

**The Complementary Exponentiated Inverted Weibull Power Series Family of Distributions
and its Applications**

Abstract

The statistical literature has several distributions for modeling lifetime data. But may if not most of these distributions lack motivation from a lifetime context. Recently, attempts have been made to define new families of probability distributions that extended well known families of distributions and at the same time provide great flexibility in modeling data in practice. In this paper, a new family of distributions is introduced by compounding exponentiated inverted Weibull and power series distributions. The properties of the proposed family are discussed, including quantiles, moments and moment generating function. The estimation of the model parameters is performed by maximum likelihood method. The new family generalizes some new lifetime distributions. In particular, two special models of the new family are studied in some details. Applications to real data sets are given to show the flexibility and potentiality of the proposed family of distributions.

Keywords: Exponentiated inverted Weibull ditribution, power series distribution; Order statistics; hazard function.

1 INTRODUCTION

In recent years, compound distributions arise and applied in several areas, such as public health, economics, engineering, and industrial reliability. New several compound models of distributions have been introduced follows same way that was previously carried out by [1]. Many researchers have proposed a series of new compounding distributions by mixing the distribution when the lifetime can be expressed as the minimum or maximum of a sequence of independent and identical distributed, random variable which represents the failure time of the system.

Several authors have proposed a new compounding distributions by mixing the distribution of a maximum of a fixed number of independent for continuous lifetime distributed random variables and discrete type. The exponential truncated Poisson with increasing failure rate introduced by [2]. A new lifetime distribution, which is called the exponentiated Weibull-geometric distribution was introduced by [3]. This new distribution obtained by compounding the exponentiated Weibull and geometric distributions. For more about another compounding distribution; see ([4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and [14]. Furthermore, several authors derived new compounding distribution by mixing continuous distribution with power series distribution for both schemes (minimum or maximum). [15] defined the Weibull power series family which includes as sub-models the exponential power series distributions. The generalized exponential power series distribution was introduced by [16]. A new class of extended Weibull power series distributions was introduced by [17]. Two compound families, namely the max-Erlang power series distribution and min-Erlang power series distribution are proposed by [18]. A new family of Burr XII power series models introduced by [19]. A new class of lifetime distributions called the Lindley power series is proposed by [20]. A new life time with decreasing failure rate called inverse Weibull power series distribution is introduced by [21]

In this article, a new lifetime family of distributions is introduced by compounding exponentiated inverted Weibull and power series distributions. The density, cumulative, survival and hazard rate functions of the new family are introduced in Section 2. In Section 3, some mathematical properties of the new family are obtained such as, quantiles and moments. In the same section some distributions of order statistics are derived. In Section 4, some special sub-models and mathematical properties of the complementary exponentiated inverted Weibull power series distributions are given. In Section 5, maximum likelihood estimator of the unknown parameters for the family is obtained. In section 6, three illustrative examples based on real data are provided. Finally, concluding remarks are addressed in Section 7.

2 The New Family of Distributions

The inverted Weibull distribution is one of the most popular probability distribution to analyze the life time data with some monotone failure rates. The usefulness and applications of inverse Weibull distribution in various areas including reliability, and branching processes can be seen in [22] and in references therein. Exponentiated (generalized) inverted Weibull distribution is a generalization to the inverted Weibull distribution through adding a new shape parameter. The probability density function (pdf) of exponentiated inverted Weibull (EIW) with two shape parameters; takes the following form

$$g(y; \theta, \beta) = \theta \beta y^{-(\beta+1)} \left(e^{-y^{-\beta}} \right)^{\theta}, y, \theta, \beta > 0, \quad (1)$$

The corresponding cumulative distribution (cdf) is

$$G(y; \theta, \beta) = \left(e^{-y^{-\beta}} \right)^{\theta}, y, \theta, \beta > 0, \quad (2)$$

for $\theta = 1$, it represents the standard inverted Weibull distribution, for $\beta = 1$, it represents the exponentiated standard inverted exponential distribution (see[23]).

Let $Y = \max\{Y_i\}_{i=1}^Z$, be independent identically distribution random variables having the exponentiated inverted Weibull distribution with shape parameters $\theta > 0$ and $\beta > 0$, and Z is a discrete random variable has the following power series probability mass function (truncated at zero)

$$P(Z = z; \lambda) = \frac{a_z \lambda^z}{A(\lambda)}, z = 1, 2, \dots \quad (3)$$

The coefficient a_z 's depends only on z , $A(\lambda) = \sum_{i=1}^{\infty} a_z \lambda^z$, $\lambda > 0$ is such that $A(\lambda)$ is finite.

Table (1): Useful quantities of some Power series distributions

Distributions	a_z	$A(\lambda)$	$A'(\lambda)$	$A''(\lambda)$	$A(\lambda)^{-1}$	λ
Poisson	$z!^{-1}$	$e^{\lambda} - 1$	e^{λ}	e^{λ}	$\log(\lambda + 1)$	$\lambda \in (0, \infty)$
Logarithm	z^{-1}	$-\log(1 - \lambda)$	$(1 - \lambda)^{-1}$	$(1 - \lambda)^{-2}$	$1 - e^{-\lambda}$	$\lambda \in (0, 1)$
Geometric	1	$\lambda(1 - \lambda)^{-1}$	$(1 - \lambda)^{-2}$	$2(1 - \lambda)^{-3}$	$\lambda(\lambda + 1)^{-1}$	$\lambda \in (0, 1)$
Binomial	$\binom{m}{z}$	$(\lambda + 1)^m - 1$	$m(\lambda + 1)^{m-1}$	$\frac{m(m-1)}{(\lambda+1)^{2-m}}$	$(\lambda - 1)^{\frac{1}{m}} - 1$	$\lambda \in (0, \infty)$

[6] introduced a compound class of distributions which is obtained by mixing the maximum of a sequence of identically independent any lifetime distributed random variables with power series random variable. The cumulative distribution function of this class is given by

$$F(y; \lambda, \underline{\phi}) = \frac{A[\lambda G(y)]}{A(\lambda)}, y > 0, \lambda > 0, \underline{\phi} = (\phi_1, \phi_2, \dots, \phi_l). \quad (4)$$

In this study, assume that $G(y)$ is the cdf of EIW distribution. Therefore, the distribution function of the complementary exponentiated inverted Weibull power series (CEIWPS) family of distributions is obtained by substituting cdf (2) in cdf (4) as follows

$$F(y; \theta, \beta, \lambda) = \frac{A\left[\lambda \left(e^{-y^{-\beta}}\right)^{\theta}\right]}{A(\lambda)}, y > 0, \theta, \beta, \lambda > 0. \quad (5)$$

The pdf corresponding to (5) is given by,

$$f(y; \theta, \beta, \lambda) = \frac{\lambda \theta \beta y^{-(\beta+1)} \left(e^{-y^{-\beta}}\right)^{\theta} A'\left[\lambda \left(e^{-y^{-\beta}}\right)^{\theta}\right]}{A(\lambda)}, y > 0, \theta, \beta, \lambda > 0. \quad (6)$$

A random variable Y with distribution function (5) shall be denoted by $Y \sim \text{CEIWPS}(\underline{\psi})$, where $\underline{\psi} \equiv (\theta, \beta, \lambda)$ is an unknown vector of parameters. Furthermore, the reliability and hazard rate functions take the following forms

$$R(y; \underline{\psi}) = 1 - \frac{A[\lambda(e^{-y-\beta})^\theta]}{A(\lambda)}, y > 0, \theta, \beta, \lambda > 0, \quad (7)$$

and

$$h(y; \underline{\psi}) = \frac{\lambda \theta \beta y^{-(\beta+1)} (e^{-y-\beta})^\theta A'[\lambda(e^{-y-\beta})^\theta]}{A(\lambda) - A[\lambda(e^{-y-\beta})^\theta]}, y > 0, \theta, \beta, \lambda > 0. \quad (8)$$

3 Properties of the Family

In this section, some statistical properties of the new family, including quantiles, moments and moment generating function are obtained. In addition, some distributions of order statistics are studied.

(3.1) A useful expansion

Some properties of the cdf (5) and pdf (6) will be studied through the following two proportions.

Proposition (1)

The EIW distribution is limiting special case of EIWPS family of distributions when $\lambda \rightarrow 0$.

Proof

By substituting $A(\lambda) = \sum_{z=1}^{\infty} a_z \lambda^z$ in cdf (5), then it can be written as

$$F(y; \theta, \beta, \lambda) = \frac{\sum_{z=1}^{\infty} a_z [\lambda(e^{-y-\beta})^\theta]^z}{\sum_{z=1}^{\infty} a_z \lambda^z},$$

by taking the limits for both sides,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} F(y; \theta, \beta, \lambda) &= \frac{a_1 (e^{-y-\beta})^\theta + \sum_{z=2}^{\infty} z a_z [\lambda(e^{-y-\beta})^\theta]^{z-1} (e^{-y-\beta})^\theta}{a_1 + \sum_{z=2}^{\infty} z a_z \lambda^{z-1}}, \\ &= (e^{-y-\beta})^\theta, \end{aligned}$$

which is the cdf of EIW distribution.

Proposition (2)

The pdf of the complementary exponentiated inverted Weibull power series family of distributions can be expressed as infinite mixture of exponentiated inverted Weibull with parameters $(\theta z, \beta)$ as follows

$$f(y; \theta, \beta, \lambda) = \sum_{z=1}^{\infty} P(Z = z; \lambda) g(y; \theta z, \beta),$$

Proof

Substituting $A'(\lambda) = \sum_{z=1}^{\infty} z a_z \lambda^{z-1}$ in pdf (6), then

$$\begin{aligned} f(y; \theta, \beta, \lambda) &= \frac{\lambda \theta \beta}{A(\lambda)} y^{-(\beta+1)} (e^{-y-\beta})^\theta \sum_{z=1}^{\infty} z a_z \left(\lambda (e^{-y-\beta})^\theta \right)^{z-1}, \\ &= \frac{\sum_{z=1}^{\infty} a_z \lambda^z}{A(\lambda)} z \theta \beta y^{-(\beta+1)} (e^{-y-\beta})^{\theta z}, \end{aligned}$$

$$f(y; \theta, \beta, \lambda) = \sum_{z=1}^{\infty} P(Z = z) g(y; \theta z, \beta),$$

where $g(y; \theta z, \beta)$ is the pdf of the exponentiated inverted Weibull with parameters $(\theta z, \beta)$.

3.2 Quantiles and moments

Quantile functions are used in theoretical aspects, statistical applications and simulations. Simulations methods employ quantile function to produce simulated random variables for classical and new continuous distributions. The quantile function say $Q(u)$ of Y is given by

$$u = \frac{A\left[\lambda \left(e^{-Q(u)^{-\beta}}\right)^{\theta}\right]}{A(\lambda)},$$

after some simplifications, it reduces to the following form

$$Q(u) = \left[-\frac{1}{\theta} \ln \left[\frac{1}{\lambda} \left[A^{-1}[A(\lambda).u]\right]\right]^{\frac{1}{\beta}}\right]^{\frac{-1}{\beta}}, \quad (9)$$

where u is a uniform random variable on the unit interval (0,1) and $A^{-1}(\cdot)$ is the inverse function of $A(\cdot)$. The r th moment of Y about the origin follows from proportion (2) as follows

$$\mu_r = E(Y^r) = \int_0^{\infty} \sum_{z=1}^{\infty} P(Z=z) g(y; \theta z, \beta) y^{r-(\beta+1)} dy.$$

Since, the r th moment for random variable Y' has EIW distribution with parameters θ and β is $\theta^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right)$

Then, the r th moment of Y has CEIWPS with parameters $(\theta z, \beta)$ is given by

$$\mu_r = \sum_{z=1}^{\infty} P(Z=z) (\theta z)^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right), r = 1, 2, 3, \dots \quad (10)$$

Also, it is easy to show that

$$M_y(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r,$$

where μ_r is the r th raw moment. Then by using (10), the moment generating function of Y takes the following form

$$M_y(t) = \sum_{r=0}^{\infty} \sum_{z=1}^{\infty} \frac{t^r}{r!} P(Z=z) (\theta z)^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right).$$

3.3 Some distributions of order statistics

Let $Y_{1:n} < Y_{2:n} < \dots < Y_{n:n}$ be the order statistics of a random sample of size n from exponentiated inverted Weibull power series family of distributions with parameters θ, β , and λ , the pdf of the i th order statistic can be written as follows

$$\begin{aligned} f_{i:n}(y; \underline{\psi}) &= \frac{f(y; \underline{\psi})}{B(i, n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j [F(y; \underline{\psi})]^{i+j-1}, 0 < y < \infty, \\ f_{i:n}(y; \underline{\psi}) &= \frac{\frac{\lambda \theta \beta y^{-(\beta+1)} (e^{-y^{-\beta}})^{\theta} A'[\lambda (e^{-y^{-\beta}})^{\theta}]}{B(i, n-i+1) [A(\lambda)]^{i+j}}}{\sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left[A[\lambda (e^{-y^{-\beta}})^{\theta}] \right]^{i+j-1}} \end{aligned} \quad (11)$$

The expansion for $A[\lambda (e^{-y^{-\beta}})^{\theta}]^{i+j-1}$ is given by

$$\begin{aligned} A[\lambda (e^{-y^{-\beta}})^{\theta}]^{i+j-1} &= \left[\sum_{z=1}^{\infty} a_z \lambda^z (e^{-y^{-\beta}})^{\theta z} \right]^{i+j-1}, \\ &= \left[a_1 \lambda (e^{-y^{-\beta}})^{\theta} \left[1 + \frac{a_2}{a_1} \lambda (e^{-y^{-\beta}})^{\theta} + \frac{a_3}{a_1} \lambda^2 (e^{-y^{-\beta}})^{2\theta} + \dots \right] \right]^{i+j-1}, \\ &= a_1^{i+j-1} \lambda^{i+j-1} (e^{-y^{-\beta}})^{\theta(i+j-1)} \left[\sum_{m=0}^{\infty} b_m \lambda^m (e^{-y^{-\beta}})^{\theta m} \right]^{i+j-1}, \end{aligned}$$

where $b_m = \frac{a_{m+1}}{a_1}$ for $m = 1, 2, 3, \dots$

By using the identity $(\sum_{m=0}^{\infty} b_m z^m)^j = \sum_{m=0}^{\infty} d_{j:m} z^m$ for positive integer j (see [24]), then

$$A[\lambda (e^{-y^{-\beta}})^{\theta}]^{i+j-1} = \sum_{m=0}^{\infty} d_{i+j-1:m} a_1^{i+j-1} \lambda^{i+j+m-1} (e^{-y^{-\beta}})^{\theta(i+j+m-1)}. \quad (12)$$

By using the expansion for $A'(\lambda) = \sum_{z=1}^{\infty} z a_z \lambda^{z-1}$, then

$$\begin{aligned} A' \left[\lambda \left(e^{-y^{(n)}} \right)^{-\beta} \right] &= \sum_{k=1}^{\infty} k a_k \lambda^{k-1} \left(e^{-y^{-\beta}} \right)^{\theta(k-1)} \\ &= a_1 \sum_{k=0}^{\infty} (k+1) b_k \lambda^k \left(e^{-y^{-\beta}} \right)^{\theta k}, \end{aligned} \quad (13)$$

substituting (12) and (13) in (11), then

$$\begin{aligned} f_{i:n} (y; \underline{\psi}) &= \frac{\theta \beta y^{-(\beta+1)}}{B(i, n-i+1) A(\lambda)^{j+1}} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \binom{n-i}{j} (-1)^j b_k (k+1) \\ &\quad \times d_{i+j-1:m} a_1^{i+j} \lambda^{i+j+m+k} \left(e^{-y^{-\beta}} \right)^{\theta(i+j+m+k)}, \\ f_{i:n} (y; \underline{\psi}) &= \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \omega_{i,j,k,m} y^{-(\beta+1)} \left(e^{-y^{-\beta}} \right)^{\theta(i+j+m+k)}, \end{aligned} \quad (14)$$

$$\text{where, } \omega_{i,j,k,m} = \frac{\theta \beta \binom{n-i}{j} (-1)^j b_k (k+1) d_{i+j-1:m} a_1^{i+j} \lambda^{i+j+m+k}}{B(i, n-i+1) A(\lambda)^{j+1}}.$$

The distribution of the smallest order statistic is given by substitute $i = 1$ in (14) as follows

$$f_{1:n} (y; \underline{\psi}) = \sum_{j=0}^{n-1} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \omega_{1,j,k,m} y^{-(\beta+1)} \left(e^{-y^{-\beta}} \right)^{\theta(j+m+k+1)}, 0 < y < \infty$$

where

$$\omega_{1,j,k,m} = \frac{n \theta \beta \binom{n-1}{j} (-1)^j b_k (k+1) d_{j:m} a_1^{j+1} \lambda^{j+m+k+1}}{A(\lambda)^{j+1}}.$$

The distribution of the largest order statistic is obtained by substitute $i = n$ in (14) as follows

$$f_{n:n} (y; \underline{\psi}) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \omega_{n,j,k,m} y^{-(\beta+1)} \left(e^{-y^{-\beta}} \right)^{\theta(n+j+m+k)}, 0 < y < \infty$$

where

$$\omega_{n,j,k,m} = \frac{n \theta \beta (-1)^j b_k (k+1) d_{n+j-1:m} a_1^{n+j} \lambda^{n+j+m+k}}{A(\lambda)^{j+1}}.$$

4 Special Models of the Family

In this section some special sub models of the family for complementary exponentiated inverted power series family of distributions are studied. The complementary exponentiated inverted Weibull Poisson (CEIWP) and the complementary exponentiated inverted Weibull logarithmic (CEIWL) distributions are discussed in some details.

The sub models are considered as follows:

1- For $A(\lambda) = (e^\lambda - 1)$ with $\lambda \in (0, \infty)$, the CEIWPS family of distributions reduces to the CEIWP distribution with the following cdf

$$F(y; \theta, \beta, \lambda) = \frac{\exp \left[\lambda \left(e^{-y^{-\beta}} \right)^{\theta} \right] - 1}{(e^\lambda - 1)}, y, \theta, \beta > 0. \quad (15)$$

2- For $A(\lambda) = -\log(1 - \lambda)$ with $\lambda \in (0, 1)$, the CEIWPS family of distributions reduces to the CEIWL distribution with the following cdf

$$F(y; \theta, \beta, \lambda) = \frac{-\log \left(1 - \lambda \left(e^{-y^{-\beta}} \right)^{\theta} \right)}{-\log(1 - \lambda)}, y, \theta, \beta > 0. \quad (16)$$

3- For $A(\lambda) = \lambda(1 - \lambda)^{-1}$ with $\lambda \in (0, 1)$, the CEIWPS family of distributions reduces to the complementary exponentiated inverted Weibull geometric distribution with the following cdf

$$F(y; \theta, \beta, \lambda) = \frac{(1 - \lambda)}{(1 - \lambda (e^{-y^{-\beta}})^{\theta})} \left(e^{-y^{-\beta}} \right)^{\theta}, y, \theta, \beta > 0.$$

4- For $A(\lambda) = (\lambda + 1)^m - 1$ with $\lambda \in (0, \infty)$, the CEIWPS family of distributions reduces to the complementary exponentiated inverted Weibull binomial distribution with the following cdf

$$F(y; \theta, \beta, \lambda) = \frac{\left(\lambda(e^{-y-\beta})^\theta + 1\right)^m - 1}{(\lambda + 1)^{m-1}}, y, \theta, \beta > 0.$$

5- For $\theta = 1$, $A(\lambda) = (e^\lambda - 1)$ with $\lambda \in (0, \infty)$, the CEIWPS family of distributions reduces to the complementary standard inverted Weibull Poisson distribution with the following cdf

$$F(y; \beta, \lambda) = \frac{\exp\left[\lambda(e^{-y-\beta})\right] - 1}{(e^\lambda - 1)}, y, \beta > 0.$$

6- For $\theta = 1$, $A(\lambda) = -\log(1 - \lambda)$ with $\lambda \in (0, 1)$, the CEIWPS family of distributions reduces to the complementary standard inverted Weibull logarithmic distribution with the following cdf

$$F(y; \beta, \lambda) = \frac{-\log(1 - \lambda e^{-y-\beta})}{-\log(1 - \lambda)}, y, \beta, \lambda > 0.$$

7- For $\theta = 1$, $A(\lambda) = \lambda(1 - \lambda)^{-1}$ with $\lambda \in (0, 1)$, the CEIWPS family of distributions reduces to the complementary standard inverted Weibull geometric distribution with the following cdf

$$F(y; \beta, \lambda) = \frac{(1 - \lambda)e^{-y-\beta}}{(1 - \lambda e^{-y-\beta})}, y, \beta, \lambda > 0.$$

8- For $\theta = 1$, $A(\lambda) = (1 + \lambda)^m - 1$ with $\lambda \in (0, \infty)$, the CEIWPS family of distributions reduces to the complementary standard inverted Weibull binomial distribution with the following cdf

$$F(y; \beta, \lambda) = \frac{\left(\lambda(e^{-y-\beta}) + 1\right)^m - 1}{(\lambda + 1)^{m-1}}, y, \beta, \lambda > 0.$$

9- For $\beta = 1$, $A(\lambda) = (e^\lambda - 1)$ with $\lambda \in (0, \infty)$, the CEIWPS family of distributions reduces to the complementary exponentiated standard inverted exponential Poisson distribution with the following cdf

$$F(y; \theta, \lambda) = \frac{\exp\left[\lambda\left(e^{-\frac{\theta}{y}}\right)\right] - 1}{(e^\lambda - 1)}, y, \theta, \lambda > 0.$$

10- For $\beta = 1$, $A(\lambda) = -\log(1 - \lambda)$ with $\lambda \in (0, 1)$, the CEIWPS family of distributions reduces to the complementary exponentiated standard inverted exponential logarithmic distribution with the following cdf

$$F(y; \theta, \lambda) = \frac{-\log\left(1 - \lambda e^{-\frac{\theta}{y}}\right)}{-\log(1 - \lambda)}, y, \theta > 0.$$

11- For $\beta = 1$, $A(\lambda) = (1 - \lambda)^{-1}$ with $\lambda \in (0, 1)$, the CEIWPS family of distributions reduces to the complementary exponentiated standard inverted exponential geometric distribution with the following cdf

$$F(y; \theta, \lambda) = \frac{e^{-\frac{\theta}{y}(1-\lambda)}}{\left(1 - \lambda e^{-\frac{\theta}{y}}\right)}, y, \theta > 0.$$

12- For $\beta = 1$, $A(\lambda) = (1 + \lambda)^m - 1$ with $\lambda \in (0, \infty)$, the CEIWPS family of distributions reduces to the complementary exponentiated standard inverted exponential binomial distribution with the following cdf

$$F(y; \theta, \lambda) = \frac{\left(\lambda e^{-\frac{\theta}{y}} + 1\right)^m - 1}{(\lambda + 1)^{m-1}}, y, \theta > 0.$$

4.1 The complementary exponentiated inverted Weibull Poisson distribution

In this subsection, the pdf, reliability, hazard function, quantile and moments for complementary exponentiated inverted Weibull Poisson distribution will be studied in more details.

The pdf of CEIWP corresponding to (15) is as follows

$$f(y; \theta, \beta, \lambda) = \frac{\theta \beta \lambda}{(e^{\lambda} - 1)} y^{-(\beta+1)} \left(e^{-y^{-\beta}} \right)^{\theta} \exp \left[\lambda \left(e^{-y^{-\beta}} \right)^{\theta} \right], y, \theta, \beta > 0 \quad (17)$$

Figures (1) and (2) represent the pdf and the cdf of the CEIWP distribution for selected values of parameters (θ, β, λ) , (4,6,0.5), (6,4,0.5), (6,6,0.5), (0.5,3,1), (1,3,1), (2,3,1), (5,8,6), (8,6,5), (6,5,8), (5,7,6), (6,5,3) and (7,6,5) respectively.

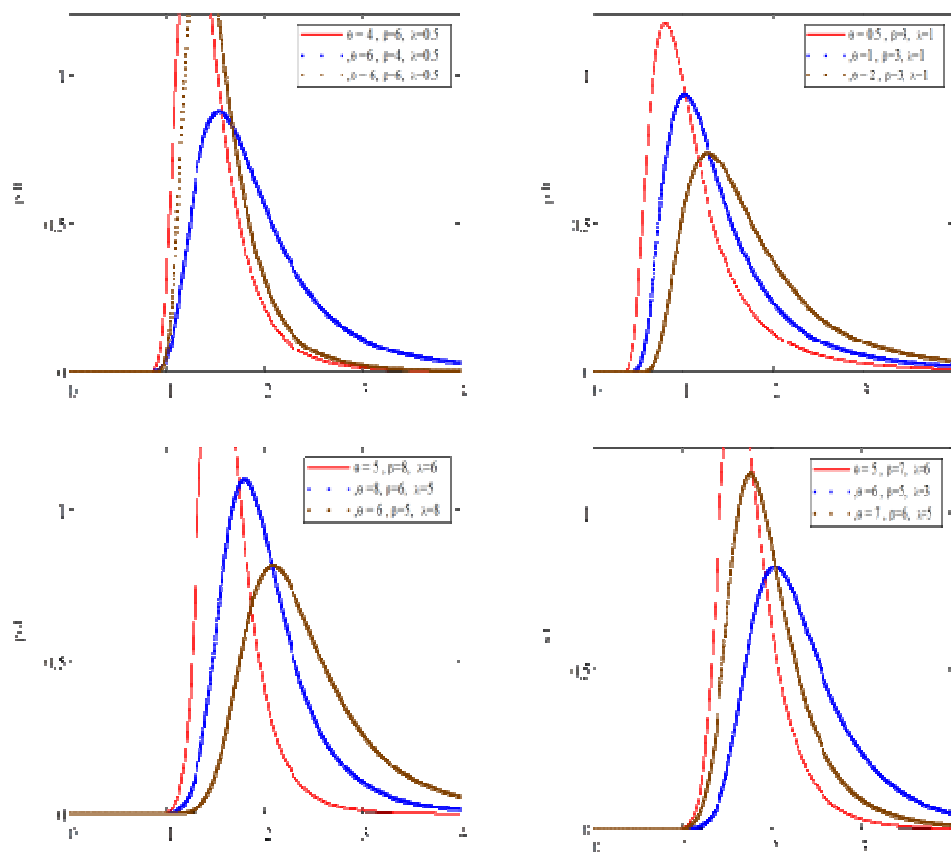
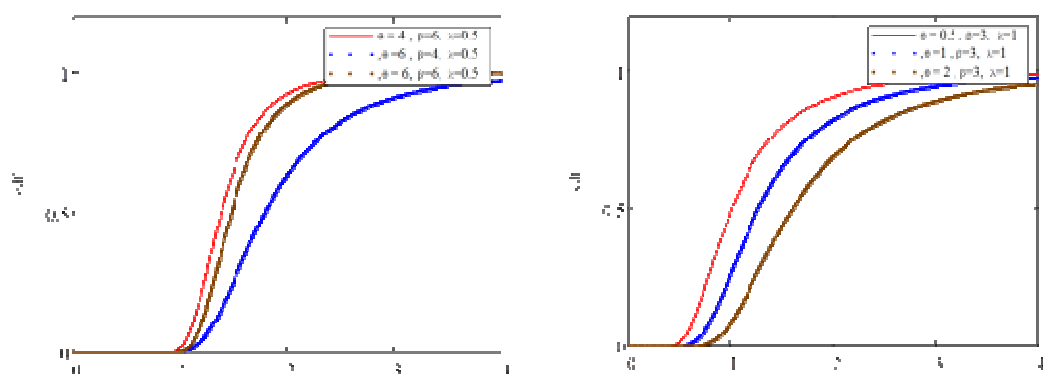


Figure (1): Plots of the CEIWP densities for some parameter values



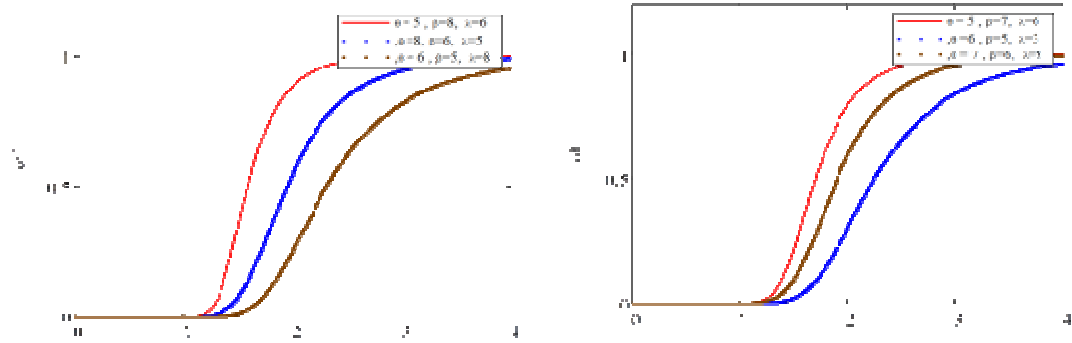


Figure (2): Plots of the CEIWP distribution function for some parameter values

The reliability and hazard rate functions of the CEIWP are obtained respectively, as follows

$$R(y; \theta, \beta, \lambda) = \frac{e^{\lambda - \exp \lambda (e^{-y^{-\beta}})^{\theta}}}{(e^{\lambda} - 1)}, \quad (18)$$

and,

$$h(y; \theta, \beta, \lambda) = \frac{\theta \beta \lambda y^{-(\beta+1)} (e^{-y^{-\beta}})^{\theta} \exp \lambda (e^{-y^{-\beta}})^{\theta}}{e^{\lambda - \exp \lambda (e^{-y^{-\beta}})^{\theta}}}. \quad (19)$$

Figures (3) and (4) illustrate reliability and hazard rate functions of the CEIWP distribution for selected values of parameters $(\theta, \beta, \lambda) = (4, 6, 0.5), (6, 4, 0.5), (6, 6, 0.5), (0.5, 3, 1), (1, 3, 1), (2, 3, 1), (5, 8, 6), (8, 6, 5), (6, 5, 8), (5, 7, 6), (6, 5, 3)$ and $(7, 6, 5)$ respectively.

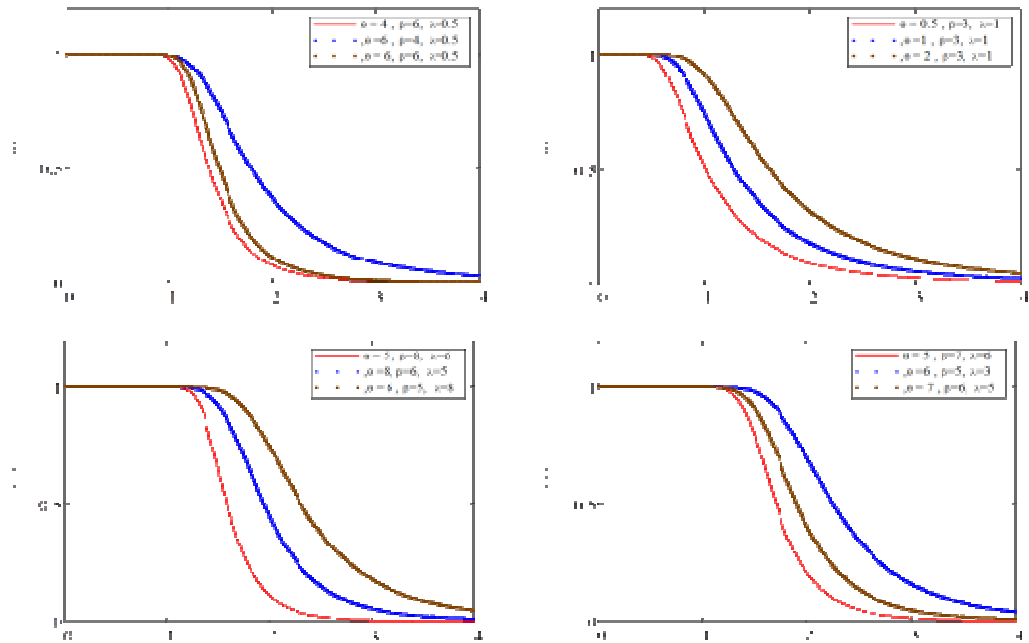


Figure (3): Plots of CEIWP reliability functions for some parameter values

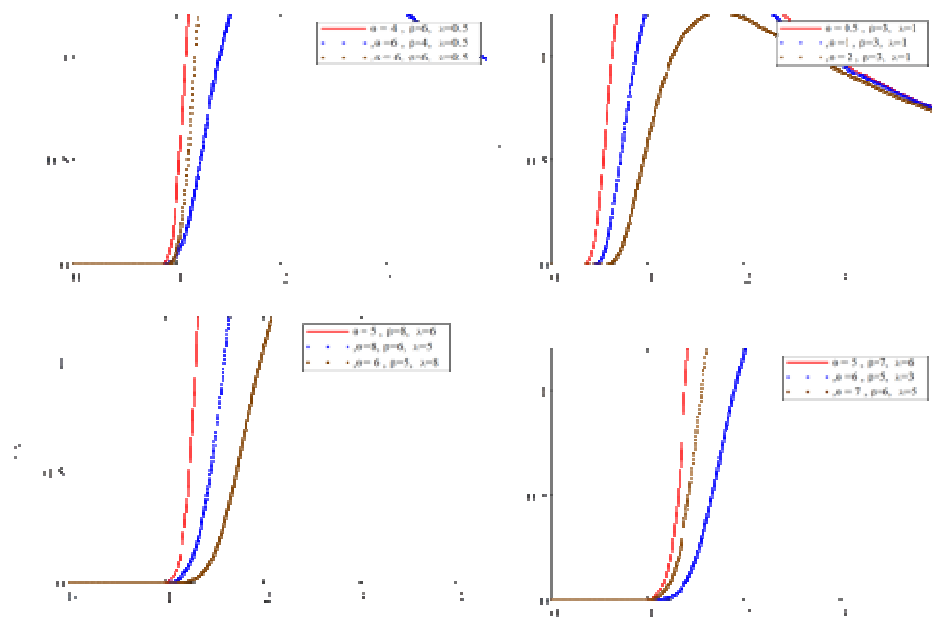


Figure (4): Plots of the CEIWP hazard rates functions for some parameter values

It is clear from Figures (4) that the hazard function takes different forms according to different values of the parameters.

The quantile function of the CEIWP distribution is given by substituting $A(\lambda) = (e^\lambda - 1)$, and $A^{-1}(\lambda) = \log(\lambda + 1)$ in (9), as follows

$$Q(u) = \left[-\frac{1}{\theta} \ln \left[\left[\frac{1}{\lambda} \ln [u(e^\lambda - 1) + 1] \right] \right] \right]^{-\frac{1}{\beta}}, \quad (20)$$

where u is a uniform random variable on the unit interval (0,1).

In particular the median of the CEIWP distribution, say Q_2 , is obtained by substituting, $u = \frac{1}{2}$ in (20) as follows

$$Q_2 = \left[-\frac{1}{\theta} \ln \left[\frac{1}{\lambda} \ln \left[\frac{e^\lambda + 1}{2} \right] \right] \right]^{-\frac{1}{\beta}}. \quad (21)$$

The r th raw moment of the CEIWP distribution about the origin is given by substituting $A(\lambda) = (e^\lambda - 1)$ in (10) as follows

$$\mu_r = \sum_{z=1}^{\infty} \frac{a_z \lambda^z}{(e^\lambda - 1)} (\theta z)^{\frac{r}{\beta}} \Gamma \left(1 - \frac{r}{\beta} \right), r = 1, 2, 3, \dots, \quad (22)$$

In particular, setting $r = 1$ in (22), the mean of CEIWP distribution is

$$E(Y) = \sum_{z=1}^{\infty} \frac{a_z \lambda^z}{(e^\lambda - 1)} (\theta z)^{\frac{1}{\beta}} \Gamma \left(1 - \frac{1}{\beta} \right). \quad (23)$$

The variance of the CEIWP distribution takes the following form

$$var(Y) = \sum_{z=1}^{\infty} \frac{a_z \lambda^z}{(e^\lambda - 1)} (\theta z)^{\frac{2}{\beta}} \Gamma \left(1 - \frac{2}{\beta} \right) - \left[\sum_{z=1}^{\infty} \frac{a_z \lambda^z}{(e^\lambda - 1)} (\theta z)^{\frac{1}{\beta}} \Gamma \left(1 - \frac{1}{\beta} \right) \right]^2.$$

4.2 The complementary exponentiated inverted Weibull logarithmic distribution

In this subsection, the pdf, reliability, hazard function, quantile, and moments for complementary exponentiated inverted Weibull logarithmic distribution will be discussed.

The probability density function of the CEIWL distribution corresponding to (16) is given by

$$f(y; \theta, \beta, \lambda) = \frac{\theta \beta \lambda}{-\log(1-\lambda)(1-\lambda(e^{-y^{-\beta}})^{\theta})} y^{-(\beta+1)} (e^{-y^{-\beta}})^{\theta}, y, \theta, \beta > 0, \quad (24)$$

Figures (5) and (6) represent the pdf and the cdf of the CEIWL distribution for selected values of parameters $(\theta, \beta, \lambda) = (0.7, 3, 0.1), (1.4, 0.1), (2.5, 0.1), (0.5, 5, 0.3), (1.5, 0.3), (2.5, 0.3), (0.5, 3, 0.5), (2.3, 0.5), (1.4, 0.8), (1.5, 0.8),$ and $(1.7, 0.8)$ respectively.

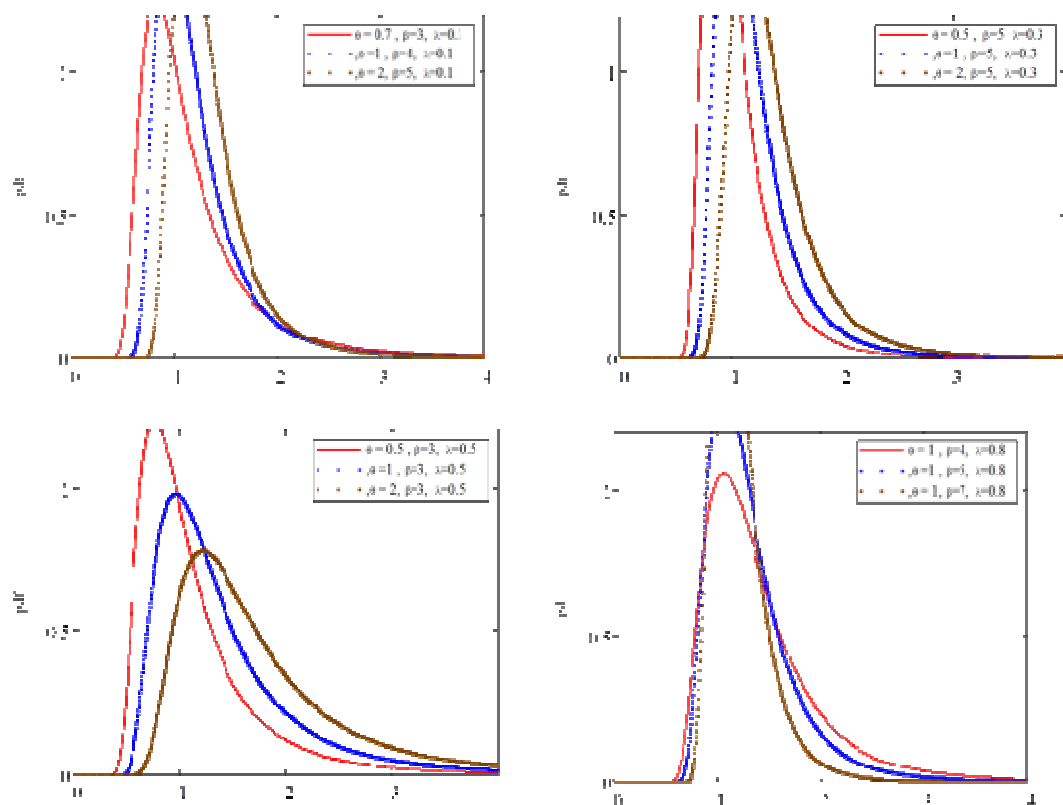
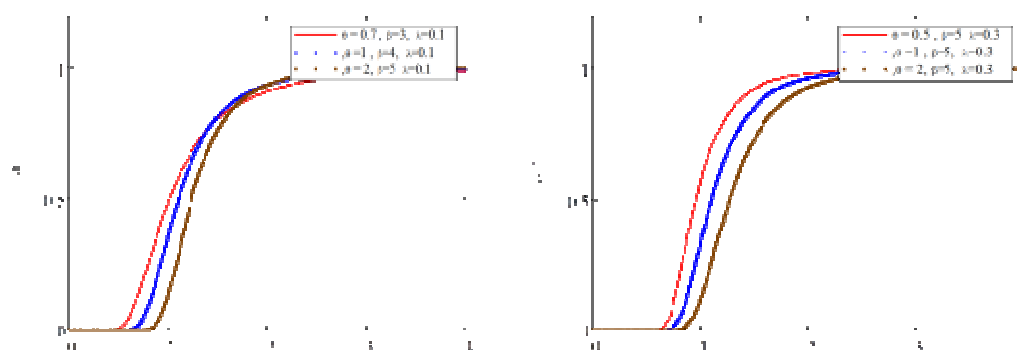


Figure (5): Plots of the CEIWL densities for some parameter values



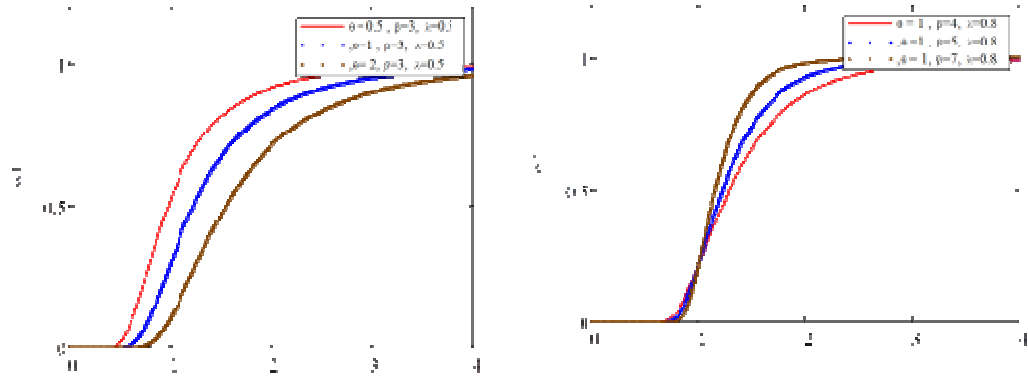


Figure (6): Plots of the cdf of CEIWL for some parameter values

The reliability and hazard rate functions of the CEIWL distribution are obtained respectively, as follows

$$R(y; \theta, \beta, \lambda) = 1 - \frac{\log(1 - \lambda(e^{-y^{-\beta}})^{\theta})}{\log(1 - \lambda)}, \quad (25)$$

and,

$$h(y; \theta, \beta, \lambda) = \frac{\theta \beta \lambda y^{-(\beta+1)} (e^{-y^{-\beta}})^{\theta} (1 - \lambda(e^{-y^{-\beta}})^{\theta})^{-1}}{-\log(1 - \lambda) + \log(1 - \lambda(e^{-y^{-\beta}})^{\theta})}, \quad (26)$$

Figures (7) and (8) illustrate reliability and hazard rate functions of the CEIWL for the parameters $(\theta, \beta, \lambda) = (0.7, 3, 0.1), (1, 4, 0.1), (2, 5, 0.1), (1, 3, 0.5), (2, 3, 0.5), (1, 4, 0.8), (1, 5, 0.8), (0.5, 5, 0.3), (1, 5, 0.3), (2, 5, 0.3), (0.5, 3, 0.5),$ and $(1, 7, 0.8)$ respectively.

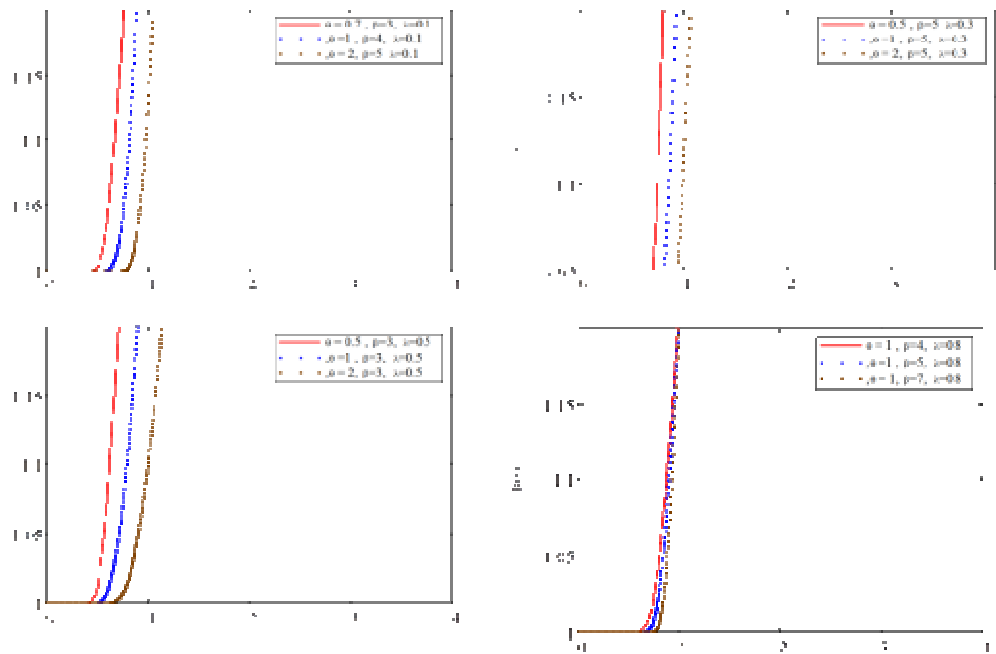


Figure (7): Plots of CEIWL reliability functions for some parameter values

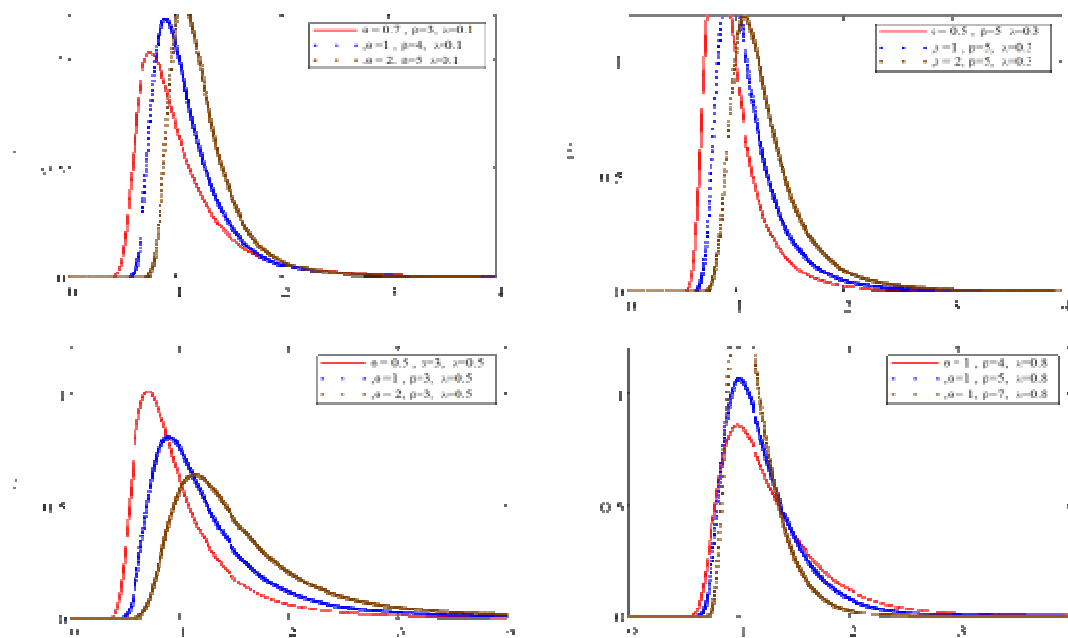


Figure (8): Plots of the CEIWL hazard rates functions for some parameter values

The quantile function of CEIWL distribution is given by substituting $A(\lambda) = -\log(1 - \lambda)$, and $A^{-1}(\lambda) = (1 - e^{-\lambda})$ in (9), as follows

$$Q(u) = \left[-\frac{1}{\theta} \ln \left[\frac{1}{\lambda} [1 - (1 - \lambda) \cdot u] \right] \right]^{-\frac{1}{\beta}}, \quad (27)$$

where u is a uniform random variable on the unit interval $(0,1)$.

In particular the median of the CEIWL distribution, say Q_2 , is obtained by substituting, $u = \frac{1}{2}$ in (27), as follows

$$Q_2 = \left[-\frac{1}{\theta} \ln \left[\frac{1}{\lambda} \left[\frac{1+\lambda}{2} \right] \right] \right]^{-\frac{1}{\beta}},$$

The r th raw moment of the CEIWL distribution about the origin is given by substituting $A(\lambda) = -\log(1 - \lambda)$ in (10) as follows

$$\mu'_r = \sum_{z=1}^{\infty} -\frac{a_z \lambda^z}{\log(1-\lambda)} (\theta z)^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right), r = 1, 2, 3, \dots \quad (28)$$

In particular, setting $r = 1$ in (28), the mean of Y is as follows

$$E(Y) = \sum_{z=1}^{\infty} -\frac{a_z \lambda^z}{\log(1-\lambda)} (\theta z)^{\frac{1}{\beta}} \Gamma\left(1 - \frac{1}{\beta}\right). \quad (29)$$

The variance of the CEIWL distribution takes the following form

$$var(Y) = \sum_{z=1}^{\infty} -\frac{a_z \lambda^z}{\log(1-\lambda)} (\theta z)^{\frac{2}{\beta}} \Gamma\left(1 - \frac{2}{\beta}\right) - \left[\sum_{z=1}^{\infty} -\frac{a_z \lambda^z}{\log(1-\lambda)} (\theta z)^{\frac{1}{\beta}} \Gamma\left(1 - \frac{1}{\beta}\right) \right]^2.$$

5 Parameter Estimation of the Family

In this section estimation of the model parameters of CEIWPS family of distributions is obtained by using the maximum likelihood method.

Let $Y_1, Y_2, \dots, Y_n$ be a random sample from the family with parameters θ, β and λ . The likelihood function based on the observed random sample of size n is given by

$$L(y; \psi) = \lambda^n \theta^n \beta^n (A(\lambda))^{-n} \prod_{i=1}^n y_i^{-(\beta+1)} \prod_{i=1}^n \left(e^{-y_i^{-\beta}} \right)^{\theta} \prod_{i=1}^n A' \left[\lambda \left(e^{-y_i^{-\beta}} \right)^{\theta} \right].$$

The natural logarithm of the likelihood function, $\ln L \equiv \ln L(y; \psi)$, is given by

$$\ln L = n \ln \lambda + n \ln \theta + n \ln \beta - n \ln A(\lambda) - (\beta + 1) \sum_{i=1}^n \ln y_i - \theta \sum_{i=1}^n y_i^{-\beta} + \sum_{i=1}^n \ln A' \left[\lambda \left(e^{-y_i^{-\beta}} \right)^{\theta} \right].$$

The maximum likelihood estimators of θ, β, λ , say $\hat{\theta}, \hat{\beta}, \hat{\lambda}$, are obtained by setting the first partial derivatives of $\ln L$ to be zero with respect to θ, β and λ as follows,

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n y_i^{-\beta} - \lambda \sum_{i=1}^n \frac{A'' \left[\lambda \left(e^{-y_i^{-\beta}} \right)^{\theta} \right] \left(e^{-y_i^{-\beta}} \right)^{\theta} y_i^{-\beta}}{A' \left[\lambda \left(e^{-y_i^{-\beta}} \right)^{\theta} \right]}, \quad (30)$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^n \ln y_i + \sum_{i=1}^n y_i^{-\beta} \ln y_i + \lambda \theta \sum_{i=1}^n \frac{A'' \left[\lambda \left(e^{-y_i^{-\beta}} \right)^{\theta} \right] \left(e^{-y_i^{-\beta}} \right)^{\theta-1} y_i e^{-y_i^{-\beta}}}{A' \left[\lambda \left(e^{-y_i^{-\beta}} \right)^{\theta} \right]}, \quad (31)$$

and

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \frac{n A'(\lambda)}{A(\lambda)} + \sum_{i=1}^n \frac{A'' \left[\lambda \left(e^{-y_i^{-\beta}} \right)^{\theta} \right] \left(e^{-y_i^{-\beta}} \right)^{\theta}}{A' \left[\lambda \left(e^{-y_i^{-\beta}} \right)^{\theta} \right]}. \quad (32)$$

The maximum likelihood estimate of θ, β and λ , say $\hat{\theta}, \hat{\beta}$ and $\hat{\lambda}$ is obtained by setting $\frac{\partial \ln L}{\partial \theta} = 0, \frac{\partial \ln L}{\partial \beta} = 0$ and $\frac{\partial \ln L}{\partial \lambda} = 0$, and solving the non- linear system numerically.

6 Applications

In this section, the flexibility and potentiality of some special models of EIWPS family are examined using three real data sets. Applications of the CEIWP distributions based on their sub-models; namely, CEIWP, CEIWL and EIW are considered.

6.1 The first data set

The first data set is given by [25], the vinyl chloride data obtained from clean upgradient ground –water monitoring wells in mg/L; the data are as follows

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

The estimator of the unknown parameters of each distribution is obtained by the maximum-likelihood method. The values for some models of the following statistics: Kolmogorov-Smirnov (K-S), Akaike information criterion (AIC), Bayesian information criterion (BIC), and the consistent Akaike information criterion (CAIC) are used to compare the candidate distributions. The best distribution corresponds to lower -2logL (-2LL), AIC, BIC, and CAIC statistics values.

Table (2) shows of the maximum-likelihood estimates (MLEs), AIC values, BIC values, CAIC values and Kolmogorov- Smirnov statistics for the first data.

The Model	MLEs			Measures				
	θ	β	λ	K-S	-2LL	AIC	CAIC	BIC
CEIWP	0.228	1.084	3.259	0.090	122.801	128.801	129.601	127.395
CEIWL	0.608	0.902	0.181	0.114	195.045	201.045	201.845	199.639
EIW	0.654	0.880		0.113	172.152	176.152	176.539	175.215

Table (2) MLEs, AIC values, BIC values, CAIC values and K- S statistics for the first data set
Based on the values of AIC, CAIC, BIC and K-S in Table (2), the CEIWP model provide better fit than the CEIWL and EIW models.

Figures (9)and (10) shows the plots of estimated cumulative and estimated densities of the fitted CEIWP, CEIWL and EIW models for the first data set.

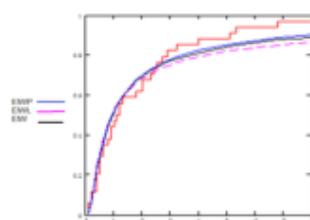


Figure (9) : Estimated cumulative densities for the first data

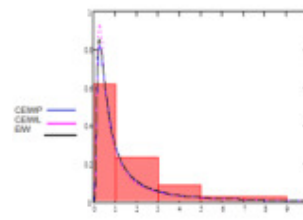


Figure (10): Estimated densities of models for the first data

It is clear from Figure (10), that the fitted density for CEIWP model is closer to the empirical histogram than CEIWL and EIW models.

6.2 The second data set

The second data set is obtained from [26]. It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data are as follows
0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

The following table shows the MLEs, AIC values, BIC values CAIC values and K- S statistics for the second data.

The Model	MLEs			Measures				
	θ	β	λ	K-S	-2LL	AIC	CAIC	BIC
CEIWP	0.376	1.916	3.474	0.117	85.627	91.627	92.550	90.058
CEIWL	1.019	1.553	0.020	0.153	233.526	239.526	240.326	238.120
EIW	1.025	1.550		0.152	140.186	144.186	144.573	143.249

Table (3): MLEs, AIC values, BIC values, CAIC values and K- S statistics for the second data set.

In order to assess if the CEIWP models is appropriate, the histograms of the data, the plots of the fitted CEIWP, CEIWL and EIW density functions are displayed in Figure (11). From figure it concluded that the CEIWP distribution is very suitable to this data.

Figures (11) and (12) give the plots of estimated cumulative and estimated densities of the fitted CEIWP, CEIWL and EIW models for the second data set.

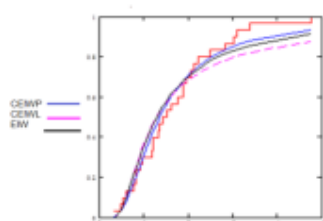
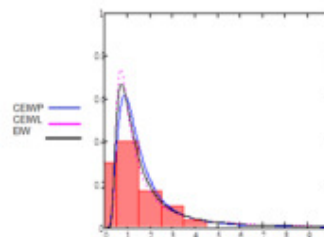


Figure (11): Estimated cumulative densities for the second data



Figure(12): Estimated densities for the second data

6.3 The third data set

The third data set is given by [27]. The data refer the time between failures for repairable item, the data are as follows

1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

The following table shows MLEs, AIC values, BIC values, CAIC values and K- S statistics for the third data.

The Model	MLEs			Measures				
	θ	β	λ	K-S	-2LL	AIC	CAIC	BIC
CEIWP	0.680	1.104	0.279	0.1334	101.368	107.368	108.292	105.800
CEIWL	0.747	1.075	0.017	0.1339	221.411	227.411	228.334	225.842
EIW	0.752	1.073		0.1337	127.816	131.816	132.260	130.770

Table (4): MLEs, AIC values, BIC values, CAIC values and K- S statistics for the third data.

The values in Table (4), indicate that the complementary exponentiated inverted Weibull Poisson model is a strong competitor to other distributions used here for fitting data set.

Figures (13) and (14) show the plots of estimated cumulative of the fitted CEIWP, CEIWL and EIW models for the third data set.

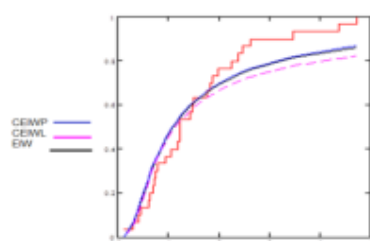


Figure (13): Estimated cumulative densities for the third data

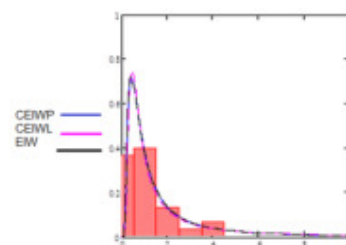


Figure (14): Estimated densities of the third data

A density plot compares the fitted densities of the model with the empirical histogram of the observed data as shown in Figure (14). Based on plots in Figure (14), one can see that the fitted density for CEIWP model is closer to the empirical histogram than the corresponding CEIWL and EIW models.

7 Conclusions

In this paper a new family of lifetime distributions called the complementary exponentiated inverted Weibull power series is introduced, which is obtained by compounding the exponentiated inverted Weibull and power series distributions. The properties of the proposed family are discussed, including quantile, moments and moment generating functions. Some distributions of order statistics are obtained. The estimation of the model parameters is performed by maximum likelihood method. Two special sub-models of the new family are investigated, namely, the complementary exponentiated inverted Weibull Poisson distribution and the complementary exponentiated inverted Weibull logarithmic distribution. Some mathematical properties of new distributions are discussed. Finally the complementary exponentiated inverted Weibull Power series models are fitted to real data sets to show the flexibility and potentiality of the proposed family of distributions.

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