

New fourth-order Schroder-type methods for finding zeros of nonlinear equations having unknown multiplicity

Abstract

In this paper we define two new fourth-order Schroder-type methods for finding zeros of nonlinear equations having unknown multiplicity. In terms of computational cost the new iterative methods requires six evaluations of functions per iteration. It is proved that the new methods have a convergence of order four. Numerical comparisons are included to demonstrate exceptional convergence speed of the proposed methods.

Keywords: Schroder-type method; Root-finding; Nonlinear equations; Multiple roots; Order of convergence; Efficiency index.

Subject Classifications: AMS (MOS): 65H05, 41A25.

1 Introduction

Solving nonlinear equations is one of the most important problem in science and engineering [1,3,10]. In this paper, we present a new fourth-order iterative method to find multiple roots of the nonlinear equation $f(x) = 0$, where $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval I is a scalar function. The root-solver is of great practical importance since it overcomes theoretical limits of one-point methods concerning convergence order and computational efficiency. In recent years, many modifications of the Newton-type methods for simple roots have been proposed and analysed [3] and little work has been done on multiple roots. In this paper, we are interested in the case that α is a root of multiplicity $m > 1$ of a nonlinear equation, that is $f^k(\alpha) = 0$, $k = 0, 1, 2, \dots, m-1$ and $f^m(\alpha) \neq 0$. Therefore, the prime motive of this study is to develop a new class of iterative method for finding multiple roots of nonlinear equations of a higher order than the existing iterative methods [3-9]. The purpose of this paper is to show further development of the third-order methods [9]. In addition, the new iterative methods have a better precision than the classical Schroder method and Soleymani et al. methods [5-7]. Hence, the proposed fourth-order method is significantly better when compared with these established methods.

The remaining sections of the paper are organized as follows. Some basic definitions relevant to the present work are presented in section 2. In section 3, we construct two new iterative methods and verify their convergence order. In section 4, two well-established methods are stated, which will demonstrate the effectiveness of the new fourth-order iterative method. Finally, in section 5, numerical comparisons are made to demonstrate the performance of the presented method.

2 Preliminaries

In order to establish the order of convergence of the new fourth-order method, we use the following definitions.

Definition 1 Let $f(x)$ be a real-valued function with a root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α . The order of convergence p is given by

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \zeta \neq 0, \quad (1)$$

where $p \in \mathbb{R}^+$ and ζ is the asymptotic error constant [1,3,10].

Definition 2 Let $e_k = x_k - \alpha$ be the error in the k th iteration, then the relation

$$e_{k+1} = \zeta e_k^p + O(e_k^{p+1}), \quad (2)$$

is the error equation. If the error equation exists, then p is the order of convergence of the iterative method [1,3,10].

Definition 3 Let r be the number of function evaluations of the method. The efficiency of the method is measured by the concept of efficiency index and defined as

$$\sqrt[r]{p}, \quad (3)$$

where p is the order of convergence of the method [1,3,10].

Definition 4 Suppose that x_{n-2}, x_{n-1} and x_n are three successive iterations closer to the root α . Then the computational order of convergence may be approximated by

$$\text{COC} \approx \frac{\ln |\delta_n \div \delta_{n-1}|}{\ln |\delta_{n-1} \div \delta_{n-2}|}, \quad (4)$$

where $\delta_i = f(x_i) \div f'(x_i)$, [9].

3 Development of the methods and convergence analysis

In this section we define two new fourth-order iterative methods for finding multiple roots of a nonlinear equation. In order to construct new iterative methods we require total of six function evaluations. The new methods are actually based on the Schroder second-order method and Thukral third-order method given [5,9], respectively.

3.1 Method 1

The first of our new scheme is actually based on the classical Schroder method. We use this method as our first step and repeat the process at an improved point, therefore the new scheme is expressed as

$$y_n = x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)}, \quad (5)$$

$$x_{n+1} = y_n - \frac{f(y_n)f'(y_n)}{f'(y_n)^2 - f(y_n)f''(y_n)}, \quad (6)$$

where $n \in \mathbb{N}$, x_0 is the initial guess and provided that the denominator of (6) is not equal to zero.

Theorem 1

Let $\alpha \in I$ be a multiple zero of a sufficiently differentiable function $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval I with multiplicity m , which includes x_0 as an initial guess of α . Then the iterative method defined by scheme (6) has fourth-order convergence.

Proof

82 Let α be a multiple root of multiplicity m of a sufficiently differentiable function $f(x)$,
 83 $e = x - \alpha$, $\hat{e} = y - \alpha$,
 84 Using the Taylor series expansion of $f(x), f'(x), f''(x), f'''(x), f(y), f'(y), f''(y)$, about α ,
 85 we have
 86

$$87 \quad f(x_n) = \left(\frac{f^{(m)}(\alpha)}{m!} \right) e_n^m \left[1 + \sum_{i=1}^{\infty} A_i e_n^i \right], \quad (7)$$

$$88 \quad f'(x_n) = \left(\frac{f^{(m)}(\alpha)}{(m-1)!} \right) e_n^{m-1} \left[1 + \sum_{i=1}^{\infty} B_i e_n^i \right], \quad (8)$$

$$90 \quad f''(x_n) = \left(\frac{f^{(m)}(\alpha)}{(m-2)!} \right) e_n^{m-2} \left[1 + \sum_{i=1}^{\infty} C_i e_n^i \right], \quad (9)$$

$$92 \quad f'''(x_n) = \left(\frac{f^{(m)}(\alpha)}{(m-3)!} \right) e_n^{m-3} \left[1 + \sum_{i=1}^{\infty} D_i e_n^i \right], \quad (10)$$

$$94 \quad f(y_n) = \left(\frac{f^{(m)}(\alpha)}{m!} \right) \hat{e}_n^m \left[1 + \sum_{i=1}^{\infty} A_i \hat{e}_n^i \right], \quad (11)$$

$$96 \quad f'(y_n) = \left(\frac{f^{(m)}(\alpha)}{(m-1)!} \right) \hat{e}_n^{m-1} \left[1 + \sum_{i=1}^{\infty} B_i \hat{e}_n^i \right], \quad (12)$$

$$98 \quad f''(y_n) = \left(\frac{f^{(m)}(\alpha)}{(m-2)!} \right) \hat{e}_n^{m-2} \left[1 + \sum_{i=1}^{\infty} C_i \hat{e}_n^i \right], \quad (13)$$

100 where $n \in \mathbb{N}$ and
 101

$$102 \quad T_k = \frac{f^{(m+k)}(\alpha)}{f^{(m)}(\alpha)}, \quad (14)$$

$$103 \quad A_k = \frac{m! T_k}{(m+k)!}, \quad (15)$$

$$104 \quad B_k = \frac{(m-1)! T_k}{(m+k-1)!}, \quad (16)$$

$$105 \quad C_k = \frac{(m-2)! T_k}{(m+k-2)!}, \quad (17)$$

$$106 \quad D_k = \frac{(m-3)! T_k}{(m+k-3)!}. \quad (18)$$

107 Dividing (7) by (8), we get

$$\frac{f(x_n)}{f'(x_n)} = \frac{e}{m} - \frac{T_1 e^2}{m^2(m+1)} + \frac{e^3(T_1^2(m+2) - m^2 T_2)}{m^3(m+1)(m+2)} + \dots, \quad (19)$$

and dividing (9) by (8) yields

$$\frac{f''(x_n)}{f'(x_n)} = \frac{m-1}{e} - \frac{T_1}{m} + \frac{e(T_1^2(m+1) - T_2 m(m-1))}{m^2(m+1)} + \dots, \quad (20)$$

Product of (19) and (20), we obtain

$$\left(\frac{f(x_n)}{f'(x_n)} \right) \left(\frac{f''(x_n)}{f'(x_n)} \right) = \frac{m-1}{m} - \frac{2T_1 e}{m^2(m+1)} + \frac{(3T_1^2(m+2) - 2T_2 m(m-1))e^2}{m^3(m+1)(m+2)} + \dots, \quad (21)$$

Substituting appropriate expressions in (5) and simplifying, we have

$$y_n - \alpha = x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)} = -\frac{T_1 e^2}{m(m+1)} + \dots, \quad (22)$$

Now we need to expand $f(y_n), f'(y_n), f''(y_n)$ about α . Substituting appropriate expressions in (6),

$$e_{n+1} = y_n - \frac{f(y_n)f'(y_n)}{f'(y_n)^2 - f(y_n)f''(y_n)}, \quad (23)$$

Simplifying (23), we obtain the asymptotic error constant

$$e_{n+1} = \left(\frac{T_1^3}{m^3(m+1)^2} \right) e_n^4. \quad (24)$$

The expression (24) establishes the asymptotic error constant for the fourth-order of convergence for the new Schroder-type method defined by (6). $\square$

3.2 Method 2

In this sub-section we define another new fourth-order iterative method for finding multiple roots of a nonlinear equation. In order to construct the new iterative method we use Thukral third-order method as our first step and the second step is combination of Newton-Schroder principle. Hence, the second new scheme is given as

$$y_n = x_n - \frac{2(f(x_n)f'(x_n)^2 - f(x_n)^2 f''(x_n))}{2f'(x_n)^3 - 3f(x_n)f'(x_n)f''(x_n) + f'''(x_n)f(x_n)^2}, \quad (25)$$

$$x_{n+1} = y_n - \left(\frac{f'(x_n)^2}{f'(x_n)^2 - f(x_n)f''(x_n)} \right) \left(\frac{f(y_n)}{f'(y_n)} \right), \quad (26)$$

where $n \in \mathbb{N}$, x_0 is the initial guess and provided that the denominator of (26) is not equal to zero.

Theorem 2

Let $\alpha \in I$ be a multiple zero of a sufficiently differentiable function $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval I with multiplicity m , which includes x_0 as an initial guess of α . Then the iterative method defined by scheme (26) has fourth-order convergence.

Proof

Using appropriate expressions in the proof of the theorem 1 and substituting them into (26), we obtain the asymptotic error constant

$$e_{n+1} = 2T_1 \left(\frac{(2(m+1)T_2 + (m+2)T_1^2)}{m^2(m+1)^3(m+2)} \right) e_n^4. \quad (27)$$

The expression (19) establishes the asymptotic error constant for the fourth-order of convergence for the new Schroder-type method defined by (26). $\square$

4 The established methods

For the purpose of comparison, two iterative methods presented in [6,7] are considered. Since these methods are well established, the essential formulas are used to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new fourth-order methods.

The first of fourth-order method presented by Soleymani et al. [6], is expressed as

$$y_n = x_n - \left(\frac{2}{3} \right) \left(\frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)} \right), \quad (28)$$

$$x_{n+1} = x_n - 4 \left[\frac{1 + \left(\frac{3}{4} \right)^2 (w_n - 1)^2}{(4 - u_n - 3v_n)} \right] \left(\frac{f(x_n)}{f'(x_n)} \right), \quad (29)$$

where

$$u_n = \frac{f(x_n)f''(x_n)}{f'(x_n)^2}, \quad (30)$$

$$v_n = \frac{f(y_n)f''(y_n)}{f'(y_n)^2}, \quad (31)$$

$$w_n = \frac{1 - v_n}{1 - u_n}, \quad (32)$$

The second of fourth-order method is by Soleymani et al. [7], and is given as

$$y_n = x_n - \left(\frac{2}{3} \right) \left(\frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)} \right), \quad (33)$$

$$x_{n+1} = y_n - \left[2^{-3} (17 - 16w_n + 7w_n^2) \left(\left(\frac{f(x_n)(2f'(x_n) + t_n f''(x_n))}{f'(x_n)^2} \right) + t_n (v_n - 2) \right) \right] (2 - 2v_n)^{-1}, \quad (34)$$

where $t_n = x_n - y_n$ and u_n, v_n, w_n are given by (30), (31), (32) respectively.

5 Numerical examples

In this section, we shall present the numerical results obtained by employing the iterative methods (6), (26) to solve some nonlinear equations with unknown multiplicity m . To demonstrate the performance of the new fourth order iterative methods, we use ten particular nonlinear equations. The consistency and stability of the results are determined by examining the convergence of the new iterative methods. The findings are generalised by illustrating the effectiveness of the fourth order methods for

determining the multiple root of a nonlinear equation. Consequently, the errors obtained by each of the methods are displayed in the following tables. In fact, the errors displayed are of absolute value and insignificant approximations by the various methods have been omitted in the following tables. The numerical computations listed in the tables were performed on an algebraic system called Maple.

The new fourth-order method requires six function evaluations and has the order of convergence four. To determine the efficiency index of the new method, definition 3 will be used. Hence, the efficiency index of the new methods given by (6) and (26) is $\sqrt[6]{4} = \sqrt[3]{2}$, which is equivalent to the Soleymani et al methods (29) and (34). The efficiency index of the third-order method (25) $\sqrt[4]{3} \approx 1.316$, whereas the efficiency index of the second-order method (5) is given by $\sqrt[3]{2} \approx 1.260$. We can see that the efficiency index of the new fourth-order methods has not been improved, however we have constructed new simple and effective methods when compared to the established fourth order methods [6,7]. The test functions, multiplicity m and their exact root α are displayed in table 1. The difference between the root α and the approximation x_n for test functions with initial guess x_0 are displayed in table 2. Table 2 shows the absolute errors obtained by each of iterative methods described, we see that the new fourth order method is producing better results than the established methods. Furthermore, the computational order of convergence (COC) are displayed in table 3. From the table 3, we observe that the COC perfectly coincides with the theoretical result. In addition, the approximation $\hat{m}$ based on the present methods (5) are displayed in table 4. In table 4 we see a remarkable precision of the multiplicity m . As expected, we see remarkable precision of the multiplicity m based on new Thukral third-order formula (5). In fact, x_n is calculated by using the same total number of function evaluations (TNFE) for all methods.

Table 1 Test functions, multiplicity m , root α and initial guess x_0 .

Functions	m	Roots	Initial guess
$f_1(x) = (x^6 - 1)^m$	$m = 2$	$\alpha = -1$	$x_0 = -1.2$
$f_2(x) = (x^3 + 4x - 10)^m$	$m = 3$	$\alpha = 1.365230...$	$x_0 = 1.1$
$f_3(x) = ((x - 1)^3 - 1)^m$	$m = 4$	$\alpha = 2$	$x_0 = 2.2$
$f_4(x) = (\exp(x^2 + 7x - 30) - 1)^m$	$m = 5$	$\alpha = 2.842438...$	$x_0 = 3$
$f_5(x) = (\cos(x) + x)^m$	$m = 6$	$\alpha = -0.739085...$	$x_0 = -1$
$f_6(x) = (\sin(x)^2 - x^2 + 1)^m$	$m = 7$	$\alpha = 1.404491...$	$x_0 = 1.7$
$f_7(x) = (e^{-x^2} - e^{x^2} - x^8 + 10)^m$	$m = 8$	$\alpha = 1.239417...$	$x_0 = 1.5$
$f_8(x) = (6x^5 + 5x^4 - 4x^3 + 3x^2 - 2x + 1)^m$	$m = 9$	$\alpha = -1.57248...$	$x_0 = -1.8$
$f_9(x) = (\tan(x) - e^x - 1)^m$	$m = 10$	$\alpha = 1.371045...$	$x_0 = 1.2$
$f_{10}(x) = (\ln(x^2 + 3x + 5) - 2x + 7)^m$	$m = 11$	$\alpha = 5.469012...$	$x_0 = 5$

Table 2 Comparison of iterative methods

f_i	(5)	(25)	(6)	(26)	(29)	(34)
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f_1	0.433e-5	0.410e-71	0.141e-79	0.777e-145	0.233e-37	0.153e-83
f_2	0.396e-13	0.493e-81	0.815e-219	0.271e-220	0.118e-198	0.241e-241
f_3	0.211e-11	0.393e-76	0.399e-93	0.460e-147	0.411e-41	0.106e-60
f_4	0.302e-17	0.307e-106	0.367e-142	0.478e-220	0.214e-62	0.395e-71
f_5	0.704e-21	0.129e-76	0.154e-173	0.335e-189	0.652e-79	0.322e-75
f_6	0.245e-11	0.347e-56	0.232e-93	0.694e-115	0.925e-48	0.311e-56
f_7	0.499e-3	0.773e-64	0.235e-23	0.304e-89	0.582e-2	-
f_8	0.765e-6	0.927e-39	0.231e-46	0.417e-65	0.323e-16	0.343e-33
f_9	0.310e-4	0.135e-26	0.176e-30	0.122e-47	0.269e-12	0.880e-15
f_{10}	0.219e-34	0.284e-40	0.554e-291	0.157e-190	0.135e-123	0.432e-125

Table 3 Performance of COC

f_i	(5)	(25)	(6)	(26)	(29)	(34)
f_1	1.9601	3.0000	4.0000	4.0000	3.9988	4.0000
f_2	2.0000	2.9888	4.0000	4.0000	4.0000	4.0000
f_3	1.9998	2.9872	4.0000	4.0000	4.0000	4.0000
f_4	2.0000	3.0000	4.0000	4.0000	4.0000	4.0000
f_5	2.0000	2.9874	4.0000	4.0000	4.0000	4.0000
f_6	1.9999	2.0768	4.0000	4.0000	4.0000	4.0000
f_7	1.7889	3.0000	3.9997	4.0000	3.0373	-
f_8	1.9840	3.0000	4.0000	4.0000	4.0000	4.0000
f_9	1.8481	2.9994	4.0000	4.0000	3.9995	3.9999
f_{10}	3.0000	3.0000	4.0000	4.0000	4.0000	4.0000

Table 4 Performance of multiplicity m , based on the approximants of (25)

f_i	(5)	(25)	(6)	(26)	(29)	(34)
f_1	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
f_2	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
f_3	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
f_4	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
f_5	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000
f_6	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000
f_7	7.9835	8.0000	8.0000	8.0000	7.9999	-
f_8	9.0000	9.0000	9.0000	9.0000	9.0000	9.0000

f_9	9.9967	10.000	10.000	10.000	10.000	10.000
f_{10}	11.000	11.000	11.000	11.000	11.000	11.000

6 Conclusions

In this paper, two new fourth-order iterative methods for solving nonlinear equations with multiple roots has been introduced. Convergence analysis proves that the new methods preserve the order of convergence. The prime motive for presenting the new methods were to establish a higher order of convergence method than the existing third order methods [9]. The effectiveness of the new fourth-order methods are examined by showing the accuracy of the multiple roots of several nonlinear equations. After an extensive experimentation, it can be concluded that the convergence of the new fourth-order methods is remarkably fast. The main purpose of demonstrating the new methods for different types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative method. We have shown numerically, and verified, that the new iterative methods converge to the order four. Empirically, we have found that the results obtained by the new fourth order methods are producing better approximation of the zeros than the other well established methods, namely the Soleymani et al. methods (29) and (34). Finally, the new methods are simple, very effective and robust.

References

- [1] W. Gautschi, Numerical Analysis: an Introduction, Birkhauser, 1997.
- [2] J. L. Lagouanelle, Sur une mtdode de calcul de l'ordre de multiplicity des zros d'un polynme, C. R. Acad. Sci. Paris Sr. A 262 (1966) 626-627.
- [3] M. S. Petkovic, B. Neta, L. D. Petkovic, J. Dzunic, Multipoint methods for solving nonlinear equations, Elsevier 2012.
- [4] M. A. Noor, F. A. Shah, A family of iterative schemes for finding zeros of nonlinear equations having unknown multiplicity, Appl. Math. Inf. Sci. 8 (2014) 2367-2373.
- [5] E. Schroder, Uber unendlich viele Algorithmen zur Auflosung der Gleichungen, Math. Ann. 2 (1870) 317-365.
- [6] F. Soleymani, D. K. R. Babajee, Computing multiple zeros using a class of quartically convergent methods, Alex. Eng. 52 (2013) 531-541.
- [7] F. Soleymani, S. Shateyi, G. Ozkum, An iterative solver in the presence and absence of multiplicity for nonlinear equations, The Scientic World Journal, vol 2013, article ID 837243.
- [8] R. Thukral, New variants of the Schroder method for finding zeros of nonlinear equations having unknown multiplicity, J. Adv. Math. 8 (3) (2014) 1675-1683.
- [9] R. Thukral, New third-order Schroder-type method for solving nonlinear equations having unknown multiplicity, submitted AJCAM 2015.
- [10] J. F. Traub, Iterative Methods for solution of equations, Chelsea publishing company, New York 1977.