

Original Research Article

Convergence of Cosine Trigonometric Series

Abstract

We obtain a necessary and sufficient condition for L^1 -convergence of a modified cosine sum and a theorem of Telyakovskii [1] concerning convergence behavior of cosine series with monotonic decreasing coefficients has been deduced as a corollary.

Keywords : L^1 -convergence, modified cosine sum, class S.

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1 Introduction

Consider the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx . \quad (1.1)$$

Let $S_n(x)$ denote the partial sum of (1.1) and let $f(x) = \lim_{n \rightarrow \infty} S_n(x)$.

The problem of L^1 – convergence, via Fourier coefficients, consists of finding the properties of Fourier coefficients such that the necessary and sufficient condition for $\| S_n(x) - f(x) \| = o(1)$, $n \rightarrow \infty$, is given in the form $a_n \log n = o(1)$, $n \rightarrow \infty$, where $\| \cdot \|$ denotes the L^1 -norm.

The following definitions are related to this paper.

Convex sequence. [2,vol.II,p.202] A sequence $\{a_k\}$ is said to be convex if $\Delta^2 a_k \geq 0$ for every k where $\Delta^2 a_k = \Delta a_k - \Delta a_{k+1}$ and $\Delta a_k = a_k - a_{k+1}$.

Quasi- Convex sequence [2,vol.II,p.202]. A sequence $\{a_k\}$ is said to be quasi-convex if $\sum_{k=1}^{\infty} k |\Delta^2 a_k| < \infty$.

The class of all such sequences is an extension of the class of convex null sequences. The class of quasi-convex sequences is a subclass of BV class

($\sum_{k=1}^{\infty} |\Delta a_k| < \infty$), the class of all null sequences of bounded variation.

Telyakovskii [3] generalized the notion of quasi- convexity .

Let $\{a_k\}$ be a sequence satisfying

$$a_k \rightarrow 0 \text{ as } k \rightarrow \infty ; \quad (1.2)$$

$$S_1 = \sum_{k=0}^{\infty} |\Delta a_k| < \infty ; \quad (1.3)$$

$$S_2 = \sum_{m=2}^{\infty} \left| \sum_{k=1}^{[M/2]} \frac{\Delta a_{M-k} - \Delta a_{M+k}}{k} \right| < \infty . \quad (1.4)$$

It has been established [3] that a quasi-convex null sequence satisfies the conditions (1.2) – (1.4) and imply $\lim_{n \rightarrow \infty} S_n(x)$ exists where $S_n(x)$ is the partial sum of (1.1).

Concerning L^1 -convergence of the cosine series (1.1), the following theorem is known :

Theorem A [2]. If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (1.1) in the metric space L^1 , it is necessary and sufficient that $a_n \log n = o(1), n \rightarrow \infty$.

Teljakovskii [4] generalized Theorem A for the cosine series (1.1) with coefficients $\{a_k\}$ satisfying conditions (1.2) – (1.4) and established the following Theorem :

Theorem B [4]. Let the coefficients $\{a_k\}$ of the series (1.1) satisfy the condition (1.2) – (1.4). If $\lim_{n \rightarrow \infty} a_n \log n = 0$, then the cosine series (1.1) converges in the L^1 -metric space.

Teljakovskii [3] has also shown that under the conditions (1.2) – (1.4), the series (1.1) is a Fourier series and

$$\int_0^{\pi} \left| \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \right| dx \leq C(S_1 + S_2)$$

where C is a positive constant.

Sidon generalized the concept of quasi-convexity as follows :

The class S[5]. A null sequence $\{a_k\}$ belongs to the class S if there exists a sequence $\{A_k\}$ such that (i) $A_k \downarrow 0, k \rightarrow \infty$, (ii) $\sum_{k=0}^{\infty} A_k < \infty$, and (iii) $|\Delta a_k| \leq A_k$ for all k .

The class S is the extension of the class of quasi-convex sequences. Since a quasi-convex null sequence satisfies conditions of the class S , if we choose $A_n = \sum_{m=n}^{\infty} |\Delta^2 a_m|$.

Teljakovskii generalized Theorem A by establishing the following theorem :

Theorem C[1]. Let $\{a_k\}$ belong to the class S . Then the cosine series (1.1) is the Fourier series of its sum f and $\|S_n(x) - f(x)\| = o(1), n \rightarrow \infty$

61 if and only if $a_n \log n = o(1)$, $n \rightarrow \infty$.

62 Teljakovskii, thus showed that the class S is also a class of L^1 -
63 convergence which in turn led to numerous, more general results.

64 Rees and Stanojevic [6,7] introduced a new type of cosine sum

$$65 \quad h_n(x) = \frac{1}{2} \sum_{k=0}^{\infty} \Delta a_k + \sum_{k=1}^n \sum_{j=k}^n \Delta a_j \cos kx \quad (1.5)$$

66 and obtained a necessary and sufficient for its integrability.

67 Regarding the L^1 - convergence of (1.5) to a cosine trigonometric
68 series belonging to the class S , Ram proved the following result :

69 *Theorem D. [8].* If $\{a_k\}$ belongs to the class S , then

$$70 \quad \|f(x) - h_n(x)\| = o(1), \quad n \rightarrow \infty.$$

71 Kumari and Ram [9] introduced a new modified cosine sum

$$72 \quad f_n(x) = \frac{a_0}{2} + \sum_{k=1}^n \sum_{j=k}^n \Delta \left(\frac{a_j}{j} \right) k \cos kx, \quad (1.6)$$

73 and proved the result :

74 *Theorem E.* Let $\{a_k\}$ belong to the class S .

75 If $\lim_{n \rightarrow \infty} |a_{n+1}| \log n = 0$, then $\|f(x) - f_n(x)\| = o(1)$, $n \rightarrow \infty$.

76 Hooda *et al.* [10] introduced new modified cosine sum

$$77 \quad g_n(x) = \left(\frac{1}{2} \right) \left[a_1 + \sum_{k=0}^n \Delta^2 a_k \right] + \sum_{k=1}^n \left[a_{n+1} + \sum_{j=k}^n \Delta^2 a_j \right] \cos kx, \quad (1.7)$$

78 and studied the necessary and sufficient conditions for the L^1 -
79 convergence and integrability of the limit of (1.7) under the
80 conditions (1.2) – (1.4).

81 In recent years, significant results have been developed by various
82 authors [11-16] by imposing different conditions on the coefficients a_k
83 of trigonometric series (1.1). The aim of this paper is to study the L^1 -
84 convergence of (1.7) under class S on the coefficients a_k and deduce
85 Theorem C as a corollary of our result.

86

87 **2 Lemma**

88 The following lemma is required for the proof of our result.

89 *Lemma 1.[17].* If $|c_k| \leq 1$, then

$$90 \quad \int_0^\pi \left| \sum_{k=0}^n c_k D_k(x) \right| dx \leq C(n+1),$$

91 where C is a positive constant.

92 3 Results

93 *Theorem 1.* Let $\{a_k\}$ belong to the class S , then

94 $\|f(x) - g_n(x)\| = o(1), n \rightarrow \infty$ if and only if $a_n \log n = o(1), n \rightarrow \infty$.

95 *Corollary 1.* Let $\{a_k\}$ belong to the class S , then

96 $\|S_n(x) - f(x)\| = o(1), n \rightarrow \infty$ if and only if $a_n \log n = o(1), n \rightarrow \infty$.

97 This is nothing but theorem C.

98 *Proof of Theorem 1.* We have

$$\begin{aligned}
 99 \quad g_n(x) &= \left(\frac{1}{2}\right) \left[a_1 + \sum_{k=0}^n \Delta^2 a_k \right] + \sum_{k=1}^n \left[a_{n+1} + \sum_{j=k}^n \Delta^2 a_j \right] \cos kx \\
 100 \quad &= (1/2)[a_0 - a_{n+1} + a_{n+2}] + \sum_{k=1}^n [a_k - a_{n+1} + a_{n+2}] \cos kx \\
 101 \quad & \\
 102 \quad &= (a_0/2) - (1/2)\Delta a_{n+1} + \sum_{k=1}^n a_k \cos kx - \Delta a_{n+1} \sum_{k=1}^n \cos kx \\
 103 \quad & \\
 104 \quad &= (a_0/2) + \sum_{k=1}^n a_k \cos kx - \Delta a_{n+1} \sum_{k=1}^n \left[\cos ks + \frac{1}{2} \right] \\
 105 \quad & \\
 106 \quad &= S_n(x) - \Delta a_{n+1} D_n(x),
 \end{aligned}$$

107 where $D_n(x) = (1/2) + \cos x + \dots + \cos nx$ represents Dirichlet's kernel.

108 Using Abel's transformation, we get

$$\begin{aligned}
 109 \quad g_n(x) &= \sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - \Delta a_{n+1} D_n(x) \quad (3.1) \\
 110 \quad & \\
 111 \quad &= \sum_{k=0}^n \Delta a_k D_k(x) + a_{n+2} D_n(x).
 \end{aligned}$$

112 Now,

113

$$114 \quad f(x) - g_n(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+2} D_n(x).$$

115

116 Abel transformation with lemma1 yield,

$$117 \quad \int_0^{\pi} |f(x) - g_n(x)| dx$$

$$118 \quad \leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^{\pi} |a_{n+2} D_n(x)| dx$$

$$\begin{aligned}
 &= \int_0^\pi \left| \sum_{k=n+1}^\infty A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx + \int_0^\pi |a_{n+2} D_n(x)| dx \\
 &= \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta A_k \sum_{j=0}^k \frac{\Delta a_j}{A_j} D_j(x) \right| dx + \int_0^\pi |a_{n+2} D_n(x)| dx \\
 &\leq C \sum_{k=n+1}^\infty (k+1) \Delta A_k + \int_0^\pi |a_{n+2} D_n(x)| dx .
 \end{aligned}$$

Now, $\int_0^\pi |a_{n+2} D_n(x)|$ behaves like $a_n \log n$ and under the assumed hypothesis $a_n \log n = o(1)$, $n \rightarrow \infty$ as well as $\sum_{k=n+1}^\infty (k+1) \Delta A_k$ converges, the right hand side tends to zero as $n \rightarrow \infty$ and this gives

$$\lim_{n \rightarrow \infty} \int_0^\pi |f(x) - g_n(x)| dx = 0.$$

On the other hand

$$a_{n+2} D_n(x) = \sum_{k=n+1}^\infty \Delta a_k D_k(x) - [f(x) + g_n(x)] ,$$

and so

$$\int_0^\pi |a_{n+2} D_n(x)| dx \leq \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta a_k D_k(x) \right| dx + \int_0^\pi |f(x) - g_n(x)| dx .$$

Using the hypothesis of the theorem along with above estimates , the right hand side tends to zero as $n \rightarrow \infty$.

This completes the proof of our theorem.

Proof of Corollary 1. We have

$$\begin{aligned}
 \int_0^\pi |f(x) - S_n(x)| dx &= \int_0^\pi |f(x) - g_n(x) + g_n(x) - S_n(x)| dx \\
 &\leq \int_0^\pi |f(x) - g_n(x)| dx + \int_0^\pi |g_n(x) - S_n(x)| dx \\
 &\leq \int_0^\pi |f(x) - f_n(x)| dx + \int_0^\pi |\Delta a_{n+1} D_n(x)| dx
 \end{aligned}$$

whereas

$$\int_0^{\pi} |\Delta_{n+1} D_n(x)| dx \leq \int_0^{\pi} |f(x) - f_n(x)| dx + \int_0^{\pi} |f(x) - S_n(x)| dx .$$

Since $\int_0^{\pi} |D_n(x)| dx$ behave like $\log n$ for large values of n and by the hypothesis of our result the corollary follows.

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