

EVALUATION OF SOME CONVOLUTION SUMS AND REPRESENTATION NUMBERS OF QUADRATIC FORMS OF DISCRIMINANT -135.

ABSTRACT. We evaluate the convolution sums $\sum_{l+135m=n} \sigma(l) \sigma(m)$, $\sum_{3l+45m=n} \sigma(l) \sigma(m)$, $\sum_{5l+27m=n} \sigma(l) \sigma(m)$, $\sum_{9l+15m=n} \sigma(l) \sigma(m)$, $\sum_{l+45m=n} \sigma(l) \sigma(m)$, $\sum_{5l+9m=n} \sigma(l) \sigma(m)$, and $\sum_{3l+15m=n} \sigma(l) \sigma(m)$ for all $n \in \mathbb{N}$ using the theory of quasimodular forms and use some of these convolution sums to determine the number of representations of a positive integer n by some direct sum of the forms $x_1^2 + x_1x_2 + 34x_2^2$, $5x_1^2 + 5x_1x_2 + 8x_2^2$, $4x_1^2 \pm 3x_1x_2 + 9x_2^2$, $2x_1^2 \pm x_1x_2 + 17x_2^2$ of discriminant -135

1. INTRODUCTION

Let $\sigma(m)$ be the sum of positive divisors of a positive integer m . It is well known that divisor function σ appears in a number of remarkable identities, including relationship on the Riemann zeta function and the Eisenstein series of modular forms. It was studied by Ramanujan [14], who has found a number of important congruences and identities. It was also used in the counting of the number of nonisomorphic branched coverings of surfaces of genus g with a given ramification type σ , and in the orbitwise counting of $\mathfrak{H}(2)$, see [?]. On the other hand, the work on representation number $r(Q, n)$ of quadratic forms has been started by Fermat in 1640 on $Q = x^2 + y^2$. Later the formula $r(Q, n) = 4 \left(\sum_{d|n} d \text{ is odd} (-1)^{\frac{d-1}{2}} \right)$ has been proved by Euler. Afterwards it was advanced by Jacobi, see [?] with the proof of

$$r(Q, n) = 8 \left(\sum_{d|n} d \right), Q = x^2 + y^2 + z^2 + t^2.$$

It would be nice to obtain such simple formulas for other positive definite quadratic forms so that we would be able to understand the number of solutions of the equation $Q = n$ for any positive integer.

Here, we want to study some convolutions of divisor functions

$$W_N(n) := \sum_{\substack{m < n/N \\ m, n \in \mathbb{N}}} \sigma(m) \sigma(n - Nm),$$

$$W_{a,b}(n) := \sum_{\substack{al+bm=n \\ l, m \in \mathbb{N}}} \sigma(l) \sigma(m),$$

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where N, a, b positive integers. Some of them were calculated as early as 19th century. For example, $W_1(n)$ was evaluated by [6], [?], and [14]. The convolution sums $W_N(n)$ (for $1 \leq N \leq 24$ with a few exceptions) and $W_{a,b}(n)$ for $(a, b) \in \{(2, 3), (3, 4), (3, 8), (2, 9)\}$ have been evaluated by using either elementary methods or analytic methods (see [6], [?], [14], [?], [?], [?], [10], [?], [1], [2], [?], [4], [8], [9], [11], [16], [17]).

A different important algebraic method by means of quasimodular forms was introduced by Royer in [15] and it was applied in [13], in order to get $W_{15}, W_{3,5}$. The quasimodular forms was defined in [12]. The number of representations of integers by certain quadratic forms (see [10], [4], [2], [3], [16], [17], [13]) have been found by evaluation of these convolution sums. In this article, following the method of Royer [15], and using Magma for the calculations, we evaluate the following convolution sums

$$W_{135}(n)W_{3,45}(n), W_{5,27}(n), W_{9,15}(n)$$

by using the theory of quasimodular forms.

Although, in general it may be a challenging problem to describe the integer solutions to a polynomial equation of several variables, by using the above convolution sums we will determine the formulas for the number of representations of integers of some direct sum following positive definite quadratic forms

$$F = x_1^2 + x_1x_2 + 34x_2^2, \Lambda = 5x_1^2 + 5x_1x_2 + 8x_2^2,$$

$$\Phi = 4x_1^2 + 3x_1x_2 + 9x_2^2, \Phi^{-1} = 4x_1^2 - 3x_1x_2 + 9x_2^2,$$

$$\Psi = 2x_1^2 + x_1x_2 + 17x_2^2, \Psi^{-1} = 2x_1^2 - x_1x_2 + 17x_2^2,$$

2.

3. EVALUATION OF $W_{135}(n) W_{3,45}(n), W_{5,27}(n), W_{9,15}(n), W_{45}(n)$ $W_{3,15}(n), W_{5,9}(n)$,

The vector space $M_4[\Gamma_0(135)]$ is 60 dimensional and spanned by the 12 linearly

independent Eisenstein forms

$$E_4, E_4(3z), E_4(5z), E_4(9z), E_4(15z), E_4(27z), E_4(45z), E_4(135z),$$

$$E_4^{\psi,\psi}(z) = \sum_{n=1}^{+\infty} \sigma_3^{\psi,\psi}(n)q^n = q - 9q^2 + \cdots + 0q^{135} + O(q^{136}) \in M_4[\Gamma_0(9)],$$

$$E_4^{\psi,\psi}(3z), E_4^{\psi,\psi}(5z), E_4^{\psi,\psi}(15z),$$

where ψ is the Dirichlet character mod 3 sending 2 to -1, 32 old cusp forms

$$\begin{aligned} & \Delta_{4,5}(z), \Delta_{4,5}(5z), \Delta_{4,5}(9z), \Delta_{4,5}(27z), \\ & \Delta_{4,9}(z), \Delta_{4,9}(3z), \Delta_{4,9}(5z), \Delta_{4,9}(15z), \\ & \Delta_{4,15,1}(z), \Delta_{4,15,1}(3z), \Delta_{4,15,1}(9z), \Delta_{4,15,2}(z), \Delta_{4,15,2}(3z), \Delta_{4,15,2}(9z), \\ & \Delta_{4,27,1}(z), \Delta_{4,27,1}(5z), \Delta_{4,27,2}(z), \Delta_{4,27,2}(5z), \Delta_{4,27,3}(z), \Delta_{4,27,3}(5z), \\ & \Delta_{4,27,4}(z), \Delta_{4,27,4}(5z), \\ & \Delta_{4,45,1}(z), \Delta_{4,45,1}(3z), \Delta_{4,45,2}(z), \Delta_{4,45,2}(3z), \Delta_{4,45,3}(z), \Delta_{4,45,3}(3z), \\ & \Delta_{4,45,4}(z), \Delta_{4,45,4}(3z), \Delta_{4,45,5}(z), \Delta_{4,45,5}(3z), \\ & \Delta_{4,135,1}(z), \Delta_{4,135,2}(z), \Delta_{4,135,1}(z), \Delta_{4,135,1}(z), \Delta_{4,135,1}(z), \Delta_{4,135,1}(z), \\ & \Delta_{4,135,1}(z), \Delta_{4,135,1}(z), \end{aligned}$$

and 16 newforms

$$\begin{aligned} & \Delta_{4,135,1}(z), \Delta_{4,135,2}(z), \Delta_{4,135,3}(z), \Delta_{4,135,4}(z), \Delta_{4,135,5}(z), \\ & \Delta_{4,135,6}(z), \Delta_{4,135,7}(z), \Delta_{4,135,8}(z), \Delta_{4,135,9}(z), \Delta_{4,135,10}(z), \\ & \Delta_{4,135,11}(z), \Delta_{4,135,12}(z), \Delta_{4,135,13}(z), \Delta_{4,135,14}(z), \Delta_{4,135,15}(z), \Delta_{4,135,16}(z), \end{aligned}$$

The vector space $M_2[\Gamma_0(135)]$ is 24 dimensional and spanned by 11 Eisenstein series

$$\Phi_{1,3}, \Phi_{1,5}, \Phi_{1,9}, \Phi_{1,15}, \Phi_{1,27}, \Phi_{1,45}, \Phi_{1,135},$$

$$\begin{aligned} E_2^{\psi,\psi}(z) &= \sum_{n=1}^{+\infty} \sigma^{\psi,\psi}(n) q^n = q - 3q^2 + \cdots + 0q^{135} + O(q^{136}), \\ &E_2^{\psi,\psi}(3z), E_2^{\psi,\psi}(5z), E_2^{\psi,\psi}(15z) \end{aligned}$$

and old cusp forms

$$\Delta_{2,15}(z), \Delta_{2,15}(3z), \Delta_{2,15}(9z), \Delta_{2,27}(z), \Delta_{2,27}(5z), \Delta_{2,45}(z), \Delta_{2,45}(3z),$$

and 6 newforms

$$\Delta_{2,135,1}(z), \Delta_{2,135,2}(z), \Delta_{2,135,3}(z), \Delta_{2,135,4}(z), \Delta_{2,135,5}(z), \Delta_{2,135,6}(z).$$

Hence, by taking suitable coefficients of $q^n, n < 136$, from the equation

$$E_2(135z)E_2(z) = 1 - \sum_{n=1}^{+\infty} (24 \left(\sigma(n) + \sigma\left(\frac{n}{135}\right) \right) + 576W_{135}(n)) q^n,$$

we get

$$W_{135}(n) =$$

$$\begin{aligned}
& -1/540n\sigma(n) + 1/24\sigma(n) - 1/4n\sigma\left(\frac{n}{135}\right) + 1/24\sigma\left(\frac{n}{135}\right) + 1/50544\sigma_3(n) \\
& + 1/6318\sigma_3\left(\frac{n}{3}\right) + 25/50544\sigma_3\left(\frac{n}{5}\right) + 1/702\sigma_3\left(\frac{n}{9}\right) + 25/6318\sigma_3\left(\frac{n}{15}\right) \\
& + 3/208\sigma_3\left(\frac{n}{27}\right) + 25/702\sigma_3\left(\frac{n}{45}\right) + 75/208\sigma_3\left(\frac{n}{135}\right) - 2/5265\tau_{4,5}(n) \\
& - 25/7371\tau_{4,5}\left(\frac{n}{3}\right) - 25/819\tau_{4,5}\left(\frac{n}{9}\right) - 18/65\tau_{4,5}\left(\frac{n}{27}\right) - 2/1215\tau_{4,9}(n) \\
& - 2/135\tau_{4,9}\left(\frac{n}{3}\right) - 10/243\tau_{4,9}\left(\frac{n}{5}\right) - 10/27\tau_{4,9}\left(\frac{n}{15}\right) - 1/720\tau_{4,15,1}(n) \\
& - 1/80\tau_{4,15,1}\left(\frac{n}{3}\right) - 9/80\tau_{4,15,1}\left(\frac{n}{9}\right) - 1/324\tau_{4,15,2}(n) - 31/756\tau_{4,15,2}\left(\frac{n}{3}\right) \\
& - 1/4\tau_{4,15,2}\left(\frac{n}{9}\right) - 2/1215\tau_{4,27,1}(n) - 10/243\tau_{4,27,1}\left(\frac{n}{5}\right) \\
& + (-1/1836a6 - 7/2754)\tau_{4,27,3}(n) + (-25/1836a6 - 175/2754)\tau_{4,27,3}\left(\frac{n}{5}\right) \\
& - 1/1134\tau_{4,45,2}(n) - 1/126\tau_{4,45,2}\left(\frac{n}{3}\right) - 1/189\tau_{4,45,3}(n) - 1/21\tau_{4,45,3}\left(\frac{n}{3}\right)
\end{aligned}$$

$$\begin{aligned}
& -1/405\tau_{4,45,4}(n) - 1/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 1/918\tau_{4,135,2}(n) + 1/4565970(32a1^2 \\
& - 1206a1 - 6989)\tau_{4,135,5}(n) + (1/2282985(-16a1 - 587)a2 \\
& + 1/4565970(-32a1^2 + 32a1 - 6253))\tau_{4,135,6}(n) + (1/2282985(16a1 + 587)a2 \\
& + 1/4565970(1174a1 - 7427))\tau_{4,135,7}(n) + 1/304398(-10a1^2 \\
& + 142a1 - 517)\tau_{4,135,8}(n) + (1/152199(5a1 + 66)a2 + 1/304398(10a1^2 \\
& - 10a1 - 747))\tau_{4,135,9}(n) + (1/152199(-5a1 - 66)a2 + 1/101466(-44a1 \\
& - 205))\tau_{4,135,10}(n) + (1/11961a3^2 - 17/35883a3 - 173/71766)\tau_{4,135,11}(n) \\
& + ((-1/11961a3 - 2/35883)a4 + (-1/11961a3^2 + 5/11961a3 \\
& - 125/71766))\tau_{4,135,12}(n) + ((1/11961a3 + 2/35883)a4 + (2/35883a3 \\
& - 145/71766))\tau_{4,135,13}(n).
\end{aligned}$$

Similarly, we find

$$W_{3,45}(n) =$$

$$\begin{aligned}
& -1/180n\sigma\left(\frac{n}{3}\right) + 1/24\sigma\left(\frac{n}{3}\right) - 7/180n\sigma\left(\frac{n}{45}\right) + 1/24\sigma\left(\frac{n}{45}\right) \\
& + 1/45n\sigma\left(\frac{n}{135}\right) + 1/624\sigma_3\left(\frac{n}{3}\right) + 3/208\sigma_3\left(\frac{n}{9}\right) + 25/624\sigma_3\left(\frac{n}{15}\right) \\
& - 133/208\sigma_3\left(\frac{n}{45}\right) + \sigma_3\left(\frac{n}{135}\right) - 1/455\tau_{4,5}\left(\frac{n}{3}\right) - 9/455\tau_{4,5}\left(\frac{n}{9}\right) \\
& - 1/80\tau_{4,15,1}\left(\frac{n}{3}\right) - 1/84\tau_{4,15,2}\left(\frac{n}{3}\right)
\end{aligned}$$

$$W_{5,27}(n) =$$

$$\begin{aligned}
& -1/108n\sigma\left(\frac{n}{5}\right) + 1/24\sigma\left(\frac{n}{5}\right) - 1/20n\sigma\left(\frac{n}{27}\right) + 1/24\sigma\left(\frac{n}{27}\right) + 1/50544\sigma_3(n) \\
& + 1/6318\sigma_3\left(\frac{n}{3}\right) + 25/50544\sigma_3\left(\frac{n}{5}\right) + 1/702\sigma_3\left(\frac{n}{9}\right) + 25/6318\sigma_3\left(\frac{n}{15}\right) \\
& + 3/208\sigma_3\left(\frac{n}{27}\right) + 25/702\sigma_3\left(\frac{n}{45}\right) + 75/208\sigma_3\left(\frac{n}{135}\right) - 2/5265\tau_{4,5}(n) \\
& - 25/7371\tau_{4,5}\left(\frac{n}{3}\right) - 25/819\tau_{4,5}\left(\frac{n}{9}\right) - 18/65\tau_{4,5}\left(\frac{n}{27}\right) - 2/1215\tau_{4,9}(n) \\
& - 2/135\tau_{4,9}\left(\frac{n}{3}\right) - 10/243\tau_{4,9}\left(\frac{n}{5}\right) - 10/27\tau_{4,9}\left(\frac{n}{15}\right) + 1/720\tau_{4,15,1}(n) \\
& + 1/80\tau_{4,15,1}\left(\frac{n}{3}\right) + 9/80\tau_{4,15,1}\left(\frac{n}{9}\right) - 1/324\tau_{4,15,2}(n) - 31/756\tau_{4,15,2}\left(\frac{n}{3}\right) \\
& - 1/4\tau_{4,15,2}\left(\frac{n}{9}\right) - 2/1215\tau_{4,27,1}(n) - 10/243\tau_{4,27,1}\left(\frac{n}{5}\right) + (-1/1836a6 - \\
& 7/2754)\tau_{4,27,3}(n) + (-25/1836a6 - 175/2754)\tau_{4,27,3}\left(\frac{n}{5}\right) + (1/1836a6
\end{aligned}$$

$$\begin{aligned}
& -7/2754)\tau_{4,27,4}(n) + (25/1836a6 - 175/2754)\tau_{4,27,4}\left(\frac{n}{5}\right) + 1/1134\tau_{4,45,2}(n) \\
& + 1/126\tau_{4,45,2}\left(\frac{n}{3}\right) + 1/189\tau_{4,45,3}(n) + 1/21\tau_{4,45,3}\left(\frac{n}{3}\right) - 1/405\tau_{4,45,4}(n) \\
& - 1/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 1/918\tau_{4,135,2}(n) + 1/4565970(32a1^2 - 1206a1 \\
& - 6989)\tau_{4,135,5}(n) + (1/2282985(-16a1 - 587)a2 + 1/4565970(-32a1^2 \\
& + 32a1 - 6253))\tau_{4,135,6}(n) + (1/2282985(16a1 + 587)a2 + 1/4565970(1174a1 \\
& - 7427))\tau_{4,135,7}(n) + 1/304398(10a1^2 - 142a1 + 517)\tau_{4,135,8}(n) \\
& + (1/152199(-5a1 - 66)a2 + 1/304398(-10a1^2 + 10a1 + 747))\tau_{4,135,9}(n) \\
& + (1/152199(5a1 + 66)a2 + 1/101466(44a1 + 205))\tau_{4,135,10}(n) \\
& + (-1/11961a3^2 + 17/35883a3 + 173/71766)\tau_{4,135,11}(n) + ((1/11961a3 \\
& + 2/35883)a4 + (1/11961a3^2 - 5/11961a3 + 125/71766))\tau_{4,135,12}(n) \\
& + ((-1/11961a3 - 2/35883)a4 + (-2/35883a3 + 145/71766))\tau_{4,135,13}(n)
\end{aligned}$$

$$W_{9,15}(n) =$$

$$\begin{aligned}
& -1/60n\sigma\left(\frac{n}{9}\right) + 1/24\sigma\left(\frac{n}{9}\right) - 1/36n\sigma\left(\frac{n}{15}\right) + 1/24\sigma\left(\frac{n}{15}\right) + 1/624\sigma_3\left(\frac{n}{3}\right) \\
& + 3/208\sigma_3\left(\frac{n}{9}\right) + 25/624\sigma_3\left(\frac{n}{15}\right) + 75/208\sigma_3\left(\frac{n}{45}\right) - 1/455\tau_{4,5}\left(\frac{n}{3}\right) \\
& - 9/455\tau_{4,5}\left(\frac{n}{9}\right) + 1/80\tau_{4,15,1}\left(\frac{n}{3}\right) - 1/84\tau_{4,15,2}\left(\frac{n}{3}\right)
\end{aligned}$$

$$W_{45} =$$

$$\begin{aligned}
& -1/180n\sigma(n) + 1/24\sigma(n) - 1/4n\sigma\left(\frac{n}{45}\right) + 1/24\sigma\left(\frac{n}{45}\right) \\
& + 1/5616\sigma_3(n) + 1/702\sigma_3\left(\frac{n}{3}\right) + 25/5616\sigma_3\left(\frac{n}{5}\right) \\
& + 3/208\sigma_3\left(\frac{n}{9}\right) + 25/702\sigma_3\left(\frac{n}{15}\right) + 75/208\sigma_3\left(\frac{n}{45}\right) \\
& - 17/16380\tau_{4,5}(n) - 11/1638\tau_{4,5}\left(\frac{n}{3}\right) - \\
& 153/1820\tau_{4,5}\left(\frac{n}{9}\right) - 1/270\tau_{4,9}(n) - 5/54\tau_{4,9}\left(\frac{n}{5}\right) \\
& - 1/240\tau_{4,15,1}(n) - 3/80\tau_{4,15,1}\left(\frac{n}{3}\right) - 1/126\tau_{4,15,2}(n) \\
& - 1/14\tau_{4,15,2}\left(\frac{n}{3}\right) - 1/504\tau_{4,45,2}(n) - 1/84\tau_{4,45,3}(n) \\
& - 1/180\tau_{4,45,4}(n)
\end{aligned}$$

$$W_{3,15} =$$

$$\begin{aligned}
& -1/60n\sigma\left(\frac{n}{3}\right) + 1/24\sigma\left(\frac{n}{3}\right) - 1/12n\sigma\left(\frac{n}{15}\right) + 1/24\sigma\left(\frac{n}{15}\right) \\
& + 5/312\sigma_3\left(\frac{n}{3}\right) + 125/312\sigma_3\left(\frac{n}{15}\right) - 1/130\tau_{4,5}\left(\frac{n}{3}\right)
\end{aligned}$$

$$W_{5,9} =$$

$$\begin{aligned}
& -1/36n\sigma\left(\frac{n}{5}\right) + 1/24\sigma\left(\frac{n}{5}\right) - 1/20n\sigma\left(\frac{n}{9}\right) + 1/24\sigma\left(\frac{n}{9}\right) \\
& + 1/5616\sigma_3(n) + 1/702\sigma_3\left(\frac{n}{3}\right) + 25/5616\sigma_3\left(\frac{n}{5}\right) \\
& + 3/208\sigma_3\left(\frac{n}{9}\right) + 25/702\sigma_3\left(\frac{n}{15}\right) + 75/208\sigma_3\left(\frac{n}{45}\right) \\
& - 17/16380\tau_{4,5}(n) - 11/1638\tau_{4,5}\left(\frac{n}{3}\right) - 153/1820\tau_{4,5}\left(\frac{n}{9}\right) \\
& - 1/270\tau_{4,9}(n) - 5/54\tau_{4,9}\left(\frac{n}{5}\right) + 1/240\tau_{4,15,1}(n) \\
& + 3/80\tau_{4,15,1}\left(\frac{n}{3}\right) - 1/126\tau_{4,15,2}(n) - 1/14\tau_{4,15,2}\left(\frac{n}{3}\right) \\
& + 1/504\tau_{4,45,2}(n) + 1/84\tau_{4,45,3}(n) - 1/180\tau_{4,45,4}(n)
\end{aligned}$$

Obviously, the formula for $W_{3,45}(n)$ and $W_{9,15}(n)$ can also be obtained by the formula of $W_{15}(n)$, $W_{3,5}(n)$ (see [13]) since

$$W_{3,45}(n) = W_{15}\left(\frac{n}{3}\right), W_{9,15}(n) = W_{3,5}\left(\frac{n}{3}\right).$$

4. APPLICATION TO QUADRATIC FORMS

$$\begin{aligned} F &= x_1^2 + x_1x_2 + 34x_2^2, \Lambda = 5x_1^2 + 5x_1x_2 + 8x_2^2, \\ \Phi &= 4x_1^2 + 3x_1x_2 + 9x_2^2, \Phi^{-1} = 4x_1^2 - 3x_1x_2 + 9x_2^2, \\ \Psi &= 2x_1^2 + x_1x_2 + 17x_2^2, \Psi^{-1} = 2x_1^2 - x_1x_2 + 17x_2^2, \end{aligned}$$

Consider the following 21 quadratic forms of 4 variables

$$F_2, \Lambda_2, \Phi_2, \Psi_2, F \oplus \Lambda, F \oplus \Phi, F \oplus \Psi, \Lambda \oplus \Phi, \Lambda \oplus \Psi, \Phi \oplus \Psi,$$

where $F_2 := F \oplus F$. Their theta functions are in $M_2(\Gamma_0(135))$, so we can determine the representation numbers by

$$r_{F_2}(n) =$$

$$\begin{aligned} &2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right) - 40/9\sigma\left(\frac{n}{15}\right) - 6\sigma\left(\frac{n}{27}\right) \\ &+ 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) + 2/9\tau_{2,15}(n) + 2/9\tau_{2,15}\left(\frac{n}{3}\right) + 2\tau_{2,15}\left(\frac{n}{9}\right) \\ &+ 8/9\tau_{2,27}(n) + 40/9\tau_{2,27}\left(\frac{n}{5}\right) + 8/9\tau_{2,135,2}(n) + (16/117a5 + 20/39)\tau_{2,135,3}(n) \\ &+ (-16/117a5 + 44/117)\tau_{2,135,4}(n) + (-16/117a5 + 44/117)\tau_{2,135,5}(n) \\ &+ (16/117a5 + 20/39)\tau_{2,135,6}(n) \end{aligned}$$

$$r_{\Lambda_2}(n) =$$

$$\begin{aligned} &2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right) - 40/9\sigma\left(\frac{n}{15}\right) \\ &- 6\sigma\left(\frac{n}{27}\right) + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) - 2/9\tau_{2,15}(n) - 2/9\tau_{2,15}\left(\frac{n}{3}\right) \\ &- 2\tau_{2,15}\left(\frac{n}{9}\right) + 8/9\tau_{2,27}(n) + 40/9\tau_{2,27}\left(\frac{n}{5}\right) - 8/9\tau_{2,135}\left(\frac{n}{2}\right) \\ &+ (-16/117a5 - 20/39)\tau_{2,135,3}(n) + (16/117a5 - 44/117)\tau_{2,135,4}(n) \\ &+ (-16/117a5 + 44/117)\tau_{2,135,5}(n) + (16/117 * a5 + 20/39)\tau_{2,135,6}(n) \end{aligned}$$

$$r_{\Phi_2}(n) =$$

$$\begin{aligned} &2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right) - 40/9\sigma\left(\frac{n}{15}\right) \\ &- 6\sigma\left(\frac{n}{27}\right) + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) + 2/9\tau_{2,15}(n) + 2/9\tau_{2,15}\left(\frac{n}{3}\right) \\ &+ 2\tau_{2,15}\left(\frac{n}{9}\right) - 4/9\tau_{2,27}(n) - 20/9\tau_{2,27}\left(\frac{n}{5}\right) - 4/9\tau_{2,135,2}(n) \\ &+ (-14/117a5 + 2/39)\tau_{2,135,3}(n) + (14/117a5 + 20/117)\tau_{2,135,4}(n) \\ &+ (-22/117a5 + 2/117)\tau_{2,135,5}(n) + (22/117a5 + 8/39)\tau_{2,135,6}(n) \end{aligned}$$

$$r_{\psi_2}(n) =$$

$$\begin{aligned}
& 2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right) - 40/9\sigma\left(\frac{n}{15}\right) - 6\sigma\left(\frac{n}{27}\right) \\
& + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) - 2/9\tau_{2,15}(n) - 2/9\tau_{2,15}\left(\frac{n}{3}\right) - 2\tau_{2,15}\left(\frac{n}{9}\right) \\
& - 4/9\tau_{2,27}(n) - 20/9\tau_{2,27}\left(\frac{n}{5}\right) + 4/9\tau_{2,135,2}(n) + (14/117a5 - 2/39)\tau_{2,135,3}(n) \\
& + (-14/117a5 - 20/117)\tau_{2,135,4}(n) + (-22/117a5 + 2/117)\tau_{2,135,5}(n) \\
& + (22/117a5 + 8/39)\tau_{2,135,6}(n)
\end{aligned}$$

$$r_{F\oplus\Lambda}(n) =$$

$$\begin{aligned}
& 1/6\sigma(n) + 1/3\sigma\left(\frac{n}{3}\right) - 5/6\sigma\left(\frac{n}{5}\right) + \sigma\left(\frac{n}{9}\right) - 5/3\sigma\left(\frac{n}{15}\right) + 9/2\sigma\left(\frac{n}{27}\right) \\
& - 5\sigma\left(\frac{n}{45}\right) - 45/2\sigma\left(\frac{n}{135}\right) + 17/18\tau_{2,15}(n) - 1/18\tau_{2,15}\left(\frac{n}{3}\right) + 17/2\tau_{2,15}\left(\frac{n}{9}\right) \\
& + 4/9\tau_{2,135,2}(n) + (-4/117a5 + 8/39)\tau_{2,135,3}(n) + (4/117a5 + 28/117)\tau_{2,135,4}(n)
\end{aligned}$$

$$r_{F\oplus\Phi}(n) =$$

$$\begin{aligned}
& 2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right) - 40/9\sigma\left(\frac{n}{15}\right) \\
& - 6\sigma\left(\frac{n}{27}\right) + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) + 2/9\tau_{2,15}(n) + 2/9\tau_{2,15}\left(\frac{n}{3}\right) \\
& + 2\tau_{2,15}\left(\frac{n}{9}\right) + 2/9\tau_{2,27}(n) + 10/9\tau_{2,27}\left(\frac{n}{5}\right) + 2/9\tau_{2,135,2}(n) \\
& + (-17/117a5 + 8/39)\tau_{2,135,3}(n) + (17/117a5 + 41/117)\tau_{2,135,4}(n) \\
& + (-1/117a5 + 32/117)\tau_{2,135,5}(n) + (1/117a5 + 11/39)\tau_{2,135,6}(n)
\end{aligned}$$

$$r_{F\oplus\Psi}(n) =$$

$$\begin{aligned}
& 1/6\sigma(n) + 1/3\sigma\left(\frac{n}{3}\right) - 5/6\sigma\left(\frac{n}{5}\right) + \sigma\left(\frac{n}{9}\right) - 5/3\sigma\left(\frac{n}{15}\right) \\
& + 9/2\sigma\left(\frac{n}{27}\right) - 5\sigma\left(\frac{n}{45}\right) - 45/2\sigma\left(\frac{n}{135}\right) + 5/18\tau_{2,15}(n) \\
& + 23/18\tau_{2,15}\left(\frac{n}{3}\right) + 5/2\tau_{2,15}\left(\frac{n}{9}\right) + 2/3\tau_{2,45}(n) + 2\tau_{2,45}\left(\frac{n}{3}\right) \\
& + 4/9\tau_{2,135,2}(n) + (-1/117a5 + 2/39)\tau_{2,135,3}(n) + (1/117a5 \\
& + 7/117)\tau_{2,135,4}(n) + (-1/117a5 + 2/39)\tau_{2,135,5}(n) + (1/117a5 \\
& + 7/117)\tau_{2,135,6}(n)
\end{aligned}$$

$$r_{\Lambda\oplus\Phi}(n) =$$

$$\begin{aligned}
& 1/6\sigma(n) + 1/3\sigma\left(\frac{n}{3}\right) - 5/6\sigma\left(\frac{n}{5}\right) + \sigma\left(\frac{n}{9}\right) - 5/3\sigma\left(\frac{n}{15}\right) \\
& + 9/2\sigma\left(\frac{n}{27}\right) - 5\sigma\left(\frac{n}{45}\right) - 45/2\sigma\left(\frac{n}{135}\right) + 5/18\tau_{2,15}(n) \\
& + 23/18\tau_{2,15}\left(\frac{n}{3}\right) + 5/2\tau_{2,15}\left(\frac{n}{9}\right) - 2/3\tau_{2,45}(n) - 2\tau_{2,45}\left(\frac{n}{3}\right) \\
& + 4/9\tau_{2,135,2}(n) + (-1/117a5 + 2/39)\tau_{2,135,3}(n) + (1/117a5 \\
& + 7/117)\tau_{2,135,4}(n) + (1/117a5 - 2/39)\tau_{2,135,5}(n) + (-1/117a5 \\
& - 7/117)\tau_{2,135,6}(n)
\end{aligned}$$

$$r_{\Lambda\oplus\Psi}(n) =$$

$$\begin{aligned}
& 2/9\sigma(n) - 8/9\sigma\left(\frac{n}{3}\right) + 10/9\sigma\left(\frac{n}{5}\right) + 8/3\sigma\left(\frac{n}{9}\right) - 40/9\sigma\left(\frac{n}{15}\right) \\
& - 6\sigma\left(\frac{n}{27}\right) + 40/3\sigma\left(\frac{n}{45}\right) - 30\sigma\left(\frac{n}{135}\right) - 2/9\tau_{2,15}(n) - 2/9\tau_{2,15}\left(\frac{n}{3}\right) \\
& - 2\tau_{2,15}\left(\frac{n}{9}\right) + 2/9\tau_{2,27}(n) + 10/9\tau_{2,27}\left(\frac{n}{5}\right) - 2/9\tau_{2,135,2}(n) \\
& + (17/117a5 - 8/39)\tau_{2,135,3}(n) + (-17/117a5 - 41/117)\tau_{2,135,4}(n) \\
& + (-1/117a5 + 32/117)\tau_{2,135,5}(n) + (1/117a5 + 11/39)\tau_{2,135,6}(n)
\end{aligned}$$

$$r_{\Phi \oplus \Psi}(n) =$$

$$\begin{aligned}
& 1/6\sigma(n) + 1/3\sigma\left(\frac{n}{3}\right) - 5/6\sigma\left(\frac{n}{5}\right) + \sigma\left(\frac{n}{9}\right) - 5/3\sigma\left(\frac{n}{15}\right) \\
& + 9/2\sigma\left(\frac{n}{27}\right) - 5\sigma\left(\frac{n}{45}\right) - 45/2\sigma\left(\frac{n}{135}\right) - 7/18\tau_{2,15}(n) \\
& - 25/18\tau_{2,15}\left(\frac{n}{3}\right) - 7/2\tau_{2,15}\left(\frac{n}{9}\right) + 4/9\tau_{2,135,2}(n) + (2/117a5 \\
& - 4/39)\tau_{2,135,3}(n) + (-2/117a5 - 14/117)\tau_{2,135,4}(n)
\end{aligned}$$

Now, consider the following 21 quadratic forms of 8 variables

$$F_2 := F \oplus F \oplus F \oplus F,$$

$$\begin{aligned}
& F_4, \Lambda_4, \Phi_4, \Psi_4, F_3 \oplus \Lambda, F_3 \oplus \Phi, F_3 \oplus \Psi, \\
& \Lambda_3 \oplus F, \Lambda_3 \oplus \Phi, \Lambda_3 \oplus \Psi, \Phi_3 \oplus F, \Phi_3 \oplus \Psi, \\
& \Psi_3 \oplus F, \Psi_3 \oplus \Lambda, \Psi_3 \oplus \Phi, F_2 \oplus \Lambda_2, F_2 \oplus \Phi_2 \\
& F_2 \oplus \Psi_2, \Lambda_2 \oplus \Phi_2, \Lambda_2 \oplus \Psi_2, \Phi_2 \oplus \Psi_2.
\end{aligned}$$

Their theta functions are in $M_4(\Gamma_0(135))$, so we can determine the representation numbers by

$$r_{F_4}(n) =$$

$$\begin{aligned}
& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
& + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160\sigma_3\left(\frac{n}{27}\right) \\
& + 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) \\
& - 64/1053\tau_{4,5}(n) - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) - 4000/819\tau_{4,5}\left(\frac{n}{9}\right) \\
& - 576/13\tau_{4,5}\left(\frac{n}{27}\right) + 64/135\tau_{4,9}(n) + 64/15\tau_{4,9}\left(\frac{n}{3}\right) \\
& + 320/27\tau_{4,9}\left(\frac{n}{5}\right) + 320/3\tau_{4,9}\left(\frac{n}{15}\right) + 4/9\tau_{4,15,1}(n) \\
& + 4\tau_{4,15,1}\left(\frac{n}{3}\right) + 36\tau_{4,15,1}\left(\frac{n}{9}\right) + 8/9\tau_{4,15,2}(n) \\
& + 248/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 72\tau_{4,15,1}\left(\frac{n}{9}\right) + 64/135\tau_{4,27,1}(n) \\
& + 320/27\tau_{4,27,1}\left(\frac{n}{5}\right) + (56/459a6 + 8/17)\tau_{4,27,3}(n) + (1400/459a6 \\
& + 200/17)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-56/459a6 + 8/17)\tau_{4,27,4}(n) \\
& + (-1400/459a6 + 200/17)\tau_{4,27,4}\left(\frac{n}{5}\right) + 32/27\tau_{4,45,3}(n) \\
& + 32/3\tau_{4,45,3}\left(\frac{n}{3}\right) + 128/405\tau_{4,45,4}(n) + 128/45\tau_{4,45,4}\left(\frac{n}{3}\right) \\
& + 112/459\tau_{4,135,2}(n) + 1/2282985(-35848a1^2 \\
& + 106184a1 + 1067376)\tau_{4,135,5}(n) + (1/2282985(35848a1
\end{aligned}$$

$$\begin{aligned}
& + 70336)a2 + 1/2282985(35848a1^2 - 35848a1 \\
& + 242872))\tau_{4,135,6}(n) + (1/2282985(-35848a1 - 70336)a2 \\
& + 1/2282985(-70336a1 + 313208))\tau_{4,135,7}(n) \\
& (1/50733(-104a1^2 - 4536a1 + 23184))\tau_{4,135,8}(n) \\
& + (1/50733(104a1 - 4640)a2 + 1/50733(104a1^2 - 104a1 \\
& + 20792))\tau_{4,135,9}(n) + (1/50733(-104a1 + 4640)a2 \\
& + 1/50733(4640a1 + 16152))\tau_{4,135,10}(n) + (-1928/35883a3^2 \\
& + 6200/35883a3 + 10064/11961)\tau_{4,135,11}(n) \\
& + ((1928/35883a3 - 3440/35883)a4 + 1928/35883a3^2 \\
& - 9640/35883a3 + 14768/35883)\tau_{4,135,12}(n) + ((-1928/35883a3 \\
& + 3440/35883)a4 + 3440/35883a3 - 2432/35883)\tau_{4,135,13}(n)
\end{aligned}$$

$$r_{\Lambda_4}(n) =$$

$$\begin{aligned}
& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
& + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) \\
& + 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) \\
& - 64/1053\tau_{4,5}(n) - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) - 4000/819\tau_{4,5}\left(\frac{n}{9}\right) \\
& - 576/13\tau_{4,5}\left(\frac{n}{27}\right) + 64/135\tau_{4,9}(n) + 64/15\tau_{4,9}\left(\frac{n}{3}\right) \\
& + 320/27\tau_{4,9}\left(\frac{n}{5}\right) + 320/3\tau_{4,9}\left(\frac{n}{15}\right) - 4/9\tau_{4,15,1}(n) \\
& - 4\tau_{4,15,1}\left(\frac{n}{3}\right) - 36\tau_{4,15,1}\left(\frac{n}{9}\right) + 8/9\tau_{4,15,2}(n) \\
& + 248/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 72\tau_{4,15,1}\left(\frac{n}{9}\right) + 64/135\tau_{4,27,1}(n) \\
& + 320/27\tau_{4,27,1}\left(\frac{n}{5}\right) + (56/459a6 + 8/17)\tau_{4,27,3}(n) \\
& + (1400/459a6 + 200/17)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-56/459a6 \\
& + 8/17)\tau_{4,27,4}(n) + (-1400/459a6 + 200/17)\tau_{4,27,4}\left(\frac{n}{5}\right) \\
& - 32/27\tau_{4,45,3}(n) - 32/3\tau_{4,45,3}\left(\frac{n}{3}\right) + 128/405\tau_{4,45,4}(n) \\
& + 128/45\tau_{4,45,4}\left(\frac{n}{3}\right) + \tau_{4,135,1}(n) + 112/459\tau_{4,135,2}(n)
\end{aligned}$$

$$\begin{aligned}
& + 1/2282985(-35848a1^2 + 106184a1 + 1067376)\tau_{4,135,5}(n) \\
& + (1/2282985(35848a1 + 70336)a2 + 1/2282985(35848a1^2 \\
& - 35848a1 + 242872))\tau_{4,135,6}(n) + (1/2282985(-35848a1 \\
& - 70336)a2 + 1/2282985(-70336a1 + 313208))\tau_{4,135,7}(n) \\
& (1/50733(104a1^2 + 4536a1 - 23184))\tau_{4,135,8}(n) \\
& + (1/50733(-104a1 + 4640)a2 + 1/50733(-104a1^2 \\
& + 104a1 - 20792))\tau_{4,135,9}(n) + (1/50733(104a1 - 4640)a2 \\
& + 1/50733(-4640a1 - 16152))\tau_{4,135,10}(n) \\
& + (1928/35883a3^2 - 6200/35883a3 - 10064/11961)\tau_{4,135,11}(n) \\
& ((-1928/35883a3 + 3440/35883)a4 - 1928/35883a3^2 \\
& + 9640/35883a3 - 14768/35883)\tau_{4,135,12}(n) + ((1928/35883a3 \\
& - 3440/35883)a4 - 3440/35883a3 + 2432/35883)\tau_{4,135,13}(n)
\end{aligned}$$

$$r_{\Phi_4}(n) =$$

$$\begin{aligned}
 & 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
 & + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) \\
 & + 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) - 64/1053\tau_{4,5}(n) \\
 & - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) - 4000/819\tau_{4,5}\left(\frac{n}{9}\right) - 576/13\tau_{4,5}\left(\frac{n}{27}\right) \\
 & - 32/135\tau_{4,9}(n) - 32/15\tau_{4,9}\left(\frac{n}{3}\right) - 160/27\tau_{4,9}\left(\frac{n}{5}\right) \\
 & - 160/3\tau_{4,9}\left(\frac{n}{15}\right) - 4/3\tau_{4,15,1}\left(\frac{n}{3}\right) + 8/7\tau_{4,15,2}\left(\frac{n}{3}\right) \\
 & + (28/459a6 + 4/17)\tau_{4,27,3}(n) + (700/459a6 + 100/17)\tau_{4,27,3}\left(\frac{n}{5}\right) \\
 & + (-28/459a6 + 4/17)\tau_{4,27,4}(n) + (-700/459a6 \\
 & + 100/17)\tau_{4,27,4}\left(\frac{n}{5}\right) - 4/21\tau_{4,45,2}(n) - 12/7\tau_{4,45,2}\left(\frac{n}{3}\right) \\
 & + 64/189\tau_{4,45,3}(n) + 64/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 56/405\tau_{4,45,4}(n) \\
 & + 56/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 80/459\tau_{4,135,2}(n) + (1/456597(7708a1^2
 \end{aligned}$$

$$\begin{aligned}
 & + 280a1 - 143400))\tau_{4,135,5}(n) + (1/456597(-7708a1 + 7988)a2 \\
 & + 1/456597(-7708a1^2 + 7708a1 + 33884))\tau_{4,135,6}(n) \\
 & + (1/456597(7708a1 - 7988)a2 + 1/456597(-7988a1 \\
 & + 41872))\tau_{4,135,7}(n) + (1/152199(1828a1^2 - 6416a1 \\
 & - 18984))\tau_{4,135,8}(n) + (1/152199(-1828a1 - 4588)a2 \\
 & + 1/152199(-1828a1^2 + 1828a1 + 23060))\tau_{4,135,9}(n) \\
 & + (1/152199(1828a1 + 4588)a2 + 1/152199(4588a1 \\
 & + 18472))\tau_{4,135,10}(n) + (-4/35883a3^2 + 1204/35883a3 \\
 & - 1832/11961)\tau_{4,135,11}(n) + ((4/35883a3 + 1184/35883)a4 \\
 & + 4/35883a3^2 - 20/35883a3 - 5528/35883)\tau_{4,135,12}(n) \\
 & + ((-4/35883a3 - 1184/35883)a4 - 1184/35883a3 \\
 & + 392/35883)\tau_{4,135,13}(n)
 \end{aligned}$$

$$r_{\Psi_4}(n) =$$

$$\begin{aligned}
& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
& + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) \\
& + 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) - 64/1053\tau_{4,5}(n) \\
& - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) - 4000/819\tau_{4,5}\left(\frac{n}{9}\right) - 576/13\tau_{4,5}\left(\frac{n}{27}\right) \\
& - 32/135\tau_{4,9}(n) - 32/15\tau_{4,9}\left(\frac{n}{3}\right) - 160/27\tau_{4,9}\left(\frac{n}{5}\right) \\
& - 160/3\tau_{4,9}\left(\frac{n}{15}\right) + 4/3\tau_{4,15,1}\left(\frac{n}{3}\right) + 8/7\tau_{4,15,2}\left(\frac{n}{3}\right) \\
& + (28/459a6 + 4/17)\tau_{4,27,3}(n) + (700/459a6 \\
& + 100/17)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-28/459a6 + 4/17)\tau_{4,27,4}(n) \\
& + (-700/459a6 + 100/17)\tau_{4,27,4}\left(\frac{n}{5}\right) + 4/21\tau_{4,45,2}(n) \\
& + 12/7\tau_{4,45,2}\left(\frac{n}{3}\right) - 64/189\tau_{4,45,3}(n) - 64/21\tau_{4,45,3}\left(\frac{n}{3}\right) \\
& + 56/405\tau_{4,45,4}(n) + 56/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 80/459\tau_{4,135,2}(n)
\end{aligned}$$

$$\begin{aligned}
& + (1/456597(7708a1^2 + 280a1 - 143400))\tau_{4,135,5}(n) \\
& + (1/456597(-7708a1 + 7988)a2 + 1/456597(-7708a1^2 \\
& + 7708a1 + 33884))\tau_{4,135,6}(n) + (1/456597(7708a1 \\
& - 7988)a2 + 1/456597(-7988a1 + 41872))\tau_{4,135,7}(n) \\
& + (1/152199(-1828a1^2 + 6416a1 + 18984))\tau_{4,135,8}(n) \\
& + (1/152199(1828a1 + 4588)a2 + 1/152199(1828a1^2 \\
& - 1828a1 - 23060))\tau_{4,135,9}(n) + (1/152199(-1828a1 \\
& - 4588)a2 + 1/152199(-4588a1 - 18472))\tau_{4,135,10}(n) \\
& + (4/35883a3^2 - 1204/35883a3 + 1832/11961)\tau_{4,135,11}(n) \\
& + ((-4/35883a3 - 1184/35883)a4 - 4/35883a3^2 \\
& + 20/35883a3 + 5528/35883)\tau_{4,135,12}(n) + ((4/35883a3 \\
& + 1184/35883)a4 + 1184/35883a3 - 392/35883)\tau_{4,135,13}(n)
\end{aligned}$$

$$r_{F_3 \oplus \Lambda_1}(n) =$$

$$\begin{aligned}
& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) \\
& - 6000/15552\sigma_3\left(\frac{n}{5}\right) + 1200/864\sigma_3\left(\frac{n}{9}\right) \\
& + 30000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right) \\
& - 30000/864\sigma_3\left(\frac{n}{45}\right) + 18000/64\sigma_3\left(\frac{n}{135}\right) - 8/567\tau_{4,5}(n) \\
& - 164/567\tau_{4,5}\left(\frac{n}{3}\right) - 164/63\tau_{4,5}\left(\frac{n}{9}\right) - 72/7\tau_{4,5}\left(\frac{n}{27}\right) \\
& + 32/405\tau_{4,9}(n) + 88/45\tau_{4,9}\left(\frac{n}{3}\right) + 440/81\tau_{4,9}\left(\frac{n}{5}\right) \\
& + 160/9\tau_{4,9}\left(\frac{n}{15}\right) + 19/36\tau_{4,15,1}(n) - 17/12\tau_{4,15,1}\left(\frac{n}{3}\right) \\
& + 171/4\tau_{4,15,1}\left(\frac{n}{9}\right) + 2/7\tau_{4,15,2}(n) + 62/21\tau_{4,15,2}\left(\frac{n}{3}\right) \\
& + 162/7\tau_{4,15,2}\left(\frac{n}{9}\right) + 4/15\tau_{4,27,1}(n) + 20/3\tau_{4,27,1}\left(\frac{n}{5}\right) \\
& + 68/405\tau_{4,27,2}(n) - 340/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (106/1377a6 \\
& + 112/459)\tau_{4,27,3}(n) + (2650/1377a6 + 2800/459)\tau_{4,27,3}\left(\frac{n}{5}\right) \\
& + (-106/1377a6 + 112/459)\tau_{4,27,4}(n) + (-2650/1377a6 \\
& + 2000/459)\tau_{4,27,4}\left(\frac{n}{5}\right) + 4/9\tau_{4,45,3}(n) + 4\tau_{4,45,3}\left(\frac{n}{3}\right) \\
& + 8/81\tau_{4,45,4}(n) + 8/9\tau_{4,45,4}\left(\frac{n}{3}\right) + 16/135\tau_{4,45,5}(n) \\
& - 16/15\tau_{4,45,5}\left(\frac{n}{3}\right) 116/459\tau_{4,135,2}(n) + (1/2282985(-41252a1^2
\end{aligned}$$

$$\begin{aligned}
& + 86716a1 + 978144))\tau_{4,135,5}(n) + (1/2282985(41252a1 \\
& + 45464)a2 + 1/2282985(41252a1^2 - 41252a1 \\
& + 29348))\tau_{4,135,6}(n) + (1/2282985(-41252a1 - 45464)a2 \\
& + 1/2282985(-45464a1 + 74812))\tau_{4,135,7}(n) \\
& + (1/760995(-15844a1^2 - 44088a1 + 596128))\tau_{4,135,8}(n) \\
& + (1/760995(15844a1 - 59932)a2 + 1/760995(15844a1^2 \\
& - 15844a1 + 231716))\tau_{4,135,9}(n) + (1/760995(-15844a1 \\
& + 59932)a2 + 1/760995(59932a1 + 171784))\tau_{4,135,10}(n) \\
& + (-1262/35883a3^2 + 2426/35883a3 + 3436/3987)\tau_{4,135,11}(n) \\
& + ((1262/35883a3 - 3884/35883)a4 + 1262/35883a3^2 \\
& - 6310/35883a3 + 20828/35883)\tau_{4,135,12}(n) + ((-1262/35883a3 \\
& + 3884/35883)a4 + 3884/35883a3 + 1408/35883)\tau_{4,135,13}(n)
\end{aligned}$$

$$r_{F_3 \oplus \Phi_1}(n) =$$

$$\begin{aligned}
& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
& + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) \\
& + 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) - 64/1053\tau_{4,5}(n) \\
& - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) - 4000/819\tau_{4,5}\left(\frac{n}{9}\right) - 576/13\tau_{4,5}\left(\frac{n}{27}\right) \\
& + 8/27\tau_{4,9}(n) + 8/3\tau_{4,9}\left(\frac{n}{3}\right) + 200/27\tau_{4,9}\left(\frac{n}{5}\right) + 200/3\tau_{4,9}\left(\frac{n}{15}\right) \\
& + 1/2\tau_{4,15,1}(n) - 5/3\tau_{4,15,1}\left(\frac{n}{3}\right) + 81/2\tau_{4,15,1}\left(\frac{n}{9}\right) + 1/2\tau_{4,15,2}(n) \\
& + 29/7\tau_{4,15,2}\left(\frac{n}{3}\right) + 81/2\tau_{4,15,1}\left(\frac{n}{9}\right) + 16/45\tau_{4,27,1}(n) \\
& + 80/9\tau_{4,27,1}\left(\frac{n}{5}\right) + (2/27a6 + 1/3)\tau_{4,27,3}(n) + (50/27a6 \\
& + 25/3)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-2/27a6 + 1/3)\tau_{4,27,4}(n) + (-50/27a6 \\
& + 25/3)\tau_{4,27,4}\left(\frac{n}{5}\right) + 5/18\tau_{4,45,1}(n) - 5/2\tau_{4,45,1}\left(\frac{n}{3}\right) + 1/42\tau_{4,45,2}(n) \\
& + 3/14\tau_{4,45,2}\left(\frac{n}{3}\right) + 104/189\tau_{4,45,3}(n) + 104/21\tau_{4,45,3}\left(\frac{n}{3}\right) \\
& + 13/81\tau_{4,45,4}(n) + 13/9\tau_{4,45,4}\left(\frac{n}{3}\right) + 1/9\tau_{4,45,5}(n) - \tau_{4,45,5}\left(\frac{n}{3}\right) \\
& + 8/27\tau_{4,135,2}(n) + (1/2282985(-30316a1^2 + 73853)a1 \\
& + 828012)\tau_{4,135,5}(n) + (1/2282985(30316a1 + 43537)a2
\end{aligned}$$

$$\begin{aligned}
& + 1/2282985(30316a1^2 - 30316a1 + 130744))\tau_{4,135,6}(n) \\
& + (1/2282985(-30316a1 - 43537)a2 + 1/2282985(-43537a1 \\
& + 174281))\tau_{4,135,7}(n) + (1/50733(-500a1^2 - 2295a1 \\
& + 23004))\tau_{4,135,8}(n) + (1/50733(500a1 - 2795)a2 \\
& + 1/50733(500a1^2 - 500a1 + 11504))\tau_{4,135,9}(n) \\
& + (1/50733(-500a1 + 2795)a2 + (1/50733(2795a1 \\
& + 8709))\tau_{4,135,10}(n) + (-665/35883a3^2 + 815/35883a3 \\
& + 6416/11961)\tau_{4,135,11}(n) + ((665/35883a3 - 2510/35883)a4 \\
& + 665/35883a3^2 - 3325/35883a3 + 13928/35883)\tau_{4,135,12}(n) \\
& + ((-665/35883a3 + 2510/35883)a4 + 2510/35883a3 \\
& + 1378/35883)\tau_{4,135,13}(n)
\end{aligned}$$

$$r_{F_3 \oplus \Psi_1}(n) =$$

$$\begin{aligned}
& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right) \\
& + 1200/864\sigma_3\left(\frac{n}{9}\right) + 30\,000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right) \\
& - 30000/864\sigma_3\left(\frac{n}{45}\right) + 18000/64\sigma_3\left(\frac{n}{135}\right) + 64/567\tau_{4,5}(n) \\
& + 340/567\tau_{4,5}\left(\frac{n}{3}\right) + 340/63\tau_{4,5}\left(\frac{n}{9}\right) + 576/7\tau_{4,5}\left(\frac{n}{27}\right) \\
& + 32/405\tau_{4,9}(n) + 32/9\tau_{4,9}\left(\frac{n}{3}\right) + 800/81\tau_{4,9}\left(\frac{n}{5}\right) \\
& + 160/9\tau_{4,9}\left(\frac{n}{15}\right) + 7/36\tau_{4,15,1}(n) + 7/12\tau_{4,15,1}\left(\frac{n}{3}\right) \\
& + 63/4\tau_{4,15,1}\left(\frac{n}{9}\right) + 17/63\tau_{4,15,2}(n) + 100/21\tau_{4,15,2}\left(\frac{n}{3}\right) \\
& + 153/7\tau_{4,15,2}\left(\frac{n}{9}\right) + 32/135\tau_{4,27,1}(n) + 160/27\tau_{4,27,1}\left(\frac{n}{5}\right) \\
& - 16/405\tau_{4,27,2}(n) + 80/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (124/1377a6 \\
& + 208/459)\tau_{4,27,3}(n) + (3100/1377a6 \\
& + 5200/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-124/1377a6 \\
& + 208/459)\tau_{4,27,4}(n) + (-3100/1377a6 \\
& + 5200/459)\tau_{4,27,4}\left(\frac{n}{5}\right) - 1/9\tau_{4,45,1}(n) + \tau_{4,45,1}\left(\frac{n}{3}\right) \\
& + 13/63\tau_{4,45,2}(n) + 13/7\tau_{4,45,2}\left(\frac{n}{3}\right) \\
& + 4/7\tau_{4,45,3}(n) + 36/7\tau_{4,45,3}\left(\frac{n}{3}\right) + 94/405\tau_{4,45,4}(n) \\
& + 94/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 2/135\tau_{4,45,5}(n) + 2/15\tau_{4,45,5}\left(\frac{n}{3}\right) \\
& + 128/459\tau_{4,135,2}(n) + (1/2282985(-30674a1^2
\end{aligned}$$

$$\begin{aligned}
 &+93217a1+1086468))\tau_{4,135,5}(n)+(1/2282985(30674a1 \\
 &+62543)a2+1/2282985(30674a1^2-30674a1 \\
 &+380966))\tau_{4,135,6}(n)+(1/2282985(-30674a1-62543)a2 \\
 &+1/2282985(-62543a1+443509))\tau_{4,135,7}(n) \\
 &+(1/760995(-1322a1^2-49999a1+338644))\tau_{4,135,8}(n) \\
 &+(1/760995(1322a1-51321)a2+1/760995(1322a1^2 \\
 &-1322a1+308238))\tau_{4,135,9}(n)+(1/760995(-1322a1 \\
 &+51321)a2+1/253665(17107a1+85639))\tau_{4,135,10}(n) \\
 &+(-518/35883a3^2+1754/35883a3+1840/3987)\tau_{4,135,11}(n) \\
 &+((518/35883a3-836/35883)a4+518/35883a3^2 \\
 &-2590/35883a3+12416/35883)\tau_{4,135,12}(n) \\
 &+((-518/35883a3+836/35883)a4+836/35883a3 \\
 &+8236/35883)\tau_{4,135,13}(n)
 \end{aligned}$$

$$r_{\Lambda_3\oplus F_1}(n)=$$

$$\begin{aligned}
& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right) \\
& + 1200/864\sigma_3\left(\frac{n}{9}\right) + 30000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right) \\
& - 30000/864\sigma_3\left(\frac{n}{45}\right) + 18000/64\sigma_3\left(\frac{n}{135}\right) + 8/567\tau_{4,5}(n) \\
& + 164/567\tau_{4,5}\left(\frac{n}{3}\right) + 164/63\tau_{4,5}\left(\frac{n}{9}\right) + 72/7\tau_{4,5}\left(\frac{n}{27}\right) \\
& - 88/405\tau_{4,9}(n) - 32/45\tau_{4,9}\left(\frac{n}{3}\right) - 160/81\tau_{4,9}\left(\frac{n}{5}\right) \\
& - 440/9\tau_{4,9}\left(\frac{n}{15}\right) + 19/36\tau_{4,15,1}(n) - 17/12\tau_{4,15,1}\left(\frac{n}{3}\right) \\
& + 171/4\tau_{4,15,1}\left(\frac{n}{9}\right) - 2/7\tau_{4,15,2}(n) - 62/21\tau_{4,15,2}\left(\frac{n}{3}\right) \\
& - 162/7\tau_{4,15,2}\left(\frac{n}{9}\right) - 4/15\tau_{4,27,1}(n) - 20/3\tau_{4,27,1}\left(\frac{n}{5}\right) \\
& + 68/405\tau_{4,27,2}(n) - 340/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (-106/1377a6 \\
& - 112/459)\tau_{4,27,3}(n) + (-2650/1377a6 \\
& - 2800/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (106/1377a6 - 112/459)\tau_{4,27,4}(n) \\
& + (2650/1377a6 - 2800/459)\tau_{4,27,4}\left(\frac{n}{5}\right) + 4/9\tau_{4,45,3}(n) \\
& + 4\tau_{4,45,3}\left(\frac{n}{3}\right) - 8/81\tau_{4,45,4}(n) - 8/9\tau_{4,45,4}\left(\frac{n}{3}\right) \\
& + 16/135\tau_{4,45,5}(n) - 16/15\tau_{4,45,5}\left(\frac{n}{3}\right) - 116/459\tau_{4,135,2}(n)
\end{aligned}$$

$$\begin{aligned}
& + (1/2282985(41252a1^2 - 86716a1 - 978144))\tau_{4,135,5}(n) \\
& + (1/2282985(-41252a1 - 45464)a2 + 1/2282985(-41252a1^2 \\
& + 41252a1 - 29348))\tau_{4,135,6}(n) + (1/2282985(41252a1 \\
& + 45464)a2 + 1/2282985(45464a1 - 74812))\tau_{4,135,7}(n) \\
& + (1/760995(-15844a1^2 - 44088a1 + 596128))\tau_{4,135,8}(n) \\
& + (1/760995(15844a1 - 59932)a2 + 1/760995(15844a1^2 \\
& - 15844a1 + 231716))\tau_{4,135,9}(n) + (1/760995(-15844a1 \\
& + 59932)a2 + 1/760995(59932a1 + 171784))\tau_{4,135,10}(n) \\
& + (-1262/35883a3^2 + 2426/35883a3 + 3436/3987)\tau_{4,135,11}(n) \\
& ((1262/35883a3 - 3884/35883)a4 + 1262/35883a3^2 \\
& - 6310/35883a3 + 20828/35883)\tau_{4,135,12}(n) + ((-1262/35883a3 \\
& + 3884/35883)a4 + 3884/35883a3 + 1408/35883)\tau_{4,135,13}(n)
\end{aligned}$$

$$r_{\Lambda_3 \oplus \Phi_1}(n) =$$

$$\begin{aligned}
& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) \\
& -6000/15552\sigma_3\left(\frac{n}{5}\right) + 1200/864\sigma_3\left(\frac{n}{9}\right) \\
& +30000/7776\sigma_3\left(\frac{n}{15}\right) - 720/64\sigma_3\left(\frac{n}{27}\right) \\
& -30000/864\sigma_3\left(\frac{n}{45}\right) + 18000/64\sigma_3\left(\frac{n}{135}\right) \\
& -64/567\tau_{4,5}(n) - 340/567\tau_{4,5}\left(\frac{n}{3}\right) - 340/63\tau_{4,5}\left(\frac{n}{9}\right) \\
& -576/7\tau_{4,5}\left(\frac{n}{27}\right) - 32/81\tau_{4,9}(n) - 32/45\tau_{4,9}\left(\frac{n}{3}\right) \\
& -160/81\tau_{4,9}\left(\frac{n}{5}\right) - 800/9\tau_{4,9}\left(\frac{n}{15}\right) + 7/36\tau_{4,15,1}(n) \\
& +7/12\tau_{4,15,1}\left(\frac{n}{3}\right) + 63/4\tau_{4,15,1}\left(\frac{n}{9}\right) - 17/63\tau_{4,15,2}(n) \\
& -100/21\tau_{4,15,2}\left(\frac{n}{3}\right) - 153/7\tau_{4,15,2}\left(\frac{n}{9}\right) - 32/135\tau_{4,27,1}(n) \\
& -160/27\tau_{4,27,1}\left(\frac{n}{5}\right) - 16/405\tau_{4,27,2}(n) + 80/81\tau_{4,27,2}\left(\frac{n}{5}\right) \\
& +(-124/1377a6 - 208/459)\tau_{4,27,3}(n) + (-3100/1377a6 \\
& -5200/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (124/1377a6 - 208/459)\tau_{4,27,4}(n) \\
& +(3100/1377a6 - 5200/459)\tau_{4,27,4}\left(\frac{n}{5}\right) \\
& +1/9\tau_{4,45,1}(n) - \tau_{4,45,1}\left(\frac{n}{3}\right) + 13/63\tau_{4,45,2}(n) + 13/7\tau_{4,45,2}\left(\frac{n}{3}\right) \\
& +4/7\tau_{4,45,3}(n) + 36/7\tau_{4,45,3}\left(\frac{n}{3}\right) - 94/405\tau_{4,45,4}(n) \\
& -94/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 2/135\tau_{4,45,5}(n) + 2/15\tau_{4,45,5}\left(\frac{n}{3}\right) \\
& -128/459\tau_{4,135,2}(n) + (1/2282985(30674a1^2 - 93217a1 \\
& -1086468))\tau_{4,135,5}(n) + (1/2282985(-30674a1 - 62543)a2 \\
& +1/2282985(-30674a1^2 + 30674a1 - 380966))\tau_{4,135,6}(n) \\
& +(1/2282985(30674a1 + 62543)a2 + 1/2282985(62543a1 \\
& -443509))\tau_{4,135,7}(n) + (1/760995(-1322a1^2 - 49999a1 \\
& +338644))\tau_{4,135,8}(n) + (1/760995(1322a1 - 51321)a2 \\
& +1/760995(1322a1^2 - 1322a1 + 308238))\tau_{4,135,9}(n) \\
& +(1/760995(-1322a1 + 51321)a2 + 1/253665(17107a1 \\
& +85639))\tau_{4,135,10}(n) + (-518/35883a3^2 + 1754/35883a3 \\
& +1840/3987)\tau_{4,135,11}(n) + ((518/35883a3 - 836/35883)a4 \\
& +518/35883a3^2 - 2590/35883a3 + 12416/35883)\tau_{4,135,12}(n) \\
& +((-518/35883a3 + 836/35883)a4 + 836/35883a3 \\
& +8236/35883)\tau_{4,135,13}(n)
\end{aligned}$$

$$\begin{aligned}
 r_{\Lambda_3 \oplus \Psi_1}(n) = & \\
 & 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
 & + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) \\
 & + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right) \\
 & + 10800/52\sigma_3\left(\frac{n}{135}\right) - 64/1053\tau_{4,5}(n) - 4000/7371\tau_{4,5}\left(\frac{n}{3}\right) \\
 & - 4000/819\tau_{4,5}\left(\frac{n}{9}\right) - 576/13\tau_{4,5}\left(\frac{n}{27}\right) + 8/27\tau_{4,9}(n) \\
 & + 8/3\tau_{4,9}\left(\frac{n}{3}\right) + 200/27\tau_{4,9}\left(\frac{n}{5}\right) + 200/3\tau_{4,9}\left(\frac{n}{15}\right) \\
 & - 1/2\tau_{4,15,1}(n) + 5/3\tau_{4,15,1}\left(\frac{n}{3}\right) - 81/2\tau_{4,15,1}\left(\frac{n}{9}\right) \\
 & + 1/2\tau_{4,15,2}(n) + 29/7\tau_{4,15,2}\left(\frac{n}{3}\right) + 81/2\tau_{4,15,1}\left(\frac{n}{9}\right) \\
 & + 16/45\tau_{4,27,1}(n) + 80/9\tau_{4,27,1}\left(\frac{n}{5}\right) + (2/27a6 + 1/3)\tau_{4,27,3}(n) \\
 & + (50/27a6 + 25/3)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-2/27a6 + 1/3)\tau_{4,27,4}(n) \\
 & + (-50/27a6 + 25/3)\tau_{4,27,4}\left(\frac{n}{5}\right) + 5/18\tau_{4,45,1}(n) - 5/2\tau_{4,45,1}\left(\frac{n}{3}\right) \\
 & - 1/42\tau_{4,45,2}(n) - 3/14\tau_{4,45,2}\left(\frac{n}{3}\right) - 104/189\tau_{4,45,3}(n) \\
 & - 104/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 13/81\tau_{4,45,4}(n) + 13/9\tau_{4,45,4}\left(\frac{n}{3}\right) \\
 & - 1/9\tau_{4,45,5}(n) + \tau_{4,45,5}\left(\frac{n}{3}\right) + 8/27\tau_{4,135,2}(n) \\
 & - 13928/35883\tau_{4,135,12}(n) + ((665/35883a3 - 2510/35883)a4 \\
 & \\
 & + (1/2282985(-30316a1^2 + 73853a1 + 828012))\tau_{4,135,5}(n) \\
 & + (1/2282985(30316a1 + 43537)a2 + 1/2282985(30316a1^2 \\
 & - 30316a1 + 130744))\tau_{4,135,6}(n) + (1/2282985(-30316a1 \\
 & - 43537)a2 + 1/2282985(-43537a1 + 174281))\tau_{4,135,7}(n) \\
 & + (1/50733(500a1^2 + 2295a1 - 23004))\tau_{4,135,8}(n) \\
 & + (1/50733(-500a1 + 2795)a2 + 1/50733(-500a1^2 + 500a1 \\
 & - 11504))\tau_{4,135,9}(n) + (1/50733(500a1 - 2795)a2 \\
 & + 1/50733(-2795a1 - 8709))\tau_{4,135,10}(n) + (665/35883a3^2 \\
 & - 815/35883a3 - 6416/11961)\tau_{4,135,11}(n) + ((-665/35883a3 \\
 & + 2510/35883)a4 - 665/35883a3^2 + 3325/35883a3 \\
 & - 2510/35883a3 - 1378/35883)\tau_{4,135,13}
 \end{aligned}$$

$$r_{\Phi_3 \oplus F_1}(n) =$$

$$\begin{aligned}
& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
& + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) \\
& + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right) \\
& + 10800/52\sigma_3\left(\frac{n}{135}\right) + 488/7371\tau_{4,5}(n) + 2552/7371\tau_{4,5}\left(\frac{n}{3}\right) \\
& + 2552/819\tau_{4,5}\left(\frac{n}{9}\right) + 4392/91\tau_{4,5}\left(\frac{n}{27}\right) - 8/135\tau_{4,9}(n) \\
& - 8/15\tau_{4,9}\left(\frac{n}{3}\right) - 40/27\tau_{4,9}\left(\frac{n}{5}\right) - 40/3\tau_{4,9}\left(\frac{n}{15}\right) + 1/18\tau_{4,15,1}(n) \\
& - 1/3\tau_{4,15,1}\left(\frac{n}{3}\right) + 9/2\tau_{4,15,1}\left(\frac{n}{9}\right) + 19/26\tau_{4,15,2}(n) \\
& - 1/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 171/14\tau_{4,15,1}\left(\frac{n}{9}\right) + 16/135\tau_{4,27,1}(n) \\
& + 80/27\tau_{4,27,1}\left(\frac{n}{5}\right) + (20/459a6 + 11/51)\tau_{4,27,3}(n) + (500/459a6 \\
& + 275/51)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-20/459a6 + 11/51)\tau_{4,27,4}(n) \\
& + (-500/459a6 + 275/51)\tau_{4,27,4}\left(\frac{n}{5}\right) + 1/18\tau_{4,45,1}(n) \\
& - 1/2\tau_{4,45,1}\left(\frac{n}{3}\right) + 3/14\tau_{4,45,2}(n) + 27/14\tau_{4,45,2}\left(\frac{n}{3}\right) \\
& - 16/189\tau_{4,45,3}(n) - 16/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 29/405\tau_{4,45,4}(n) \\
& + 29/45\tau_{4,45,4}\left(\frac{n}{3}\right) + 1/9\tau_{4,45,5}(n) - \tau_{4,45,5}\left(\frac{n}{3}\right) + 40/459\tau_{4,135,2}(n)
\end{aligned}$$

$$\begin{aligned}
& + (1/2282985(15278a1^2 + 14921a1 - 193596))\tau_{4,135,5}(n) \\
& + (1/2282985(-15278a1 + 30199)a2 + 1/2282985(-15278a1^2 \\
& + 15278a1 + 157798))\tau_{4,135,6}(n) + (1/2282985(15278a1 \\
& - 30199)a2 + 1/2282985(-30199a1 + 187997))\tau_{4,135,7}(n) \\
& + (1/50733(-566a1^2 + 897a1 + 11700))\tau_{4,135,8}(n) \\
& + (1/50733(566a1 + 331)a2 + 1/50733(566a1^2 - 566a1 \\
& - 1318))\tau_{4,135,9}(n) + (1/50733(-566a1 - 331)a2 \\
& + 1/50733(-331a1 - 987))\tau_{4,135,10}(n) + (301/35883a3^2 \\
& - 2887/35883a3 + 2300/11961)\tau_{4,135,11}(n) + ((-301/35883a3 \\
& - 1382/35883)a4 - 301/35883a3^2 + 1505/35883a3 \\
& + 9308/35883)\tau_{4,135,12}(n) + (301/35883a3 + 1382/35883)a4 \\
& + 1382/35883a3 + 2398/35883)\tau_{4,135,13}
\end{aligned}$$

$$r_{\Phi_3 \oplus \Lambda_1}(n) =$$

$$\begin{aligned}
& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right) \\
& + 1200/864\sigma_3\left(\frac{n}{9}\right) + 30000/7776\sigma_3\left(\frac{n}{15}\right) \\
& - 720/64\sigma_3\left(\frac{n}{27}\right) - 30000/864\sigma_3\left(\frac{n}{45}\right) \\
& + 18000/64\sigma_3\left(\frac{n}{135}\right) - 8/567\tau_{4,5}(n) - 164/567\tau_{4,5}\left(\frac{n}{3}\right) \\
& - 164/63\tau_{4,5}\left(\frac{n}{9}\right) - 72/7\tau_{4,5}\left(\frac{n}{27}\right) - 112/405\tau_{4,9}(n) \\
& + 16/45\tau_{4,9}\left(\frac{n}{3}\right) + 80/81\tau_{4,9}\left(\frac{n}{5}\right) - 560/9\tau_{4,9}\left(\frac{n}{15}\right) \\
& + 1/12\tau_{4,15,1}(n) - 1/12\tau_{4,15,1}\left(\frac{n}{3}\right) + 27/4\tau_{4,15,1}\left(\frac{n}{9}\right) \\
& - 1/21\tau_{4,15,2}(n) - 64/21\tau_{4,15,2}\left(\frac{n}{3}\right) - 27/7\tau_{4,15,2}\left(\frac{n}{9}\right) \\
& - 16/135\tau_{4,27,1}(n) - 80/27\tau_{4,27,1}\left(\frac{n}{5}\right) - 16/405\tau_{4,27,2}(n) \\
& + 80/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (-62/1377a6 - 104/459)\tau_{4,27,3}(n) \\
& + (-1550/1377a6 - 2600/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (162/1377a6 \\
& - 104/459)\tau_{4,27,4}(n) + (1550/1377a6 - 2600/459)\tau_{4,27,4}\left(\frac{n}{5}\right) \\
& + 1/9\tau_{4,45,1}(n) - \tau_{4,45,1}\left(\frac{n}{3}\right) + 8/63\tau_{4,45,2}(n) + 8/7\tau_{4,45,2}\left(\frac{n}{3}\right) \\
& + 20/63\tau_{4,45,3}(n) + 20/7\tau_{4,45,3}\left(\frac{n}{3}\right) - 32/405\tau_{4,45,4}(n) \\
& - 32/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 2/135\tau_{4,45,5}(n) + 2/15\tau_{4,45,5}\left(\frac{n}{3}\right) \\
& - 64/459\tau_{4,135,2}(n) + (1/2282985(15712a1^2 - 37841a1 \\
& - 340644))\tau_{4,135,5}(n) + (1/2282985(-15712a1 - 22129)a2 \\
& + 1/2282985(-15712a1^2 + 15712a1 + 20732))\tau_{4,135,6}(n) \\
& + (1/2282985(15712a1 + 22129)a2 + 1/2282985(22129a1 \\
& - 1397))\tau_{4,135,7}(n) + (1/760995(2668a1^2 - 27739a1 \\
& + 99604))\tau_{4,135,8}(n) + (1/760995(-2668a1 - 25071)a2 \\
& + 1/760995(-2668a1^2 + 2668a1 + 160968))\tau_{4,135,9}(n) \\
& + (1/760995(2668a1 + 25071)a2 + 1/253665(8357a1 \\
& + 45299))\tau_{4,135,10}(n) + (178/35883a3^2 - 418/35883a3 \\
& + 4/3987)\tau_{4,135,11}(n) + (((-178/35883a3 + 472/35883)a4 \\
& - 178/35883a3^2 + 890/35883a3 + 1460/35883)\tau_{4,135,12}(n) \\
& + ((178/35883a3 - 472/35883)a4 - 472/35883a3 \\
& + 3820/35883)\tau_{4,135,13}
\end{aligned}$$

$$r_{\Phi_3 \oplus \Psi_1}(n) =$$

$$\begin{aligned}
& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right) \\
& + 1200/864\sigma_3\left(\frac{n}{9}\right) + 30\,000/7776\sigma_3\left(\frac{n}{15}\right) \\
& - 720/64\sigma_3\left(\frac{n}{27}\right) - 30000/864\sigma_3\left(\frac{n}{45}\right) \\
& + 18\,000/64\sigma_3\left(\frac{n}{135}\right) + 4/81\tau_{4,5}(n) + 88/567\tau_{4,5}\left(\frac{n}{3}\right) \\
& + 88/63\tau_{4,5}\left(\frac{n}{9}\right) + 36\tau_{4,5}\left(\frac{n}{27}\right) + 32/405\tau_{4,9}(n) \\
& + 16/45\tau_{4,9}\left(\frac{n}{3}\right) + 80/81\tau_{4,9}\left(\frac{n}{5}\right) + 160/9\tau_{4,9}\left(\frac{n}{15}\right) \\
& - 1/36\tau_{4,15,1}(n) - 3/4\tau_{4,15,1}\left(\frac{n}{3}\right) - 9/4\tau_{4,15,1}\left(\frac{n}{9}\right) \\
& - 10/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 8/405\tau_{4,27,2}(n) - 40/81\tau_{4,27,2}\left(\frac{n}{5}\right) \\
& + (-38/1377a6 - 44/459)\tau_{4,27,3}(n) + (-950/1377a6 \\
& - 1100/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (38/1377a6 - 44/459)\tau_{4,27,4}(n) \\
& + (950/1377a6 - 1100/459)\tau_{4,27,4}\left(\frac{n}{5}\right) - 1/7\tau_{4,45,2}(n) \\
& - 9/7\tau_{4,45,2}\left(\frac{n}{3}\right) + 16/63\tau_{4,45,3}(n) + 16/7\tau_{4,45,3}\left(\frac{n}{3}\right) \\
& - 14/405\tau_{4,45,4}(n) - 14/45\tau_{4,45,4}\left(\frac{n}{3}\right) + 4/135\tau_{4,45,5}(n) \\
& - 4/15\tau_{4,45,5}\left(\frac{n}{3}\right) + 20/459\tau_{4,135,2}(n) + (1/456597(-1510a1^2 \\
& - 4864a1 + 36552))\tau_{4,135,5}(n) + (1/456597(1510a1 - 6374)a2 \\
& + 1/456597(1510a1^2 - 1510a1 + 1822))\tau_{4,135,6}(n) \\
& + (1/456597(-1510a1 + 6374)a2 + 1/456597(6374a1 \\
& - 4552))\tau_{4,135,7}(n) + (1/760995(8086a1^2 - 38158a1 \\
& - 81392))\tau_{4,135,8}(n) + (1/760995(-8086a1 - 30072)a2 \\
& + 1/760995(-8086a1^2 + 8086a1 + 104586))\tau_{4,135,9}(n) \\
& + (1/760995(8086a1 + 30072)a2 + 1/253665(10024a1 \\
& + 24838))\tau_{4,135,10}(n) + (334/35883a3^2 + 470/35883a3 \\
& - 988/3987)\tau_{4,135,11}(n) + ((-334/35883a3 + 2140/35883)a4 \\
& - 334/35883a3^2 + 1670/35883a3 - 6220/35883)\tau_{4,135,12}(n) \\
& + (334/35883a3 - 2140/35883)a4 - 2140/35883a3 \\
& + 4480/35883)\tau_{4,135,13}
\end{aligned}$$

$$r_{\Psi_3 \oplus F_1}(n) =$$

$$\begin{aligned}
& 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right) \\
& + 1200/864\sigma_3\left(\frac{n}{9}\right) + 30000/7776\sigma_3\left(\frac{n}{15}\right) \\
& - 720/64\sigma_3\left(\frac{n}{27}\right) - 30000/864\sigma_3\left(\frac{n}{45}\right) \\
& + 18000/64\sigma_3\left(\frac{n}{135}\right) + 8/567\tau_{4,5}(n) + 164/567\tau_{4,5}\left(\frac{n}{3}\right) \\
& + 164/63\tau_{4,5}\left(\frac{n}{9}\right) + 72/7\tau_{4,5}\left(\frac{n}{27}\right) - 16/405\tau_{4,9}(n) \\
& + 112/45\tau_{4,9}\left(\frac{n}{3}\right) + 560/81\tau_{4,9}\left(\frac{n}{5}\right) - 80/9\tau_{4,9}\left(\frac{n}{15}\right) \\
& + 1/12\tau_{4,15,1}(n) - 1/12\tau_{4,15,1}\left(\frac{n}{3}\right) + 27/4\tau_{4,15,1}\left(\frac{n}{9}\right) \\
& + 1/21\tau_{4,15,2}(n) + 64/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 27/7\tau_{4,15,2}\left(\frac{n}{9}\right) \\
& + 16/135\tau_{4,27,1}(n) + 80/27\tau_{4,27,1}\left(\frac{n}{5}\right) - 16/405\tau_{4,27,2}(n) \\
& + 80/81\tau_{4,27,2}\left(\frac{n}{5}\right) + (62/1377a6 + 104/459)\tau_{4,27,3}(n) \\
& + (1550/1377a6 + 2600/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-62/1377a6 \\
& + 104/459)\tau_{4,27,4}(n) + (-1550/1377a6 + 2600/459)\tau_{4,27,4}\left(\frac{n}{5}\right) \\
& - 1/9\tau_{4,45,1}(n) + \tau_{4,45,1}\left(\frac{n}{3}\right) + 8/63\tau_{4,45,2}(n) + 8/7\tau_{4,45,2}\left(\frac{n}{3}\right) \\
& + 20/63\tau_{4,45,3}(n) + 20/7\tau_{4,45,3}\left(\frac{n}{3}\right) + 32/405\tau_{4,45,4}(n) \\
& + 32/45\tau_{4,45,4}\left(\frac{n}{3}\right) - 2/135\tau_{4,45,5}(n) + 2/15\tau_{4,45,5}\left(\frac{n}{3}\right) \\
& + 64/459\tau_{4,135,2}(n) + (1/2282985(-15712a1^2 + 37841a1 \\
& + 340644))\tau_{4,135,5}(n) + (1/2282985(15712a1 + 22129)a2 \\
& + 1/2282985(15712a1^2 - 15712a1 - 20732))\tau_{4,135,6}(n) \\
& + (1/2282985(-15712a1 - 22129)a2 + 1/2282985(-22129a1 \\
& + 1397))\tau_{4,135,7}(n) + (1/760995(2668a1^2 - 27739a1 \\
& + 99604))\tau_{4,135,8}(n) + (1/760995(-2668a1 - 25071)a2 \\
& + 1/760995(-2668a1^2 + 2668a1 + 160968))\tau_{4,135,9}(n) \\
& + (1/760995(2668a1 + 25071)a2 + 1/253665(8357a1 \\
& + 45299))\tau_{4,135,10}(n) + (178/35883a3^2 - 418/35883a3 \\
& + 4/3987)\tau_{4,135,11}(n) + ((-178/35883a3 + 472/35883)a4 \\
& - 178/35883a3^2 + 890/35883a3 + 1460/35883)\tau_{4,135,12}(n) \\
& + ((178/35883a3 - 472/35883)a4 - 472/35883a3 \\
& + 3820/35883)\tau_{4,135,13}
\end{aligned}$$

$$r_{\Psi_3 \oplus \Lambda_1}(n) =$$

$$\begin{aligned} & 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\ & + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) \\ & + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right) \\ & + 10800/52\sigma_3\left(\frac{n}{135}\right) + 488/7371\tau_{4,5}(n) \\ & + 2552/7371\tau_{4,5}\left(\frac{n}{3}\right) + 2552/819\tau_{4,5}\left(\frac{n}{9}\right) \\ & + 4392/91\tau_{4,5}\left(\frac{n}{27}\right) - 8/135\tau_{4,9}(n) - 8/15\tau_{4,9}\left(\frac{n}{3}\right) \\ & - 40/27\tau_{4,9}\left(\frac{n}{5}\right) - 40/3\tau_{4,9}\left(\frac{n}{15}\right) - 1/18\tau_{4,15,1}(n) \\ & + 1/3\tau_{4,15,1}\left(\frac{n}{3}\right) - 9/2\tau_{4,15,1}\left(\frac{n}{9}\right) \\ & + 19/26\tau_{4,15,2}(n) - 1/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 171/14\tau_{4,15,1}\left(\frac{n}{9}\right) \\ & + 16/135\tau_{4,27,1}(n) + 80/27\tau_{4,27,1}\left(\frac{n}{5}\right) + (20/459a6 \\ & + 11/51)\tau_{4,27,3}(n) + (500/459a6 + 275/51)\tau_{4,27,3}\left(\frac{n}{5}\right) \\ & + (-20/459a6 + 11/51)\tau_{4,27,4}(n) + (-500/459a6 \\ & + 275/51)\tau_{4,27,4}\left(\frac{n}{5}\right) + 1/18\tau_{4,45,1}(n) - 1/2\tau_{4,45,1}\left(\frac{n}{3}\right) \\ & - 3/14\tau_{4,45,2}(n) - 27/14\tau_{4,45,2}\left(\frac{n}{3}\right) + 16/189\tau_{4,45,3}(n) \\ & + 16/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 29/405\tau_{4,45,4}(n) + 29/45\tau_{4,45,4}\left(\frac{n}{3}\right) \\ & - 1/9\tau_{4,45,5}(n) + \tau_{4,45,5}\left(\frac{n}{3}\right) + 40/459\tau_{4,135,2}(n) \end{aligned}$$

$$\begin{aligned} & + (1/2282985(15278a1^2 + 14921a1 - 193596))\tau_{4,135,5}(n) \\ & + (1/2282985(-15278a1 + 30199)a2 + 1/2282985(-15278a1^2 \\ & + 15278a1 + 157798))\tau_{4,135,6}(n) + (1/2282985(15278a1 \\ & - 30199)a2 + 1/2282985(-30199a1 + 187997))\tau_{4,135,7}(n) \\ & + (1/50733(566a1^2 - 897a1 - 11700))\tau_{4,135,8}(n) \\ & + (1/50733(-566a1 - 331)a2 + 1/50733(-566a1^2 + 566a1 \\ & + 1318))\tau_{4,135,9}(n) + (1/50733(566a1 + 331)a2 \\ & + 1/50733(331a1 + 987))\tau_{4,135,10}(n) + (-301/35883a3^2 \\ & + 2887/35883a3 - 2300/11961)\tau_{4,135,11}(n) + ((301/35883a3 \\ & + 1382/35883)a4 + 301/35883a3^2 - 1505/35883a3 \\ & - 9308/35883)\tau_{4,135,12}(n) + ((-301/35883a3 \\ & - 1382/35883)a4 - 1382/35883a3 - 2398/35883)\tau_{4,135,13} \end{aligned}$$

$$r_{\Psi_3 \oplus \Phi_1}(n) =$$

$$\begin{aligned} & 240/15552\sigma_3(n) - 1200/7776\sigma_3\left(\frac{n}{3}\right) - 6000/15552\sigma_3\left(\frac{n}{5}\right) \\ & + 1200/864\sigma_3\left(\frac{n}{9}\right) + 30\,000/7776\sigma_3\left(\frac{n}{15}\right) \\ & - 720/64\sigma_3\left(\frac{n}{27}\right) - 30000/864\sigma_3\left(\frac{n}{45}\right) \\ & + 18\,000/64\sigma_3\left(\frac{n}{135}\right) - 4/81\tau_{4,5}(n) - 88/567\tau_{4,5}\left(\frac{n}{3}\right) \\ & - 88/63\tau_{4,5}\left(\frac{n}{9}\right) - 36\tau_{4,5}\left(\frac{n}{27}\right) - 16/405\tau_{4,9}(n) \\ & - 32/45\tau_{4,9}\left(\frac{n}{3}\right) - 160/81\tau_{4,9}\left(\frac{n}{5}\right) - 80/9\tau_{4,9}\left(\frac{n}{15}\right) \\ & - 1/36\tau_{4,15,1}(n) - 3/4\tau_{4,15,1}\left(\frac{n}{3}\right) - 9/4\tau_{4,15,1}\left(\frac{n}{9}\right) \\ & + 10/21\tau_{4,15,2}\left(\frac{n}{3}\right) + 8/405\tau_{4,27,2}(n) - 40/81\tau_{4,27,2}\left(\frac{n}{5}\right) \\ & + (38/1377a6 + 44/459)\tau_{4,27,3}(n) + (950/1377a6 \\ & + 1100/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-38/1377a6 + 44/459)\tau_{4,27,4}(n) \\ & + (-950/1377a6 + 1100/459)\tau_{4,27,4}\left(\frac{n}{5}\right) - 1/7\tau_{4,45,2}(n) \\ & - 9/7\tau_{4,45,2}\left(\frac{n}{3}\right) + 16/63\tau_{4,45,3}(n) + 16/7\tau_{4,45,3}\left(\frac{n}{3}\right) \\ & + 14/405\tau_{4,45,4}(n) + 14/45\tau_{4,45,4}\left(\frac{n}{3}\right) + 4/135\tau_{4,45,5}(n) \\ & - 4/15\tau_{4,45,5}\left(\frac{n}{3}\right) - 20/459\tau_{4,135,2}(n) + (1/456597(1510a1^2 \end{aligned}$$

$$\begin{aligned} & + 4864a1 - 36552))\tau_{4,135,5}(n) + (1/456597(-1510a1 + 6374)a2 \\ & + 1/456597(-1510a1^2 + 1510a1 - 1822))\tau_{4,135,6}(n) \\ & + (1/456597(1510a1 - 6374)a2 + 1/456597(-6374a1 \\ & + 4552))\tau_{4,135,7}(n) + (1/760995(8086a1^2 - 38158a1 \\ & - 81392))\tau_{4,135,8}(n) + (1/760995(-8086a1 - 30072)a2 \\ & + 1/760995(-8086a1^2 + 8086a1 + 104586))\tau_{4,135,9}(n) \\ & + (1/760995(8086a1 + 30072)a2 + 1/253665(10024a1 \\ & + 24838))\tau_{4,135,10}(n) + (334/35883a3^2 + 470/35883a3 \\ & - 988/3987)\tau_{4,135,11}(n) + ((-334/35883a3 + 2140/35883)a4 \\ & - 334/35883a3^2 + 1670/35883a3 - 6220/35883)\tau_{4,135,12}(n) \\ & + (334/35883a3 - 2140/35883)a4 - 2140/35883a3 \\ & + 4480/35883)\tau_{4,135,13} \end{aligned}$$

$$r_{F_2 \oplus \Lambda_2}(n) =$$

$$\begin{aligned} & 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\ & + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) \\ & + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right) \\ & + 10800/52\sigma_3\left(\frac{n}{135}\right) + 440/2457\tau_{4,5}(n) - 1888/2457\tau_{4,5}\left(\frac{n}{3}\right) \\ & - 1888/273\tau_{4,5}\left(\frac{n}{9}\right) + 11880/91\tau_{4,5}\left(\frac{n}{27}\right) + 32/135\tau_{4,9}(n) \\ & + 32/15\tau_{4,9}\left(\frac{n}{3}\right) + 160/27\tau_{4,9}\left(\frac{n}{5}\right) + 160/3\tau_{4,9}\left(\frac{n}{15}\right) \\ & + 44/63\tau_{4,15,2}(n) + 24/7\tau_{4,15,2}\left(\frac{n}{3}\right) + 396/7\tau_{4,15,1}\left(\frac{n}{9}\right) \\ & + 16/45\tau_{4,27,1}(n) + 80/9\tau_{4,27,1}\left(\frac{n}{5}\right) + (92/1377a6 \\ & + 128/459)\tau_{4,27,3}(n) + (2300/1377a6 + 3200/459)\tau_{4,27,3}\left(\frac{n}{5}\right) \\ & + (-92/1377a6 + 128/459)\tau_{4,27,4}(n) + (-2300/1377a6 \\ & + 3200/459)\tau_{4,27,4}\left(\frac{n}{5}\right) + 4/9\tau_{4,45,1}(n) - 4\tau_{4,45,1}\left(\frac{n}{3}\right) \\ & + 8/135\tau_{4,45,4}(n) + 8/15\tau_{4,45,4}\left(\frac{n}{3}\right) + 32/51\tau_{4,135,2}(n) \\ & + (1/760995(-5272a1^2 - 3304a1 + 294144))\tau_{4,135,5}(n) \\ & + (1/760995(5272a1 - 8576)a2 + 1/760995(5272a1^2 \\ & - 5272a1 + 172888))\tau_{4,135,6}(n) + (1/760995(-5272a1 \\ & + 8576)a2 + 1/760995(8576a1 + 164312))\tau_{4,135,7}(n) \end{aligned}$$

$$r_{F_2 \oplus \Phi_2}(n) =$$

$$\begin{aligned}
& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
& + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) \\
& + 2160/260\sigma_3\left(\frac{n}{27}\right) + 2400/117\sigma_3\left(\frac{n}{45}\right) \\
& + 10800/52\sigma_3\left(\frac{n}{135}\right) + 176/7371\tau_{4,5}(n) \\
& + 368/7371\tau_{4,5}\left(\frac{n}{3}\right) + 368/819\tau_{4,5}\left(\frac{n}{9}\right) + 1584/91\tau_{4,5}\left(\frac{n}{27}\right) \\
& + 16/135\tau_{4,9}(n) + 16/15\tau_{4,9}\left(\frac{n}{3}\right) + 80/27\tau_{4,9}\left(\frac{n}{5}\right) \\
& + 80/3\tau_{4,9}\left(\frac{n}{15}\right) + 8/27\tau_{4,15,1}(n) - 20/9\tau_{4,15,1}\left(\frac{n}{3}\right) \\
& + 24\tau_{4,15,1}\left(\frac{n}{9}\right) + 2/7\tau_{4,15,2}(n) + 4/7\tau_{4,15,2}\left(\frac{n}{3}\right) \\
& + 162/7\tau_{4,15,1}\left(\frac{n}{9}\right) + 32/135\tau_{4,27,1}(n) + 160/27\tau_{4,27,1}\left(\frac{n}{5}\right) \\
& + (22/459a6 + 38/153)\tau_{4,27,3}(n) + (550/459a6 \\
& + 950/153)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-22/459a6 + 38/153)\tau_{4,27,4}(n) \\
& + (-550/459a6 + 950/153)\tau_{4,27,4}\left(\frac{n}{5}\right) + 2/9\tau_{4,45,1}(n) \\
& - 2\tau_{4,45,1}\left(\frac{n}{3}\right) + 8/63\tau_{4,45,2}(n) + 8/7\tau_{4,45,2}\left(\frac{n}{3}\right) \\
& + 32/189\tau_{4,45,3}(n) + 32/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 32/405\tau_{4,45,4}(n) \\
& + 32/45\tau_{4,45,4}\left(\frac{n}{3}\right) + 4/27\tau_{4,45,5}(n) - 4/3\tau_{4,45,5}\left(\frac{n}{3}\right) \\
& + 112/459\tau_{4,135,2}(n) + (1/2282985(-2074a1^2 + 34712a1
\end{aligned}$$

$$\begin{aligned}
& + 235128))\tau_{4,135,5}(n) + (1/2282985(2074a1 + 32638)a2 \\
& + 1/2282985(2074a1^2 - 2074a1 + 187426))\tau_{4,135,6}(n) \\
& + (1/2282985(-2074a1 - 32638)a2 + 1/2282985(-32638a1 \\
& + 220064))\tau_{4,135,7}(n) + (1/456597(-3646a1^2 - 6100a1 \\
& + 138072))\tau_{4,135,8}(n) + (1/456597(3646a1 - 9746)a2 \\
& + 1/456597(3646a1^2 - 3646a1 + 54214))\tau_{4,135,9}(n) \\
& + (1/456597(-3646a1 + 9746)a2 + 1/456597(9746a1 \\
& + 44468))\tau_{4,135,10}(n) + (26/11961a3^2 - 82/1329a3 \\
& + 1276/3987)\tau_{4,135,11}(n) + ((-26/11961a3 - 608/11961)a4 \\
& - 26/11961a3^2 + 130/11961a3 + 4036/11961)\tau_{4,135,12}(n) \\
& + ((26/11961a3 + 608/11961)a4 + 608/11961a3 \\
& + 332/3987)\tau_{4,135,13}
\end{aligned}$$

$$r_{F_2 \oplus \Psi_2}(n) =$$

$$\begin{aligned} & 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\ & + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) \\ & + 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) \\ & + 16/117\tau_{4,5}(n) + 1000/819\tau_{4,5}\left(\frac{n}{3}\right) + 1000/91\tau_{4,5}\left(\frac{n}{9}\right) \\ & + 1296/13\tau_{4,5}\left(\frac{n}{27}\right) + 16/135\tau_{4,9}(n) + 16/15\tau_{4,9}\left(\frac{n}{3}\right) \\ & + 80/27\tau_{4,9}\left(\frac{n}{5}\right) + 80/3\tau_{4,9}\left(\frac{n}{15}\right) + 4/27\tau_{4,15,1}(n) \\ & + 8/9\tau_{4,15,1}\left(\frac{n}{3}\right) + 12\tau_{4,15,1}\left(\frac{n}{9}\right) + 8/27\tau_{4,15,2}(n) \\ & + 272/63\tau_{4,15,2}\left(\frac{n}{3}\right) + 24\tau_{4,15,1}\left(\frac{n}{9}\right) + 64/405\tau_{4,27,1}(n) \\ & + 320/81\tau_{4,27,1}\left(\frac{n}{5}\right) + (8/153a6 + 146/459)\tau_{4,27,3}(n) \\ & + (200/153a6 + 3650/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-8/153a6 \\ & + 146/459)\tau_{4,27,4}(n) + (-200/153a6 + 3650/459)\tau_{4,27,4}\left(\frac{n}{5}\right) \\ & + 4/21\tau_{4,45,2}(n) + 12/7\tau_{4,45,2}\left(\frac{n}{3}\right) \\ & + 104/189\tau_{4,45,3}(n) + 104/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 8/45\tau_{4,45,4}(n) \\ & + 8/5\tau_{4,45,4}\left(\frac{n}{3}\right) + 16/153\tau_{4,135,2}(n) + (1/760995(1938a1^2 \\ & + 3296a1 + 59984))\tau_{4,135,5}(n) + (1/760995(-1938a1 + 5234)a2 \\ & + 1/760995(-1938a1^2 + 1938a1 + 104558))\tau_{4,135,6}(n) \\ & + (1/760995(1938a1 - 5234)a2 + 1/760995(-5234a1 \\ & + 109792))\tau_{4,135,7}(n) + (1/456597(2086a1^2 - 16844a1 \\ & + 74400))\tau_{4,135,8}(n) + (1/456597(-2086a1 - 14758)a2 \\ & + 1/456597(-2086a1^2 + 2086a1 + 122378))\tau_{4,135,9}(n) \\ & + (1/456597(2086a1 + 14758)a2 + 1/456597(14758a1 \\ & + 107620))\tau_{4,135,10}(n) + (-446/35883a3^2 + 1346/35883a3 \\ & + 3056/11961)\tau_{4,135,11}(n) + ((446/35883a3 - 884/35883)a4 \\ & + 446/35883a3^2 - 2230/35883a3 + 5600/35883)\tau_{4,135,12}(n) \\ & + ((-446/35883a3 + 884/35883)a4 + 884/35883a3 \\ & + 1180/35883)\tau_{4,135,13} \end{aligned}$$

$$r_{\Lambda_2 \oplus \Phi_2}(n) =$$

$$\begin{aligned}
& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
& + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) \\
& + 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) \\
& + 16/117\tau_{4,5}(n) + 1000/819\tau_{4,5}\left(\frac{n}{3}\right) + 1000/91\tau_{4,5}\left(\frac{n}{9}\right) \\
& + 1296/13\tau_{4,5}\left(\frac{n}{27}\right) + 16/135\tau_{4,9}(n) + 16/15\tau_{4,9}\left(\frac{n}{3}\right) \\
& + 80/27\tau_{4,9}\left(\frac{n}{5}\right) + 80/3\tau_{4,9}\left(\frac{n}{15}\right) - 4/27\tau_{4,15,1}(n) \\
& - 8/9\tau_{4,15,1}\left(\frac{n}{3}\right) - 12\tau_{4,15,1}\left(\frac{n}{9}\right) + 8/27\tau_{4,15,2}(n) \\
& + 272/63\tau_{4,15,2}\left(\frac{n}{3}\right) + 24\tau_{4,15,1}\left(\frac{n}{9}\right) + 64/405\tau_{4,27,1}(n) \\
& + 320/81\tau_{4,27,1}\left(\frac{n}{5}\right) + (8/153a6 + 146/459)\tau_{4,27,3}(n) \\
& + (200/153a6 + 3650/459)\tau_{4,27,3}\left(\frac{n}{5}\right) \\
& + (-8/153a6 + 146/459)\tau_{4,27,4}(n) + (-200/153a6 \\
& + 3650/459)\tau_{4,27,4}\left(\frac{n}{5}\right) - 4/21\tau_{4,45,2}(n) - 12/7\tau_{4,45,2}\left(\frac{n}{3}\right) \\
& - 104/189\tau_{4,45,3}(n) - 104/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 8/45\tau_{4,45,4}(n) \\
& + 8/5\tau_{4,45,4}\left(\frac{n}{3}\right) + 16/153\tau_{4,135,2}(n) + ((446/35883a3 \\
& - 884/35883)a4 - 884/35883a3 - 1180/35883)\tau_{4,135,13}
\end{aligned}$$

$$\begin{aligned}
& + (1/760995(1938a1^2 + 3296a1 + 59984))\tau_{4,135,5}(n) \\
& + (1/760995(-1938a1 + 5234)a2 + 1/760995(-1938a1^2 \\
& + 1938a1 + 104558))\tau_{4,135,6}(n) + (1/760995(1938a1 \\
& - 5234)a2 + 1/760995(-5234a1 + 109792))\tau_{4,135,7}(n) \\
& + (1/456597(-2086a1^2 + 16844a1 + 74400))\tau_{4,135,8}(n) \\
& + (1/456597(2086a1 + 14758)a2 + 1/456597(2086a1^2 \\
& - 2086a1 + 122378))\tau_{4,135,9}(n) + (1/456597(-2086a1 \\
& - 14758)a2 + 1/456597(-14758a1 - 107620))\tau_{4,135,10}(n) \\
& + (446/35883a3^2 - 1346/35883a3 - 3056/11961)\tau_{4,135,11}(n) \\
& + ((-446/35883a3 + 884/35883)a4 - 446/35883a3^2 \\
& + 2230/35883a3 - 5600/35883)\tau_{4,135,12}(n)
\end{aligned}$$

$$r_{\Lambda_2 \oplus \Psi_2}(n) =$$

$$\begin{aligned}
& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
& + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) \\
& + 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) \\
& + 16/117\tau_{4,5}(n) + 1000/819\tau_{4,5}\left(\frac{n}{3}\right) + 1000/91\tau_{4,5}\left(\frac{n}{9}\right) \\
& + 1296/13\tau_{4,5}\left(\frac{n}{27}\right) + 16/135\tau_{4,9}(n) + 16/15\tau_{4,9}\left(\frac{n}{3}\right) \\
& + 80/27\tau_{4,9}\left(\frac{n}{5}\right) + 80/3\tau_{4,9}\left(\frac{n}{15}\right) + 4/27\tau_{4,15,1}(n) \\
& + 8/9\tau_{4,15,1}\left(\frac{n}{3}\right) + 12\tau_{4,15,1}\left(\frac{n}{9}\right) + 8/27\tau_{4,15,2}(n) \\
& + 272/63\tau_{4,15,2}\left(\frac{n}{3}\right) + 24\tau_{4,15,1}\left(\frac{n}{9}\right) + 64/405\tau_{4,27,1}(n) \\
& + 320/81\tau_{4,27,1}\left(\frac{n}{5}\right) + (8/153a6 + 146/459)\tau_{4,27,3}(n) \\
& + (200/153a6 + 3650/459)\tau_{4,27,3}\left(\frac{n}{5}\right) + (-8/153a6 \\
& + 146/459)\tau_{4,27,4}(n) + (-200/153a6 + 3650/459)\tau_{4,27,4}\left(\frac{n}{5}\right) \\
& + 3650/459\tau_{4,27,4}\left(\frac{n}{5}\right) + 4/21\tau_{4,45,2}(n) + 12/7\tau_{4,45,2}\left(\frac{n}{3}\right) \\
& + 104/189\tau_{4,45,3}(n) + 104/21\tau_{4,45,3}\left(\frac{n}{3}\right) + 8/45\tau_{4,45,4}(n) \\
& + 8/5\tau_{4,45,4}\left(\frac{n}{3}\right) + 16/153\tau_{4,135,2}(n) + (1/760995(1938a1^2
\end{aligned}$$

$$\begin{aligned}
& + 3296a1 + 59984))\tau_{4,135,5}(n) + (1/760995(-1938a1 \\
& + 5234)a2 + 1/760995(-1938a1^2 + 1938a1 \\
& + 104558))\tau_{4,135,6}(n) + (1/760995(1938a1 - 5234)a2 \\
& + 1/760995(-5234a1 + 109792))\tau_{4,135,7}(n) + \\
& (1/456597(2086a1^2 - 16844a1 - 74400))\tau_{4,135,8}(n) \\
& + (1/456597(-2086a1 - 14758)a2 + 1/456597(-2086a1^2 \\
& + 2086a1 - 122378))\tau_{4,135,9}(n) + (1/456597(2086a1 \\
& + 14758)a2 - 1/456597(14758a1 + 107620))\tau_{4,135,10}(n) \\
& + (-446/35883a3^2 + 1346/35883a3 \\
& + 3056/11961)\tau_{4,135,11}(n) + ((446/35883a3 \\
& - 884/35883)a4 + 446/35883a3^2 - 2230/35883a3 \\
& + 5600/35883)\tau_{4,135,12}(n) + ((-446/35883a3 \\
& + 884/35883)a4 + 884/35883a3 + 1180/35883)\tau_{4,135,13}
\end{aligned}$$

$$r_{\Phi_2 \oplus \Psi_2}(n) =$$

$$\begin{aligned}
& 240/21060\sigma_3(n) + 480/5265\sigma_3\left(\frac{n}{3}\right) + 1200/4212\sigma_3\left(\frac{n}{5}\right) \\
& + 480/585\sigma_3\left(\frac{n}{9}\right) + 2400/1053\sigma_3\left(\frac{n}{15}\right) + 2160/260\sigma_3\left(\frac{n}{27}\right) \\
& + 2400/117\sigma_3\left(\frac{n}{45}\right) + 10800/52\sigma_3\left(\frac{n}{135}\right) \\
& - 56/351\tau_{4,5}(n) - 2096/2457\tau_{4,5}\left(\frac{n}{3}\right) - 2096/273\tau_{4,5}\left(\frac{n}{9}\right) \\
& - 1512/13\tau_{4,5}\left(\frac{n}{27}\right) - 16/135\tau_{4,9}(n) - 16/15\tau_{4,9}\left(\frac{n}{3}\right) \\
& - 80/27\tau_{4,9}\left(\frac{n}{5}\right) - 80/3\tau_{4,9}\left(\frac{n}{15}\right) + 16/21\tau_{4,15,2}\left(\frac{n}{3}\right) \\
& + (4/81a6 + 4/27)\tau_{4,27,3}(n) + (100/81a6 + 100/27)\tau_{4,27,3}\left(\frac{n}{5}\right) \\
& + (-4/81a6 + 4/27)\tau_{4,27,4}(n) + (-100/81a6 \\
& + 100/27)\tau_{4,27,4}\left(\frac{n}{5}\right) + 16/135\tau_{4,45,4}(n) + 16/15\tau_{4,45,4}\left(\frac{n}{3}\right) \\
& + (1/152199(1228a1^2 + 2104a1 - 27456))\tau_{4,135,5}(n) \\
& + (1/152199(-1228a1 + 3332)a2 + 1/152199(-1228a1^2 \\
& + 1228a1 + 788))\tau_{4,135,6}(n) + (1/152199(1228a1 - 3332)a2 \\
& + 1/152199(-3332a1 + 4120))\tau_{4,135,7}(n)
\end{aligned}$$

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