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Original research papers

Finding Remainder on Dividing Numbers

Abstract

In this paper, we also prove an interesting method of divisibility by $7, 13, 17$ and 19. Let $N = a_m a_{m-1} a_{m-2} \cdots a_2 a_1 a_0$ and let $S = a_0 - 4a_1 + 2a_2 - a_3 + 4a_4 - 2a_5 + \cdots + (-4)^m a_m$. Then $7 | N$ if and only if $7 | S$. Let $T = a_0 - 3a_1 + 9a_2 - a_3 + 3a_4 - 9a_5 + \cdots + (-3)^m a_m$. Then $13 | N$ if and only if $13 | T$. Let $V = a_0 - 7a_1 - 2a_2 - 3a_3 + 4a_4 + 6a_5 - 8a_6 + 5a_7 - a_8 + 7a_9 + 2a_{10} + \cdots + (-7)^m a_m$. Then $17 \mid N$ if and only if $17 \mid V$. Let $W = a_0 - 9a_1 + 5a_2 - 7a_3 + 6a_4 + 3a_5 - 8a_6 - 4a_7 - 2a_8 - a_9 + \cdots + (-9)^m a_m$. Then $19 \mid N$ if and only if $19 \mid W$.

Keywords: Divisibility, Congruence

2010 Mathematics Subject Classification: Primary 11A07,11A99

1 Introduction

Divisibility by 7 can be tested by a recursive method. A number of the form $10x + y$ is divisible by 7 if and only if $x - 2y$ is divisible by 7. In other words, subtract twice the last digit from the number formed by the remaining digits. Continue to do this until a number known to be divisible by 7 is obtained. The original number is divisible by 7 if and only if the number obtained using this procedure is divisible by 7. For example, the number $371 : 37 - (2 \times 1) = 37 - 2 = 35; 3 - (2 \times 5) = 3 - 10 = -7$; thus, since -7 is divisible by 7, 371 is divisible by 7.

Another method is multiplication by 3. A number of the form $10x + y$ has the same remainder when divided by 7 as $3x + y$. One must multiply the left most digit of the original number by 3, add the next digit, take the remainder when divided by 7, and continue from the beginning: multiply by 3, add the next digit, etc. For example, the number $371 : 3 \times 3 + 7 = 16$ remainder 2, and $2 \times 3 + 1 = 7$. This method can be used to find the remainder of division by 7.

Divisibility by 13 $(1, -3, -4, -1, 3, 4,$ cycle goes on). If you are not comfortable with negative numbers, then use this sequence $(1, 10, 9, 12, 3, 4,$ cycle goes on). Multiply the right most digit of the number with the left most number in the sequence shown above and the second right most digit to the second left most digit of the number in the sequence.

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The aim of this paper is to prove an interesting method of divisibility by $7, 13, 17$ and 19. In Section 2, we will review some basic properties of congruence and special divisibility tests. In Section 3, we show the main result for special method divisibility by $7, 13, 17$ and 19. In Section 4, we show an application for the main theorems. Conclusions we show in Section 5.

2 Basic Properties of Congruence and Special Divisibility Tests

In this section, we review some basic properties of congruence and special divisibility tests, following David (2002).

2.1 Basic Properties of Congruence

If a number n measures the difference between two numbers a and b are said to be congruent with respect to n; if not, incongruent. Putting this into the form of a definition, we have Definition 2.1.

Definition 2.1. Let n be a fixed positive integer. Two integers a and b are said to be congruennt modulo n, symbolized by $a \equiv b \pmod{n}$. If n divides the difference $a - b$; that is, provided that $a - b = kn$ for some integer k.

Theorem 2.1. *For arbitrary integers a and b, $a \equiv b \pmod{n}$ if and only if a and b leave the same nonnegative remainder when divided by n.*

Theorem 2.2. *Let $n > 1$ be fixed and a, b, c, d be arbitrary integers. Then the following properties hold:*

(a) $a \equiv a \pmod{n}$
(b) If $a \equiv b \pmod{n}$*, then* $b \equiv a \pmod{n}$
(c) If $a \equiv b \pmod{n}$ *and* $b \equiv c \pmod{n}$*, then* $a \equiv c \pmod{n}$
(d) If $a \equiv b \pmod{n}$ *and* $c \equiv d \pmod{n}$*, then* $a + c \equiv b + d \pmod{n}$
(e) If $a \equiv b \pmod{n}$ *and* $c \equiv d \pmod{n}$*, then* $ac \equiv bd \pmod{n}$
(f) If $a \equiv b \pmod{n}$*, then* $a + c \equiv b + c \pmod{n}$
(g) If $a \equiv b \pmod{n}$*, then* $ac \equiv bc \pmod{n}$
(h) If $a \equiv b \pmod{n}$*, then* $a^k \equiv b^k \pmod{n}$ *for any positive integer* k*.*

In Theorem 2.2 we saw that if $a \equiv b \pmod{n}$, then $ca \equiv cb \pmod{n}$ for any integer c. The converse, however, fails to hold. With suitable precautions, cancellation can be allowed; one step in this direction, and an important one, is provided by the following theorem.

Theorem 2.3. *If* $ca \equiv cb \pmod{n}$*, then* $a \equiv b \pmod{\frac{n}{d}}$*, where* $d = gcd(c, n)$*.*

Theorem 2.3 get its maximum force when the requirement that $gcd(c, n) = 1$ is added, for then the cancellation may be accomplished without a change in modulus.

Corollary 2.4. *If* $ca \equiv cb \pmod{n}$ *and* $gcd(c, n) = 1$*, then* $a \equiv b \pmod{n}$*.*

We take the moment to record a special case of Corollary 2.4 which we shall have frequent occation to use, namely, Corollary 2.5.

Corollary 2.5. *If* $ca \equiv cb \pmod{p}$ *and* $p \nmid c$*, where* p *is a prime number, then* $a \equiv b \pmod{p}$*.*

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2.2 Special Divisibility Tests

One of the more interesting applications of congruence theory involves finding special criteria under which a given integer is divisible by another integer. At their heart, these divisibility tests depend on the notational system used to assign names to integers and, more particularly, to the fact that 10 is taken as the base for our number system. Let us, therefore, start by showing that, given an integer $b > 1$, any positive integer N can be written uniquely in terms of powers of b as

$$N = a_m b^m + a_{m-1} b^{m-1} + a_{m-2} b^{m-2} + \cdots + a_2 b^2 + a_1 b + a_0$$

where the coefficients a_k can be take on the b different valued $0, 1, 2, ..., b-1$.

Thus, the number

$$N = a_m b^m + a_{m-1} b^{m-1} + a_{m-2} b^{m-2} + \cdots + a_2 b^2 + a_1 b + a_0$$

may be replaced by the simpler symbol

$$N = (a_m a_{m-1} a_{m-2} \cdots a_2 a_1 a_0)_b.$$

We ordinarily record numbers in the decimal system of notation, where $b = 10$, omitting the 10-subscript which specifies the base. For instance, the symbol 1785 stands for the more awkward expression

$$1785 = 1 \cdot 10^3 + 7 \cdot 10^2 + 8 \cdot 10 + 5.$$

We are about ready to derive criteria for determining whether an integer is divisible by 9 or 11, without performing the actual division. For this, we need a result having to do with congruences involving polynomials with integral coefficients.

Theorem 2.6. *Let $P(x) = \sum_{k=0}^{m} c_k x^k$ be a polynomial function of x with integral coefficients c_k. If $a \equiv b \pmod{n}$, then $P(a) \equiv P(b) \pmod{n}$.*

One divisibility test that we have in mind is this. A positive integer is divisible by 9 if and only if the sum of the digits in its decimal representation is divisible by 9.

Theorem 2.7. *Let $N = a_m 10^m + a_{m-1} 10^{m-1} + a_{m-2} 10^{m-2} + \cdots + a_2 10^2 + a_1 10 + a_0$ be the decimal expansion of the positive integer $N, 0 \leq a_k < 10$, and let $A = a_0 + a_1 + a_2 + \cdots + a_m$. Then $9 \mid N$ if and only if $9 \mid A$.*

Theorem 2.6 also serves as the basis for a well-known test for divisibility by 11. We state this more precisely by Theorem 2.8 and Theorem 2.9.

Theorem 2.8. *Let $N = a_m 10^m + a_{m-1} 10^{m-1} + a_{m-2} 10^{m-2} + \cdots + a_2 10^2 + a_1 10 + a_0$ be the decimal expansion of the positive integer $N, 0 \leq a_k < 10$, and let $B = a_0 - a_1 + a_2 - \cdots + (-1)^m a_m$. Then $11 \mid N$ if and only if $11 \mid B$.*

Theorem 2.9. *Let $N = a_m 10^m + a_{m-1} 10^{m-1} + a_{m-2} 10^{m-2} + \cdots + a_2 10^2 + a_1 10 + a_0 a_m 10^m + a_{m-1} 10^{m-1} + a_{m-2} 10^{m-2} + \cdots + a_2 10^2 + a_1 10 + a_0$ be the decimal expansion of the positive integer $N, 0 \leq a_k < 10$, and let*

$$M = (100a_2 + 10a_1 + a_0) - (100a_5 + 10a_4 + a_3) + (100a_8 + 10a_7 + a_6) - \cdots .$$

Then $7, 11$, and 13 divide N if and only if $7, 11$, and 13 divide M.

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3 Main Result

Now we state the main results of this section.

Theorem 3.1. *Let* $N = a_m 10^m + a_{m-1}10^{m-1} + a_{m-2}10^{m-2} + \cdots + a_2 10^2 + a_1 10 + a_0$ *be the decimal expansion of the positive integer* $N, 0 \leq a_k < 10$*, and let* $S = a_0 - 4a_1 + 2a_2 - a_3 + 4a_4 - 2a_5 + \cdots + (-4)^m a_m$*. Then* $7 \mid N$ *if and only if* $7 \mid S$*.*

Proof. Consider $P(x) = \sum_{k=0}^{m} a_k x^k$, a polynomial with integral coefficients. Since $10 \equiv -4 \pmod 7$, we get $P(10) \equiv P(-4) \pmod 7$ by Theorem 2.6. But $P(10) = N$, where as $P(-4) = a_0 - 4a_1 + 16a_2 - 64a_3 + 256a_4 - 1024a_5 + \cdots + (-4)^m a_m \equiv a_0 - 4a_1 + 2a_2 - a_3 + 4a_4 - 2a_5 + \cdots + (-4)^m a_m = S \pmod 7$, so that $N \equiv S \pmod 7$. The implication is that either both N and S are divisible by 7 or neither is divisible by 7. □

Theorem 3.2. *Let* $N = a_m 10^m + a_{m-1}10^{m-1} + a_{m-2}10^{m-2} + \cdots + a_2 10^2 + a_1 10 + a_0$ *be the decimal expansion of the positive integer* $N, 0 \leq a_k < 10$*, and let* $T = a_0 - 3a_1 + 9a_2 - a_3 + 3a_4 - 9a_5 + \cdots + (-3)^m a_m$*. Then* $13 \mid N$ *if and only if* $13 \mid T$*.*

Proof. Consider $P(x) = \sum_{k=0}^{m} a_k x^k$, a polynomial with integral coefficients. Since $10 \equiv -3 \pmod{13}$, we get $P(10) \equiv P(-3) \pmod{13}$ by Theorem 2.6. But $P(10) = N$, where as $P(-3) = a_0 - 3a_1 + 9a_2 - 27a_3 + 81a_4 - 243a_5 + \cdots + (-3)^m a_m \equiv a_0 - 3a_1 + 9a_2 - a_3 + 3a_4 - 9a_5 + \cdots + (-3)^m a_m = T \pmod{13}$, so that $N \equiv T \pmod{13}$. It follow that $N \equiv 0 \pmod{13}$ if and only if $T \equiv 0 \pmod{13}$, which is what we wanted to prove. □

Theorem 3.3. *Let* $N = a_m 10^m + a_{m-1}10^{m-1} + a_{m-2}10^{m-2} + \cdots + a_2 10^2 + a_1 10 + a_0$ *be the decimal expansion of the positive integer* $N, 0 \leq a_k < 10$*, and let* $V = a_0 - 7a_1 - 2a_2 - 3a_3 + 4a_4 + 6a_5 - 8a_6 + 5a_7 - a_8 + 7a_9 + 2a_{10} + \cdots + (-7)^m a_m$*. Then* $17 \mid N$ *if and only if* $17 \mid V$*.*

Proof. Consider $P(x) = \sum_{k=0}^{m} a_k x^k$, a polynomial with integral coefficients. Since $10 \equiv -7 \pmod{17}$, we get $P(10) \equiv P(-7) \pmod{17}$ by Theorem 2.6. But $P(10) = N$, where as $P(-7) \equiv a_0 - 7a_1 - 2a_2 - 3a_3 + 4a_4 + 6a_5 - 8a_6 + 5a_7 - a_8 + 7a_9 + 2a_{10} + \cdots + (-7)^m a_m = V \pmod{17}$, so that $N \equiv V \pmod{17}$. The implication is that either both N and V are divisible by 17 or neither is divisible by 17. □

Theorem 3.4. *Let* $N = a_m 10^m + a_{m-1}10^{m-1} + a_{m-2}10^{m-2} + \cdots + a_2 10^2 + a_1 10 + a_0$ *be the decimal expansion of the positive integer* $N, 0 \leq a_k < 10$*, and let* $W = a_0 - 9a_1 + 5a_2 - 7a_3 + 6a_4 + 3a_5 - 8a_6 - 4a_7 - 2a_8 - a_9 + \cdots + (-9)^m a_m$*. Then* $19 \mid N$ *if and only if* $19 \mid W$*.*

Proof. Consider $P(x) = \sum_{k=0}^{m} a_k x^k$, a polynomial with integral coefficients. Since $10 \equiv -9 \pmod{19}$, we get $P(10) \equiv P(-9) \pmod{19}$ by Theorem 2.6. But $P(10) = N$, where as $P(-9) \equiv a_0 - 9a_1 + 5a_2 - 7a_3 + 6a_4 + 3a_5 - 8a_6 - 4a_7 - 2a_8 - a_9 + \cdots + (-9)^m a_m = W \pmod{19}$, so that $N \equiv W \pmod{19}$. It follow that $N \equiv 0 \pmod{19}$ if and only if $W \equiv 0 \pmod{19}$, which is what we wanted to prove. □

4 Applications

In this section, we shall utilize Theorem 3.1 - Theorem 3.4 to show some example.

Example 4.1. *To see an illustration of Theorem 3.1 and Theorem 3.2, take the integer*

$$N = 46423813177157416.$$

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Since the sum $6(1)+1(-4)+4(2)+7(-1)+5(4)+1(-2)+7(1)+7(-4)+1(2)+3(-1)+1(4)+8(-2)+3(1)+2(-4)+4(2)+6(-1)+4(4)=0$, *is divisible by* 7, *Theorem 3.1 guarantees that* 7 *divides* N. *It also can be divided by* 13; *for the sum* $6(1)+1(-3)+4(9)+7(-1)+5(3)+1(-9)+7(1)+7(-3)+1(9)+3(-1)+1(3)+8(-9)+3(1)+2(-3)+4(9)+6(-1)+4(3)=0$, *is divisible by* 13. *Therefore, Theorem 3.2 guarantees that* 13 *divides* N.

Example 4.2. *To see an illustration of Theorem 3.3 and Theorem 3.4, take the integer*

$$N = 46423813177157416.$$

Since the sum $6(1)+1(-7)+4(-2)+7(-3)+5(4)+1(6)+7(-8)+7(5)+1(-1)+3(7)+1(2)+8(3)+3(-4)+2(-6)+4(8)+6(-5)+4(1)=3$, *It follows that, the remainder of division of* 3 *by* 17 *is* 3. *Therefore, the remainder of division of* $N = 46423813177157416$ *by* 17 *is* 3. *Because the sum* $6(1)+1(-9)+4(5)+7(-7)+5(6)+1(3)+7(-8)+7(-4)+1(-2)+3(-1)+1(9)+8(-5)+3(7)+2(-6)+4(-3)+6(8)+4(4)=-58$. *It is easy to see that the remainder of division of* -58 *by* 19 *is* 18. *Therefore, the remainder of division of* $N = 46423813177157416$ *by* 19 *is* 18.

5 CONCLUSIONS

Divisibility by 7 test ($1,-4,2,-1,4,-2$, cycle goes on) and the remainder test 13 ($1,-3,9,-1,3,-9$, cycle goes on). Multiply the right most digit of the number with the left most number in the sequence shown above and the second right most digit to the second left most digit of the number in the sequence.

Divisibility by 17 test ($1,-7,-2,-3,4,6,-8,5,-1,7,2,3,-4,-6,8,-5,1$, cycle goes on) and the remainder test 19 ($1,-9,5,-7,6,3,-8,-4,-2,-1,9,-5,7,-6,-3,8,4,2,1$, cycle goes on). Multiply the right most digit of the number with the left most number in the sequence shown above and the second right most digit to the second left most digit of the number in the sequence.

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