. Cover and Hart [2] proved that $R^* \leq R_\infty \leq 2R^*(1 - R^*)$ under certain conditions, where R^* denotes the Bayes error (the minimum probability of error over all decision rules), and R_∞ is the nearest neighbor risk in the infinite-sample limit. Cover [3] has shown that $R_m = R_\infty + O(m^{-2})$ for the nearest neighbor classifier in the case one-dimensional bounded support, mixture density $f \geq c > 0$, and under some additional conditions, where R_m denotes the finite sample risk, and m is the sample size. Kulkarni and Posner [4] studied the rate of convergence for nearest neighbor estimation in terms of the covering numbers of totally bounded sets. Evans et al. [5] derived an asymptotic moments of near neighbor distance distributions. Irle and Rizk [6] found an asymptotic evaluation of the conditional risk $R_m(x)$ (the probability of error conditioned on the event that $X = x$) by using partial integration and Laplace's method. Liitiäinen et al. [7] studied a boundary corrected expansion of the moments of nearest neighbor distributions. Rizk and Ateya [8] found lower and upper bounds for the risk of nearest neighbor of generalized exponential distribution.

In this paper, we find upper and lower bounds on the expected nearest neighbor distance for distributions having unbounded support $S = (-\infty, \infty)$ for which we derive upper and lower bounds on the expected nearest neighbor distance of logistic and Laplace's distributions as typical, and evaluate the bounds of the risk for their distributions.

In pattern recognition if we have a random variable (X, θ) , such that $X \in R^d$ is an observed pattern from which we wish to predict the unobservable class θ . This class takes values in a finite set $M = \{1, 2, \dots, C\}$. If the joint distribution of (X, θ) is known, then the best classifier is the Bayes classifier, see [9], [10]. In general the joint distribution of (X, θ) will be unknown, and we have a training sequence $D_m = ((X^{(1)}, \theta^{(1)}), (X^{(2)}, \theta^{(2)}), \dots, (X^{(m)}, \theta^{(m)}))$ at our disposal, where patterns and corresponding classes are observed and we assume that $((X^{(1)}, \theta^{(1)}), (X^{(2)}, \theta^{(2)}), \dots, (X^{(m)}, \theta^{(m)}))$, the data, stem from a sequence of independent identically distributed random pairs with the same distribution as (X, θ) . The nearest neighbor rule assigns any input feature vector to the class given by the label θ' of the nearest reference vector.

The problem to be considered is the classification of a random variable θ taking values in $M = \{1, 2\}$ given a sample X in χ , with the goal of minimizing the finite-sample risk $R_m = P(\theta \neq \theta')$, where χ is a separable metric space equipped with metric ρ which we denote as the pair (χ, ρ) . Define the nearest distance at time m as $d_m = \rho(X, X')$.

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47 2.BOUNDS FOR THE EXPECTED NEAREST NEIGHBOR DISTANCE

In this section, we find upper and lower bounds for expected nearest neighbor distance for distributions having unbounded support $S = (-\infty, \infty)$, compare [6].

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51 2.1 An upper bound on the expected nearest neighbor distance

We use constants, $-\infty < K_1(m) \leq 0 \leq K_2(m) < \infty$ depending on m , to write

$$\begin{aligned} Ed_m &= \int_{-\infty}^{\infty} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \\ &= \int_{-\infty}^{K_1(m)} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \\ &\quad + \int_{K_2(m)}^{\infty} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \\ &\quad + \int_{K_1(m)}^{K_2(m)} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx. \end{aligned}$$

$$53 \quad Ed_m = L_1(m) + L_2(m) + L_3(m). \quad (2.1)$$

54 where,

$$L_1(m) = \int_{-\infty}^{K_1(m)} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx, (2.2)$$

$$L_2(m) = \int_{K_2(m)}^{\infty} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx, (2.3) L_3(m) = \int_{K_1(m)}^{K_2(m)} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx. (2.4)$$

55 **2.1.1 Bounding $L_1(m)$ and $L_2(m)$**

56 We assume for the following that $|X - x|$ has a finite moment generating function $\varphi(t, x) =$

57 $E(e^{t|X-x|})$, $x \in R$, $0 < t < 1$. By Markov's inequality for any $0 < t < 1$, we obtain:

$$\begin{aligned} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon &= \int_0^{\infty} P(e^{t|X-x|} > e^{t\varepsilon})^m d\varepsilon \\ &\leq \int_0^{\infty} \varphi(t, x)^m e^{-mt\varepsilon} d\varepsilon = \frac{1}{mt} \varphi(t, x)^m. \end{aligned}$$

58 For $t = \frac{1}{\tau m}$, $\tau \geq 1$, we have $\int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon \leq \tau \varphi\left(\frac{1}{\tau m}, x\right)^m$. It follows

$$L_1(m) \leq \tau \int_{-\infty}^{K_1(m)} \varphi\left(\frac{1}{\tau m}, x\right)^m f(x) dx, (2.5)$$

$$L_2(m) \leq \tau \int_{K_2(m)}^{\infty} \varphi\left(\frac{1}{\tau m}, x\right)^m f(x) dx. (2.6)$$

59 **2.1.2 Bounding $L_3(m)$**

60 From (2.4) we have

$$\begin{aligned} L_3(m) &= \int_{K_1(m)}^{K_2(m)} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \\ &= \int_{K_1(m)}^{K_2(m)} \int_0^{\infty} e^{-mG(x, \varepsilon)} f(x) d\varepsilon dx, (2.7) \end{aligned}$$

61 where $G(x, \varepsilon) = -\log P(|X - x| > \varepsilon)$. Assume that the following inequality holds:

62 There exists $c > 0$ such that for all x in the support of X and for all $\varepsilon > 0$

$$G(x, \varepsilon) \geq c\varepsilon f(x). (2.8)$$

63 From this inequality, we obtain

$$\begin{aligned} \int_{K_1(m)}^{K_2(m)} \int_0^{\infty} e^{-mG(x, \varepsilon)} f(x) d\varepsilon dx &\leq \int_{K_1(m)}^{K_2(m)} \int_0^{\infty} e^{-mc\varepsilon f(x)} f(x) d\varepsilon dx \\ &= \int_{K_1(m)}^{K_2(m)} \frac{1}{cm} dx = \frac{1}{cm} (K_2(m) - K_1(m)). (2.9) \end{aligned}$$

64 We can show that a sufficient condition for $G(x, \varepsilon) \geq c\varepsilon f(x)$ is given by, see[6]:

$$P(|X - x| \leq \varepsilon) \geq c\varepsilon f(x).$$

Note that this second condition will always be violated for unbounded Support letting ε tend to ∞ .

2.1 Deriving a lower bound

$$\begin{aligned} Ed_m &= \int_0^\infty P(d_m > \varepsilon) d\varepsilon = \int_{-\infty}^\infty \int_0^\infty P(d_m > \varepsilon | X = x) d\varepsilon f(x) dx \\ &= \int_{-\infty}^\infty \int_0^\infty P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \\ &\geq \int_{-\infty}^\infty \int_0^\infty P(X < x - \varepsilon)^m d\varepsilon f(x) dx \\ &= \int_{-\infty}^\infty \int_{-\infty}^x P(X < z)^m dz f(x) dx \end{aligned}$$

$$= \int_{-\infty}^\infty \int_z^\infty f(x) dx P(X < z)^m dz.$$

In the following sections we derive the upper and lower bounds for the expected nearest neighbor distance Ed_m for logistic and Laplace's distributions.

3. BOUNDS ON THE EXPECTED NEAREST NEIGHBOR DISTANCE FOR LOGISTIC DISTRIBUTION

Let X have a probability density function $f(x) = \frac{e^{-\frac{(x-a)}{b}}}{b \left(1 + e^{-\frac{(x-a)}{b}}\right)^2}$, $-\infty < x < \infty$, where a the location parameter and $b > 0$ the scale parameter. We use the method in the previous section to find the upper bound for the logistic distribution. Now, without loss of the generality, we assume that $a = 0$.

3.1 An upper bound for the logistic distribution

Using the method in (2.1), we take $-K_1(m) = K_2(m) > 0$, for $x \in R, t > 0$.

3.1.1 Bounding $L_1(m)$ and $L_2(m)$

Firstly, we evaluate $\varphi(t, x)$, i.e., we find the moment generating function of $|X - x|$. For $x \in R, 0 < t < 1$, we have

$$\begin{aligned} \varphi(x, t) &= E(e^{t|X-x|}) \leq E(e^{tx+tX}) \\ &= e^{tx} E(e^{tX}) = e^{tx} \int_{-\infty}^\infty e^{ty} \frac{e^{-\frac{y}{b}}}{b \left(1 + e^{-\frac{y}{b}}\right)^2} dy \\ &= e^{tx} \Gamma(1 - bt) \Gamma(1 + bt) = \frac{\pi bt}{\sin \pi bt}, \quad bt < 1, \end{aligned}$$

where Γ gamma function.

Hence, for $t = \frac{1}{2bm}$, we obtain $\varphi\left(x, \frac{1}{2bm}\right) \leq e^{\frac{x}{2bm}} \left(\frac{\pi}{\sin \frac{\pi}{2m}}\right)$. Therefore

$$\varphi\left(x, \frac{1}{2bm}\right)^m \leq e^{\frac{x}{2b}} \left[\left(\frac{\frac{\pi}{2m}}{\sin \frac{\pi}{2m}} \right) \right]^m.$$

84 It follows

$$\begin{aligned} L_1(m) + L_2(m) &= \int_{-\infty}^{K_1(m)} \int_0^\infty P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \\ &\quad + \int_{K_2(m)}^\infty \int_0^\infty P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \\ &\leq 4b \int_{K_2(m)}^\infty e^{\frac{x}{2b}} \left[\left(\frac{\frac{\pi}{2m}}{\sin \frac{\pi}{2m}} \right) \right]^m f(x) dx \\ &= 4b \left[\left(\frac{\frac{\pi}{2m}}{\sin \frac{\pi}{2m}} \right) \right]^m \int_{K_2(m)}^\infty \frac{e^{\frac{-x}{2b}}}{b \left(1 + e^{\frac{-x}{b}} \right)^2} dx \end{aligned}$$

85 Substituting $e^{\frac{-x}{2b}} = y$, $A = 4b \left[\left(\frac{\frac{\pi}{2m}}{\sin \frac{\pi}{2m}} \right) \right]^m$, we obtain

$$L_1(m) + L_2(m) \leq 2A \int_0^{e^{\frac{-K_2(m)}{2b}}} \frac{dy}{(1+y^2)^2}$$

86 Substituting $y = \tan \theta$, then

$$\begin{aligned} L_1(m) + L_2(m) &\leq 2A \int_0^{\tan^{-1} e^{\frac{-K_2(m)}{2b}}} \cos^2 \theta d\theta \\ &= A \int_0^{\tan^{-1} e^{\frac{-K_2(m)}{2b}}} (1 + \cos 2\theta) d\theta \\ &= A \left[\tan^{-1} e^{\frac{-K_2(m)}{2b}} + \frac{1}{2} \sin 2 \left(\tan^{-1} e^{\frac{-K_2(m)}{2b}} \right) \right] \end{aligned}$$

87 For $K_2(m) = 2b \log m$, it follows

$$L_1(m) + L_2(m) \leq 4b \left[\left(\frac{\frac{\pi}{2m}}{\sin \frac{\pi}{2m}} \right) \right]^m \left[\tan^{-1} \frac{1}{m} + \frac{1}{2} \sin \left(2 \tan^{-1} \frac{1}{m} \right) \right], \quad (3.1)$$

88 we note that $\left(\frac{\frac{\pi}{2m}}{\sin \frac{\pi}{2m}} \right) \rightarrow 1$ for large m .

89 **3.1.2 Bounding $L_3(m)$**

90 From (2.4) we have

$$\begin{aligned} L_3(m) &= \int_{K_1(m)}^{K_2(m)} \int_0^\infty P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \\ &= \int_{K_1(m)}^{K_2(m)} \int_0^\infty e^{-mG(x,\varepsilon)} f(x) d\varepsilon dx, (3.2) \end{aligned}$$

91 Where $G(x, \varepsilon) = -\log P(|X - x| > \varepsilon) \geq P(|X - x| \leq \varepsilon) = F(x + \varepsilon) - F(x - \varepsilon)$. (3.3)

92 Then we need good asymptotic estimates for $F(x + \varepsilon) - F(x - \varepsilon)$, as $(\varepsilon \rightarrow 0)$, By using the
93 Taylor expansion for the functions $F(x + \varepsilon)$ and $F(x - \varepsilon)$ we obtain

$$F(x + \varepsilon) = F(x) + \frac{f(x)\varepsilon}{1!} + \frac{f'(x)\varepsilon^2}{2!} + \frac{f''(x)\varepsilon^3}{3!} + \frac{f'''(x)\varepsilon^4}{4!} + \frac{f^{(4)}(x)\varepsilon^5}{5!} + \dots, (3.4)$$

$$F(x - \varepsilon) = F(x) - \frac{f(x)\varepsilon}{1!} + \frac{f'(x)\varepsilon^2}{2!} - \frac{f''(x)\varepsilon^3}{3!} + \frac{f'''(x)\varepsilon^4}{4!} - \frac{f^{(4)}(x)\varepsilon^5}{5!} + \dots. (3.5)$$

94 Substituting (3.4) and (3.5) in (3.3) yields

$$F(x + \varepsilon) - F(x - \varepsilon) = \frac{2f(x)\varepsilon}{1!} + \frac{2f''(x)\varepsilon^3}{3!} + \frac{2f^{(4)}(x)\varepsilon^5}{5!} + \dots \geq 2\varepsilon f(x),$$

95 since $f^{(n)}(x) \geq 0$ for $n = 0, 2, 4, \dots$, then we obtain $G(x, \varepsilon) \geq 2\varepsilon f(x)$. Hence

$$\begin{aligned} L_3(m) &\leq \int_{K_1(m)}^{K_2(m)} \int_0^\infty e^{-2m\varepsilon f(x)} f(x) d\varepsilon dx \\ &= \int_{K_1(m)}^{K_2(m)} \frac{1}{2m} dx = \frac{1}{2m} (K_2(m) - K_1(m)). \\ &= \frac{1}{2m} (2b \log m + 2b \log m) \leq \frac{2b \log m}{m}. (3.6) \end{aligned}$$

96 Substituting (3.1) and (3.6) in (2.1), we obtain the upper bound of Ed_m for the logistic
97 distribution

$$Ed_m \leq 4b \left[\left(\frac{\frac{\pi}{2m}}{\sin \frac{\pi}{2m}} \right) \right]^m \left[\tan^{-1} \frac{1}{m} + \frac{1}{2} \sin \left(2 \tan^{-1} \frac{1}{m} \right) \right] + \frac{2b \log m}{m}, (3.7)$$

98 **3.2 A lower bound for the logistic distribution**

99 Applying the method in section 2.2, we look at d_m in the case of logistic distribution with

100 density $\frac{\frac{-x}{e^{\frac{-x}{b}}}}{b \left(1 + e^{\frac{-x}{b}} \right)^2}$, $-\infty < x < \infty$. Then

$$\begin{aligned} Ed_m &= \int_0^\infty P(d_m > \varepsilon) d\varepsilon = \int_{-\infty}^\infty \int_0^\infty P(d_m > \varepsilon | X = x) d\varepsilon f(x) dx \\ &= \int_{-\infty}^\infty \int_0^\infty P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon \frac{e^{-\frac{x}{b}}}{b \left(1 + e^{-\frac{x}{b}}\right)^2} dx \quad (3.8) \\
 &\geq \int_{-\infty}^{\infty} \int_0^{\infty} P(X < x - \varepsilon)^m d\varepsilon \frac{e^{-\frac{x}{b}}}{b \left(1 + e^{-\frac{x}{b}}\right)^2} dx \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^x P(X < z)^m dz \frac{e^{-\frac{x}{b}}}{b \left(1 + e^{-\frac{x}{b}}\right)^2} dx \\
 &= \int_{-\infty}^{\infty} \int_z^{\infty} \frac{e^{-\frac{x}{b}}}{b \left(1 + e^{-\frac{x}{b}}\right)^2} dx P(X < z)^m dz \\
 &= \int_{-\infty}^{\infty} \frac{-e^{-\frac{z}{b}}}{\left(1 + e^{-\frac{z}{b}}\right)^{m+1}} dz \\
 &= \frac{b}{m}. \quad (3.9)
 \end{aligned}$$

101 Now, we can find an upper bound on the finite sample risk R_m in terms of the expected
 102 nearest neighbor distance for logistic distribution by using the following result:
 103 If, for some $\omega_1 > 0$ and $0 < \gamma \leq 1$ we have $|m(x) - m(x')| \leq \omega_1 \rho(x, x')^\gamma$, for all $x, x' \in \chi$, then
 104 for some suitable $\omega > 0$ independent of m ,

$$R_m \leq R_\infty + \omega \left[(Ed_m)^\gamma + (Ed_m^{2\gamma}) \right], \quad (3.10)$$

105 where $\omega = \max\{\omega_1, \omega_1^2\}$.
 106 This result is due to Irle and Rizk [6], for which they found an upper bound on the finite
 107 sample risk R_m in terms of the expected nearest neighbor distance. Putting $\gamma = \frac{1}{2}$ in (3.10), we
 108 obtain $R_m \leq R_\infty + \omega [\sqrt{Ed_m} + Ed_m]$. Hence, from (3.7) we have

$$\begin{aligned}
 R_m \leq R_\infty + \omega &\left[\left(4b \left[\left(\frac{\pi}{\sin \frac{\pi}{2m}} \right) \right]^m \left[\tan^{-1} \frac{1}{m} + \frac{1}{2} \sin \left(2 \tan^{-1} \frac{1}{m} \right) \right] + \frac{2b \log m}{m} \right)^{\frac{1}{2}} \right. \\
 &\left. + \left(4b \left[\left(\frac{\pi}{\sin \frac{\pi}{2m}} \right) \right]^m \left[\tan^{-1} \frac{1}{m} + \frac{1}{2} \sin \left(2 \tan^{-1} \frac{1}{m} \right) \right] + \frac{2b \log m}{m} \right) \right]. \quad (3.11)
 \end{aligned}$$

109 4. BOUNDS ON THE EXPECTED NEAREST NEIGHBOR DISTANCE FOR 110 LAPLACE'S DISTRIBUTION

111 We look at d_m in the case of the Laplace's distribution with probability density function
 112 $f(x) = \frac{1}{2\beta} e^{-\frac{|x|}{\beta}}$, where $\beta > 0, -\infty < x < \infty$.

113 4.1 An upper bound for the Laplace's distribution

114 Using the method in 2.1, we take $-K_1(m) = K_2(m) > 0$, for $x \in R, t > 0$.

115 **4.1.1 Bounding $L_1(m)$ and $L_2(m)$**

116 Now, we evaluate $\varphi(t, x)$, i.e, we find the moment generating function of $|X - x|$. For

117 $x \in R, 0 < t < 1$, we have

$$\begin{aligned}\varphi(t, x) &= E e^{t|X-x|} \leq E e^{t|X|+t|x|} \\ &= e^{t|x|} E e^{t|X|} = e^{t|x|} \int_{-\infty}^{\infty} \frac{1}{2\beta} e^{-\frac{|y|}{\beta}} e^{ty} dy \\ &= e^{t|x|} \left[\int_{-\infty}^0 \frac{1}{2\beta} e^{\frac{-(-y)}{\beta}} e^{ty} dy + \int_0^{\infty} \frac{1}{2\beta} e^{-\frac{y}{\beta}} e^{ty} dy \right] \\ &= \frac{e^{t|x|}}{2\beta} \left[\int_{-\infty}^0 e^{y(\frac{1}{\beta}+t)} dy + \int_0^{\infty} e^{-y(\frac{1}{\beta}-t)} dy \right] \\ &= \frac{e^{t|x|}}{2\beta} \left[\frac{\beta}{(1+\beta t)} + \frac{\beta}{(1-\beta t)} \right] = \frac{e^{t|x|}}{(1-\beta^2 t^2)}.\end{aligned}$$

118 Hence for $t = \frac{1}{2m} < \frac{1}{\beta}$

119 $\varphi\left(\frac{1}{2m}, x\right)^m \leq \frac{e^{\frac{|x|}{2}}}{\left(1-\frac{\beta^2}{4m^2}\right)^m} = e^{\frac{|x|}{2}} \left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m$. Then

$$\begin{aligned}L_1(m) + L_2(m) &\leq 4 \left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m \int_{K_2(m)}^{\infty} e^{\frac{x}{2}} \frac{1}{2\beta} e^{-\frac{x}{\beta}} dx \\ &= \frac{2}{\beta} \left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m \int_{K_2(m)}^{\infty} e^{-x(\frac{1}{\beta}-\frac{1}{2})} dx \\ &= 4 \left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m \frac{e^{-K_2(m)(\frac{1}{\beta}-\frac{1}{2})}}{(2-\beta)} \\ &= 4 \left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m \frac{e^{-K_2(m)(\frac{2-\beta}{2\beta})}}{(2-\beta)}\end{aligned}$$

120 For $K_2(m) = \log m^{\frac{2\beta}{2-\beta}}$, it follows

$$L_1(m) + L_2(m) \leq \frac{4}{2-\beta} \left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m \frac{1}{m} = O\left(\frac{1}{m}\right). \quad (4.1)$$

121 since $\left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m \rightarrow 1 (m \rightarrow \infty)$.

122 **4.1.2 Bounding $L_3(m)$**

123 From (2.4) we have

$$\begin{aligned}L_3(m) &= \int_{K_1(m)}^{K_2(m)} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \\ &= \int_{K_1(m)}^{K_2(m)} \int_0^{\infty} e^{-mG(x, \varepsilon)} f(x) d\varepsilon dx \quad (4.2)\end{aligned}$$

$$= 2 \int_0^{K_2(m)} \int_0^\infty e^{-mG(x,\varepsilon)} \frac{1}{2\beta} e^{-\frac{x}{\beta}} d\varepsilon dx,$$

124 where $G(x, \varepsilon) = -\log P(|X - x| > \varepsilon)$, and $f(x)$ even function. Similarly, as in the previous
 125 section we can obtain asymptotic estimates for $F(x + \varepsilon) - F(x - \varepsilon)$, as $(\varepsilon \rightarrow 0)$, by using the
 126 Taylor expansion for the functions $F(x + \varepsilon)$ and $F(x - \varepsilon)$ we obtain

$$F(x + \varepsilon) - F(x - \varepsilon) = \frac{2f(x)\varepsilon}{1!} + \frac{2f''(x)\varepsilon^3}{3!} + \frac{2f^{(4)}(x)\varepsilon^5}{5!} + \dots \geq 2\varepsilon f(x),$$

127 since $f^{(n)}(x) \geq 0$ for $n = 0, 2, 4, \dots$, then we obtain $G(x, \varepsilon) \geq 2\varepsilon f(x)$. Hence

$$\begin{aligned} L_3(m) &\leq \int_0^{K_2(m)} \int_0^\infty e^{-2m\varepsilon f(x)} f(x) d\varepsilon dx \\ &= \int_0^{K_2(m)} \frac{1}{2m} dx = \frac{K_2(m)}{2m}. \\ &= \frac{\log m^{\frac{2\beta}{2-\beta}}}{cm}. \end{aligned} \quad (4.3)$$

128 From (4.1), (4.3) and using $K_2(m) = (\log m)^{\frac{2\beta}{2-\beta}}$ we obtain the upper bound of Ed_m for
 129 Laplace distribution in the form:

$$Ed_m \leq \frac{4}{(2-\beta)m} \left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m + \frac{\log m^{\frac{2\beta}{2-\beta}}}{cm}. \quad (4.4)$$

130 Thus

$$\begin{aligned} R_m &\leq R_\infty + \lambda \sqrt{\frac{4}{(2-\beta)m} \left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m + \frac{\log m^{\frac{2\beta}{2-\beta}}}{cm}} \\ &\quad + \lambda \left[\frac{4}{(2-\beta)m} \left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m + \frac{\log m^{\frac{2\beta}{2-\beta}}}{cm} \right] \\ &\leq R_\infty + \lambda \left[\sqrt{\frac{C_3}{m} + \frac{\log m^{\frac{2\beta}{2-\beta}}}{cm}} + \frac{C_3}{m} + \frac{\log m^{\frac{2\beta}{2-\beta}}}{cm} \right] \end{aligned} \quad (4.5)$$

131 where, $C_3 = \frac{4}{(2-\beta)} \left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m$. Note that C_3 dependent on m and $\left(1 + \frac{\beta^2}{4m^2 - \beta^2}\right)^m \rightarrow 1$, as
 132 $m \rightarrow \infty$.

133 4.2 A lower bound for the Laplace's distribution

$$\begin{aligned} Ed_m &= \int_0^\infty P(d_m > \varepsilon) d\varepsilon \\ &= \int_{-\infty}^\infty \int_0^\infty P(d_m > \varepsilon | X = x) d\varepsilon f(x) dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon f(x) dx \\
 &= \int_{-\infty}^{\infty} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon \frac{1}{2\beta} e^{-\frac{|x|}{\beta}} dx \quad (4.6) \\
 &= \int_0^{\infty} \int_0^{\infty} P(|X - x| > \varepsilon)^m d\varepsilon \frac{1}{\beta} e^{-\frac{x}{\beta}} dx \\
 &\geq \int_0^{\infty} \int_0^{\infty} P(X < x - \varepsilon)^m d\varepsilon \frac{1}{\beta} e^{-\frac{x}{\beta}} dx \\
 &= \int_0^{\infty} \int_0^x P(X < z)^m dz \frac{1}{\beta} e^{-\frac{x}{\beta}} dx \\
 &= \int_0^{\infty} \int_0^x \left(1 - e^{-\frac{z}{\beta}}\right)^m dz \frac{1}{\beta} e^{-\frac{x}{\beta}} dx \\
 &= \int_0^{\infty} \int_z^{\infty} \frac{1}{\beta} e^{-\frac{x}{\beta}} dx \left(1 - e^{-\frac{z}{\beta}}\right)^m dz \\
 &= \int_0^{\infty} e^{-\frac{z}{\beta}} \left(1 - e^{-\frac{z}{\beta}}\right)^m dz \\
 &= \frac{\beta}{(m+1)}. \quad (4.7)
 \end{aligned}$$

134 5. CONCLUSION

135 We found upper and lower bounds on the expected nearest neighbor distance for
 136 distributions having unbounded supports $S = (-\infty, \infty)$, and induced lower and upper bounds
 137 for logistic and Laplace distributions, as typical. Then we found the risk of nearest neighbor of
 138 their distributions. Note that, from (3.7), (3.9) and (4.4), (4.7) the lower and upper bounds of
 139 the expected nearest neighbor distance are different in constants and the term $\log m$ and
 140 $\log m^{\frac{2\beta}{2-\beta}}$ respectively. That is, for the distributions have exponentially decaying tails there is
 141 an additional logarithmic term over the rates for compact support. This example illustrates
 142 that the expected nearest neighbor distance depends on the tails of the distribution.

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