

Branching Pattern Modeling for *Psychotria* Species and Their Deterministic Algorithm

Abstract

Back Ground: The branching patterns and the subsequent elongations of two shrub *Psychotria* species, *Psychotria rubra* and *Psychotria manillensis*, in Okinawa Island, the Ryukyu Islands, Japan, were examined by Watanabe. His results showed that Hamilton's classification for *Psychotria* species in the South American continent is also applicable to the same species in Okinawa Island. After Watanabe's research, the author of the present paper introduced branching pattern models for same species to simulate their branching patterns theoretically. He also determined the branching pattern models in earlier growth stages and calculated their occurrence probabilities. To proceed with the analyses, a systematic construction of the branching pattern models has been required.

Aims: To refine previous branching pattern models of *P. rubra* and *P. manillensis* in view of vector representation and to introduce a deterministic algorithm which produces the branching pattern models systematically.

Methodology: Introducing vector representations, we refine known branching pattern models for *P. rubra* and *P. manillensis*. We also study the properties of the branching pattern models in view of computational methods. Finally, we describe a deterministic algorithm for the branching pattern models by pseudo-code.

Results: By studying properties of branching pattern of *Psychotria* species in mathematical way, we systematize the construction of branching pattern models as a deterministic algorithm. Running this algorithm, we also obtain the third branching pattern models.

Conclusion: A systematic determination algorithm which produces the branching pattern models inductively has been successfully established.

Keywords: *Psychotria rubra*; *Psychotria manillensis*; branching patterns; mathematical modeling; Okinawa Island.

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1 Introduction

For a long time, tree architecture has been attracting many scholars. Plant body aboveground consists of reputations of module structures which called shoots ([10]). Namely, tree architecture is formed by reputations of branching and subsequent elongations of branches ([3]). So it is important for the understanding of tree architecture to know the branching patterns and elongation patterns. In this paper, we focus our attention on branching pattern of certain subtropical tree species.

Hallé et al [4] studied the structure of branching patterns of tropical trees and classified their branching patterns into twenty three patterns. Hamilton [5] showed that *Psychotria* species in the South American continent have three patterns in Hall's classification. In the point of view of taxonomy, Sohmer and Davis [9] indicated the effectiveness of the Hamilton's results for *Psychotria* species in Philippines. Watanabe [11] studied the branching patterns and the subsequent elongations of *Psychotria rubra* and *Psychotria manillensis* in Okinawa Island. His results showed that Hamilton's classification of branching pattern for *Psychotria* species is also applicable to those in Okinawa Island. To simulate branching patterns of *P. rubra* and *P. manillensis* theoretically, the author of this paper [13] introduced branching pattern models based on Watanabe's static datum. He also determined the branching pattern models in earlier growth stages and argued the properties of the branching patterns of *P. rubra* and *P. manillensis*. For further analyses, we need to construct further branching pattern models. However, there are some difficulties in the construction. Further branching pattern models are more complicated and the total number of these is so big. To overcome such difficulties, we introduce a deterministic algorithm for our branching pattern models in the present article.

This paper is organized as follows: In Section 2, we first recall fundamental features about branching patterns of *Psychotria* species including Watabe's results. Taking these facts into consideration, we introduce branching pattern models for *P. rubra* and *P. manillensis*, under ideal growth conditions. In Section3, we constructed a deterministic algorithm which produce our branching pattern models inductively. By using the algorithm, we determine the first, second and third branching pattern models in Section 4. Finally, we give the conclusions of this paper in Section 5.

2 Branching Pattern Models for *Psychotria* Species

In this section, we first recall fundamental features of branching patterns of *Psychotria* species. We also refine branching pattern models for same species which were introduced in [13].

2.1 Fundamental Features of Branching Patterns

It is known that *Psychotria* species flower at apices of their shoots. After flowering, new branches are produced directly under the apices. Hamilton [5] indicated that the three types Type 1 (Chamberlain), Type 2 (Leeuwenberg), Type 3 (Koriba) occur as the branching patterns of *Psychotria* species (see Figure 1 below). In Type 1, a new branch appears directly under the apex and grows preserving the direction of the original branch. On the other hand, in Type 2, two new branches appear and both of them grow with new directions (i.e., they do not preserve the original direction of the old branch). In Type 3, we also have two new branches and one of them keeps the original direction and the other does not.

In [11], Watanabe examined the branching patterns and the subsequent elongations of *P. rubra* and *P. manillensis* in Okinawa Island. They are two shrub *Psychotria* speices spread in the Ryukyu Islands. Though they are allied species, they possess some different features. For example, their soil palatabilities are different. *P. rubra* prefers limestone area. By contrast with this, *P. manillensis*

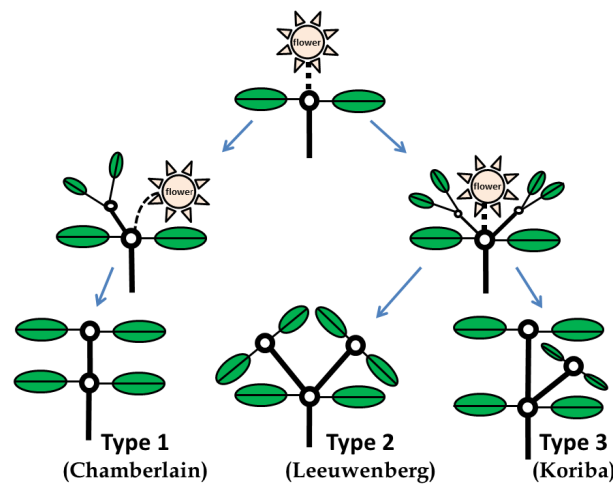


Figure 1: Three fundamental branching patterns of *Psychotria* Species

is widely distributed over non-limestone area. It is known that they are important component tree species in two different types of evergreen broad-leaved forests in the Ryukyu Islands, limestone forests and non-limestone forests, respectively (see [7]). The research in [11] was conducted at two mountains in Okinawa Island, Mt. Nago-take (26352N, 128002E, 230m a.s.l.) for *P. rubra* and Katsuudake (26372N, 27562E, 320m a.s.l.) for *P. manillensis*. For both species, ten samples were randomly selected and their branching patterns were analyzed. Watanabe concluded that Hamilton's classification of branching patterns of *Psychotria* species is applicable to two *Psychotria* species in Okinawa Island. He also found that the appearance rates of three branching patterns for two species are very different. *P. rubra* is formed by the reputations of the branching pattern Type 2. On the other hand, there is no bias in the branching patterns for *P. manillensis*.

2.2 Growth Conditions and Branching pattern models

In the construction of our mathematical models for *Psychotria* species, we only pay attention to the branching patterns and ignore the other growth factors (e.g., elongations of branches, how to grow leaves etc.). Taking account of the fundamental features of branching patterns of *Psychotria* species in Subsection 2.1, we always suppose that the growth of *Psychotria* species satisfies the following ideal conditions:

Condition and Definition 2.1 (The growth conditions for *Psychotria* species in Okinawa Island). The growth of *Psychotria* species always satisfies the followings:

1. The germination always occurs. The sprout is described as one stem (see Figure 2 below).
2. After the first flowering at the apex of the sprout, one of Type 1, Type 2 and Type 3 always occurs directly under the apex. We call plant bodies just after the first branching the first branching pattern models of *Psychotria* species.
3. Inductively, for $l \geq 2$, we always have the l th branching after the l th flowering. It also occurs directly under the latest apices of an $(l - 1)$ th model. Here latest apices are defined to be the apices of shoots arose from the $(l - 1)$ th branching. We also call plant bodies just after the l th branching the l th branching pattern models of *Psychotria* species.

4. Branches are never lost by abortion, withering and any other reasons.

Remark 2.1. In [5], Hamilton showed that the transformations of Types 1, 2 and 3 were also occurred as the branching patterns of *Psychotria* species in South American continent. However, Watanabe excluded these transformations in [11], because their definitions were obscure. We follow Watanabe's philosophy in this paper.

Remark 2.2. It is known that the frequency of flowering of *Psychotria* species depends on the regions where they live (see [12]). *P. rubra* and *P. manillensis* in Okinawa Island flower once a year. On the other hand, it was indicated that *Psychotria* species in Taiwan make flowers more than one time in a year.

The growth of *Psychotria* species in earlier stages is illustrated as in Figure 2 and 3. Note that the pictures in the figures only emphasize the number of shoots and whether new branches preserve the original directions or not. As mentioned above, the other factors of growth are all omitted. The first branching pattern models are easily determined as in Figure 2. Taking account of all possible combinations of three fundamental branching patterns and latest apices of first branching pattern models, we can also obtain all second branching models as in Figure 3.

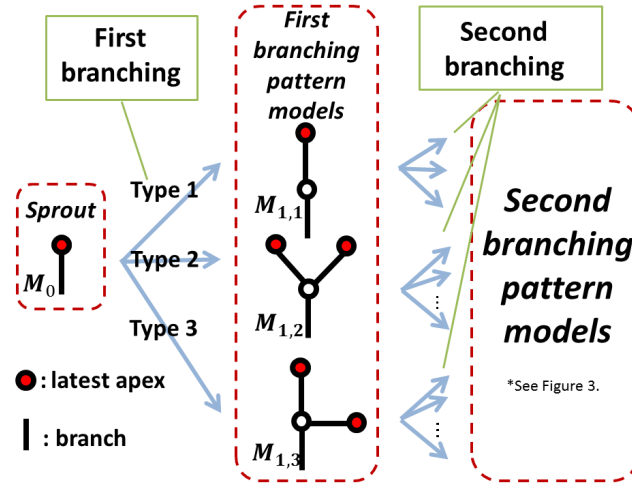


Figure 2: The growth of *Psychotria* species under the conditions 1-4.

Fix a positive integer r . Let M be an r th branching pattern model for *Psychotria* species. We denote by $m_r(M)$ the number of latest apices of M . If b_i Type i branching pattern occur in l th branching ($i = 1, 2, 3$ and $1 \leq l \leq r$), then we say the l th branching is given the vector $\mathbf{b} = (b_1, b_2, b_3)$.

Definition 2.1. We call a vector $\mathbf{b}$ which gives a branching pattern a branching vector.

Note that an r th branching pattern model is yielded by successive r branching pattern occurrences. Let M be an r th branching pattern model which is yielded by $\mathbf{b}_{i_l}$ as its l th branching ($1 \leq l \leq r$). We identify M with the set of branching vectors $\{\mathbf{b}_{i_1}, \dots, \mathbf{b}_{i_r}\}$. The growth of M is illustrated by the following sequence:

$$\{\emptyset\} \xrightarrow{\mathbf{b}_{i_1}} \{\mathbf{b}_{i_1}\} \xrightarrow{\mathbf{b}_{i_2}} \dots \xrightarrow{\mathbf{b}_{i_l}} \{\mathbf{b}_{i_1}, \dots, \mathbf{b}_{i_l}\} \xrightarrow{\mathbf{b}_{i_{l+1}}} \dots \xrightarrow{\mathbf{b}_{i_r}} M = \{\mathbf{b}_{i_1}, \dots, \mathbf{b}_{i_l}, \dots, \mathbf{b}_{i_r}\}$$

Definition 2.2. We call $M = \{\mathbf{b}_{i_1}, \dots, \mathbf{b}_{i_r}\}$ the vector representation of M . In particular, we formally express the sprout M_0 by the set $\{\emptyset\}$ which consists of an empty set.

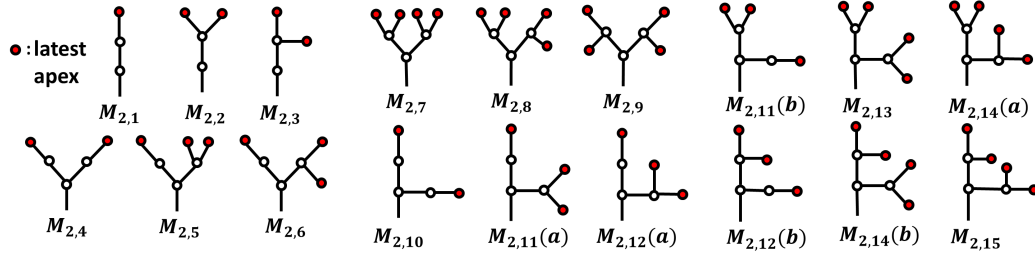


Figure 3: Second branching pattern models for *Psychotria* species.

We always consider the vector representation of our branching pattern models below.

Remark 2.3. Plenty of methods have been proposed by many scholars to model the growth processes of plants (cf. [8] and references there in). In particular, there is a very popular mathematical theory for plant development, which is called Lindenmayer system (or L-system for short). It was introduced by Aristid Lindenmayer in [6]. The essential difference between our branching pattern models and L-systems lies in the descriptions of generating branches. Our models produce new branches sequentially, whereas in L-systems branches are produced in parallel and simultaneously replaced.

3 A Deterministic Algorithm for Branching Pattern Models

In this section, we explain our computational methods.

3.1 Auxiliary Lemmas

We fix notations and prove some lemmas needed later. We consider an r th branching pattern model $M = \{b_{i_1}, \dots, b_{i_r}\}$ of *Psychotria* species. Let l be an integer with $1 \leq l \leq r$. By abuse of notations, we denote by $m_l(M)$ the number of latest apices of the l th branching pattern model $\{b_{i_1}, \dots, b_{i_l}\}$ associated with M .

Lemma 3.1. *Let $M = \{b_{i_1}, \dots, b_{i_r}\}$ be an r th branching pattern model where $b_{i_l} = (b_{i_l1}, b_{i_l2}, b_{i_l3})$ ($l = 1, \dots, r$). For any integer l with $1 \leq l \leq r$, the following relation hold:*

$$m_l(M) = b_{i_l1} + 2b_{i_l2} + 2b_{i_l3} \quad (3.1)$$

Similarly, for any integer l with $2 \leq l \leq r$, we have

$$m_{l-1}(M) = b_{i_l1} + b_{i_l2} + b_{i_l3}. \quad (3.2)$$

Proof. We first prove the relation (3.1). If a Type 1 branching occurs in the l th branching, then one latest apex will be added to the l th branching pattern model. On the hand, if one of the branching patterns Type 2 and Type 3 occurs in the l th branching, then the l th branching pattern model obtains two new apices. The relation (3.1) follows from these facts. Next we consider relation (3.2). Note that, for $b_{i_l} = (b_{i_l1}, b_{i_l2}, b_{i_l3})$, the total $b_{i_l1} + b_{i_l2} + b_{i_l3}$ must coincide with the number of the latest apices of the $(l-1)$ th branching pattern model $\{b_{i_1}, \dots, b_{i_{l-1}}\}$. Namely, the relation (3.2) holds. $\square$

Proposition 3.1. *An r th branching pattern model $M = \{b_{i_1}, \dots, b_{i_r}\}$ exists if and only if the relation (3.2) holds for any l with $2 \leq l \leq r$.*

Proof. If a branching $\mathbf{b}_{i_l} = (b_{i_l1}, b_{i_l2}, b_{i_l3})$ satisfies (3.2), then it can occur as the l th branching on the $(l-1)$ th branching pattern model $\{\mathbf{b}_{i_1}, \dots, \mathbf{b}_{i_{l-1}}\}$. The "if part" has proven. The "only if part" is trivial. $\square$

Define A_l and B_l to be the set of all branching vectors which can occur as l th branching and that of all l th branching pattern models respectively. We also set

$$C_l := \{m_l(M) | M \in B_l\}.$$

Definition 3.1. For a positive integer l with $1 \leq l \leq r$, we call the triple $\{A_l, B_l, C_l\}$ the data of the l th branching patterns. We also define $\{A_0, B_0, C_0\}$ to be the triple with $A_0 = \{\emptyset\}$, $B_0 = \{M_0 = \{\emptyset\}\}$ and $C_0 = \{1\}$.

3.2 Computational Methods

In this subsection, we introduce some computational methods. In particular, Proposition 3.3 is described by pseudo-code. pseudo-codes are not computer programs, but are more structured than usual prose. In particular, they are mixture of natural language and familiar structure from programming language. About pseudo-codes, refer to [2]. Pseudo-codes are not only used in information theory, but also in mathematics (cf. [1, Appendix B]).

Proposition 3.2. For l with $2 \leq l \leq r$, the set A_l is given by

$$A_l = \{(b_1, b_2, b_3) | b_i \in \mathbb{Z}_{\geq 0}, b_1 + b_2 + b_3 \in C_{l-1}\}$$

where $\mathbb{Z}_{\geq 0}$ is the set of all non-negative integers.

Proof. Our assertion follows from the relation (3.2) in Lemma 3.1. $\square$

Proposition 3.3. For l with $2 \leq l \leq r$, the set B_l is constructed from A_l and B_{l-1} by the following algorithm:

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INPUT  $A_l$  and  $B_{l-1}$ 
OUTPUT  $B_l$ 
DEFINE  $B_l = \emptyset$ 
FOR each pair  $(\mathbf{b} = (b_1, b_2, b_3), M) \in A_l \times B_{l-1}$ 
  IF  $m_{l-1}(M) = b_1 + b_2 + b_3$  THEN put  $B_l = B_l \cup \{M \cup \{\mathbf{b}\}\}$ 
  ELSE do nothing

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Proof. The necessary and sufficient condition for the occurrence of an l th branching pattern $\mathbf{b} = (b_1, b_2, b_3)$ on an $(l-1)$ th branching pattern model M is $m_{l-1}(M) = b_1 + b_2 + b_3$ by Proposition 3.1. In other words, an l th branching pattern model $M \cup \{\mathbf{b}\}$ exists iff $m_{l-1}(M) = b_1 + b_2 + b_3$ holds. Furthermore, this algorithm terminates in finite steps. $\square$

Proposition 3.4. We have

$$C_l = \{b_1 + 2b_2 + 2b_3 | (b_1, b_2, b_3) \in A_l\}.$$

Proof. Our assertion follows from the definition of C_l and (3.1) in Lemma 3.1. $\square$

Algorithm 3.2 (Deterministic Algorithm for the Branching Pattern Models). Let $\{A_0, B_0, C_0\}$ be the triple defined in Definition 3.1. For a given positive integer r , the data $\{A_r, B_r, C_r\}$ of the r th branching patterns is completely determined by the following finite steps:

Step l - (i) ($l = 1, \dots, r$): Determine A_l by Proposition 3.2 (i.e. collecting all triples (b_1, b_2, b_3) of non-negative integers which satisfy $b_1 + b_2 + b_3 \in C_{l-1}$).

Step l - (ii): *Determine B_l from A_l and B_{l-1} by Proposition 3.3.*

Step l - (iii): *Determine C_l by Proposition 3.4.*

Proof. Our assertion follows from Proposition 3.2, 3.3 and 3.4. □

4 Application of The Algorithm

In this section, we determine all the first, second and third branching pattern models by Algorithm 3.2. Though the first and second vector models already had obtained in [13], we revisit them in view of our new computational methods.

4.1 First, Second and Third Branching Pattern Models

For $r = 3$, we run Algorithm 3.2. We first do Step 1-(i) in Algorithm 3.2. By Proposition 3.2 with $C_0 = \{1\}$, we easily obtain $A_1 = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ where $\mathbf{b}_1 := (1, 0, 0)$, $\mathbf{b}_2 := (0, 1, 0)$ and $\mathbf{b}_3 := (0, 0, 1)$. Next we add branching pattern $\mathbf{b}_i$ ($i = 1, 2, 3$) to the sprout $M_0 = \{\emptyset\}$ as Step 1-(ii). Consequently, we obtain

$$M_{1,1} := M_0 \cup \{\mathbf{b}_1\} = \{\mathbf{b}_1\}, M_{1,2} := M_0 \cup \{\mathbf{b}_2\} = \{\mathbf{b}_2\}, M_{1,3} := M_0 \cup \{\mathbf{b}_3\} = \{\mathbf{b}_3\}.$$

Namely, we have $B_1 = \{M_{1,1}, M_{1,2}, M_{1,3}\}$. All first branching pattern models are determined. As the final part of Step 1, we do Step 1-(iii). It follows from Proposition 3.4 and $\mathbf{b}_1 = (1, 0, 0)$, $\mathbf{b}_2 = (0, 1, 0)$ and $\mathbf{b}_3 = (0, 0, 1)$ that $C_1 = \{1, 2\}$. Step 1 has finished.

Similarly, we proceed with Step 2. As Step 2-(i), we determine the set A_2 of branching vectors which can appear in second branching. Same as Step 1-(i), the number $1 \in C_1$ yields the branching vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$. On the other hand, for $2 \in C_1$, we consider all combinations of three non-negative integers b_1 , b_2 and b_3 with $b_1 + b_2 + b_3 = 2$. We obtain the following 6 branching vectors:

$$\mathbf{b}_4 := (2, 0, 0), \mathbf{b}_5 := (1, 1, 0), \mathbf{b}_6 := (1, 0, 1), \mathbf{b}_7 := (0, 2, 0), \mathbf{b}_8 := (0, 1, 1), \mathbf{b}_9 := (0, 0, 2)$$

After all, we have $A_2 = \{\mathbf{b}_i \mid i = 1, \dots, 9\}$. Next we apply Proposition 3.3 to A_2 and B_1 as Step 2-(ii). Note that the branching vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ are only elements in A_2 whose totals of all components are 1 and the branching pattern model M_1 is an only element in B_1 whose number of latest apices is 1. So we combine each of $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ and M_1 . These combinations yield the following elements of B_2 :

$$M_{2,1} := M_{1,1} \cup \{\mathbf{b}_1\} = \{\mathbf{b}_1, \mathbf{b}_1\}, M_{2,2} := M_{1,1} \cup \{\mathbf{b}_2\} = \{\mathbf{b}_1, \mathbf{b}_2\}, M_{2,3} := M_{1,1} \cup \{\mathbf{b}_3\} = \{\mathbf{b}_1, \mathbf{b}_3\}.$$

On the other hand, branching vectors $\mathbf{b}_i$ ($i = 4, \dots, 9$) have their totals of all components are 2 and two elements $M_{1,2}$ and $M_{1,3}$ in B_1 have two latest apices. Taking account of all combinations of these, we obtain the rests of elements in B_2 are determined as follows:

$$\begin{aligned} M_{2,4} &:= \{\mathbf{b}_2, \mathbf{b}_4\}, & M_{2,5} &:= \{\mathbf{b}_2, \mathbf{b}_5\}, & M_{2,6} &:= \{\mathbf{b}_2, \mathbf{b}_6\}, & M_{2,7} &:= \{\mathbf{b}_2, \mathbf{b}_7\} \\ M_{2,8} &:= \{\mathbf{b}_2, \mathbf{b}_8\}, & M_{2,9} &:= \{\mathbf{b}_2, \mathbf{b}_9\}, & M_{2,10} &:= \{\mathbf{b}_3, \mathbf{b}_4\}, & M_{2,11} &:= \{\mathbf{b}_3, \mathbf{b}_5\} \\ M_{2,12} &:= \{\mathbf{b}_3, \mathbf{b}_6\}, & M_{2,13} &:= \{\mathbf{b}_3, \mathbf{b}_7\}, & M_{2,14} &:= \{\mathbf{b}_3, \mathbf{b}_8\}, & M_{2,15} &:= \{\mathbf{b}_3, \mathbf{b}_9\} \end{aligned}$$

Hence, $B_2 = \{M_{2,i} \mid i = 1, \dots, 15\}$. We do Step 2-(iii) as the final step of Step 2. It follows from Proposition 3.4 with $\mathbf{b}_i$ ($i = 1, \dots, 9$) that $C_2 = \{1, 2, 3, 4\}$. Step 2 has finished.

Remark 4.1. In Figure 3, the branching pattern models $M_{2,11}$, $M_{2,12}$ and $M_{2,14}$ give two pictures respectively.

4.2 First, Second and Third Branching Pattern Models, continued

Similar to subsection 4.1, the procedures for Step 3 can be performed. So we only list the data of third branching patterns here. We have

$$A_3 = A_2 \cup \{b_i \mid i = 10, \dots, 34\}$$

where

$$\begin{aligned} b_{10} &:= (3, 0, 0), b_{11} := (2, 1, 0), b_{12} := (2, 0, 1), b_{13} := (1, 2, 0), b_{14} := (1, 1, 1), b_{15} := (1, 0, 2) \\ b_{16} &:= (0, 3, 0), b_{17} := (0, 2, 1), b_{18} := (0, 1, 2), b_{19} := (0, 0, 3), b_{20} := (4, 0, 0), b_{21} := (3, 1, 0) \\ b_{22} &:= (3, 0, 1), b_{23} := (2, 2, 0), b_{24} := (2, 1, 1), b_{25} := (2, 0, 2), b_{26} := (1, 3, 0), b_{27} := (1, 2, 1) \\ b_{28} &:= (1, 1, 2), b_{29} := (1, 0, 3), b_{30} := (0, 4, 0), b_{31} := (0, 3, 1), b_{32} := (0, 2, 2), b_{33} := (0, 1, 3) \\ b_{34} &:= (0, 0, 4). \end{aligned}$$

The set B_3 of all third branching pattern models consists of the following vector representations:

$$\begin{aligned} &\{b_1, b_1, b_i\} \text{ for } i = 1, 2, 3 \\ &\{b_1, b_i, b_j\} \text{ for } i = 1, 2 \text{ and } j = 4, \dots, 9 \\ &\{b_i, b_4, b_j\} \text{ for } i = 2, 3 \text{ and } j = 4, \dots, 9 \\ &\{b_i, b_j, b_k\} \text{ for } i = 2, 3, j = 5, 6 \text{ and } k = 10, \dots, 19 \\ &\{b_i, b_j, b_k\} \text{ for } i = 2, 3, j = 7, 8, 9 \text{ and } k = 20, \dots, 34 \end{aligned}$$

We also have $C_3 = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

5 Conclusions

The present study provides the followings:

- a** In Section 2, we have introduced branching pattern models for *P. rubra* and *P. manillensis*, which are the refinements of previous versions obtained in [13].
- b** In Section 3, we have constructed a deterministic algorithm to produce refined branching pattern models for *P. rubra* and *P. manillensis* systematically.
- c** In Section 4, we determined all first, second and third branching pattern models by using the algorithm.

Applying the deterministic algorithm, we have succeeded in the determinations of all first, second and third branching pattern models. The application indicated that our algorithm is effective for the systematic construction of branching pattern models for *P. rubra* and *P. manillensis*.

References

- [1] Cox, D., Little, J., and O'Shea, D. (2007) *Ideals, Varieties, And Algorithms*. An introduction to computational algebraic geometry and commutative algebra. Third edition. Undergraduate Texts in Mathematics. Springer, New York.
- [2] Goodrich, M. T. and Tamassia R. (1997). *Data Structures and Algorithms in Java*. second edition. John Wiley and Sons, Inc, New York, Chichester, Weinheim, Brisbane, Singapore, Toronto.
- [3] Hallé, F. (1978). Architectural variation at the specific level in tropical trees, in Tomlinson, B. and Zimmermann, M. H. (Eds.), *Tropical trees as living systems*, Cambridge University Press, Cambridge, U.K., 209–221.
- [4] Hallé, F., Oldeman, R. A. A. and Tomlinson, P. B. (1978). *Tropical trees and forests*. An architectural analysis. Springer, Berlin Heidelberg New York.
- [5] Hamilton, C. (1985). Architecture in neotropical *Psychotria* L.(Rubiaceae): dynamics of branching and its taxonomic significance. *American journal of botany*, 72(7), 1081–1088.
- [6] Lindenmayer, A. (1968). Mathematical models for cellular interaction in development, part I, part II. *J.Thor. Biol.* 18, 280–315.
- [7] Miyawaki, A. (1989). *NIHON SHOKUSEISHI 10 OKINAWA-OGASAWARA*. Shibundo Publishing.
- [8] Prusinkiewicz, P. (2004). Modeling plant growth and development, *Current Opinion in Plant Biology*, 7, 79–83.
- [9] Sohmer, S. H. and Davis, A. P. (2007). The genus *Psychotria* (Rubiaceae) in Philippine Archipelago. *Sida, Botanical Miscellany*, 27.
- [10] Takenaka, A. (2004). The forms of plants which receive the light, in Kouyama, T. (Ed.), *Plant ecology*. Asakura publishing Co., Ltd. Tokyo, 81–113.
- [11] Watanabe, K. (2014). The branching patterns and architectures in two subtropical tree species (*Psychotria rubra* and *P. manillensis*) in the Ryukyu Islands, Japan. *Bulletin of Okinawa National College of Technology*, 8, 53–61 (in Japanese).
- [12] Watanabe, K. (2013) The phenology of two *Psychotria* species in different regions: Relations among latitudes, species interactions and flowering. *Okinawa National College of Technology, Technical Report*, 8, 52-55. (in Japanese).
- [13] Watari, M. The branching pattern models for two subtropical tree species (*Psychotria rubra* and *Psychotria manillensis*) in Okinawa Island. *Bulletin of Okinawa National College of Technology*, 9, 21–28 (in Japanese).