

Dualities Between "Kamal & Mahgoub Integral Transforms" and "Some Famous Integral Transforms"

Abstract

Abdelilah Kamal and Mohand Mahgoub have recently introduced new integral transforms differently by the names the “Kamal transform” and “Mahgoub transform” to facilitate the solution of differential and integral equations. However, in this paper, these newly introduced transforms will closely be studied in relation to the some existing famous integral transforms that are defined in the time domain. The study will also try to establish the duality relations existing between these new integral transforms and in particular, the Laplace, Sumudu, Elzaki and Aboodh integral transforms. Further, supporting illustrations obtained from some test functions as examples are will be presented.

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Keywords: Kamal, Mahgoub, Aboodh, Elzaki, Sumudu, Laplace Transforms.

1 Introduction

Integral transforms are mathematical tools for solving differential and integral equations for centuries. However, this old area has recently got a center stage among many researchers by introducing many integral transforms among which are [1-6]. Furthermore, these transforms are studied and analyzed in order to establish relationships and connections existing among them among which are [7-9]. However, in this paper, the recently introduced integral transforms by Abdelilah Kamal [10] and Mohand Mahgoub [11] are closely studied in relation to the some existing famous integral transforms that are defined in the time domain. The study also tried to establish the duality relations existing between these newly introduced integral transforms and Laplace, Sumudu, Elzaki and Aboodh integral transforms as particular cases and further some test functions are examined as examples. Further, applications of these new integral transforms to the solution of partial differential equations are demonstrated in [13] and [14], respectively for interested readers.

The paper is organized as follows: In Section 2, we present the basic concept of Kamal and Mahgoub transforms differently. Section 3 presents Kamal transform dualities, while Section 4 presents Mahgoub transform dualities and Section 5 gives the conclusion.

2 Kamal and Mahgoub Transforms

In this section, we present the basic concept of Kamal and Mahgoub transforms.

2.1 Kamal Transform

The Kamal transform is a newly introduced integral transform similar to Laplace transform [12] and other integral transforms [1-6] that are defined in the time domain $t \geq 0$, and for functions of exponential order. We consider functions in the set B defined by:

$$B = \{ f : |f(t)| < Q e^{\frac{|t|}{k_1}} \text{ if } t \in (-1)^j \times [0, \infty), \quad j = 1, 2; (Q, k_1, k_2 > 0) \}.$$

For a given function in the set B , the constant Q must be finite number; k_1, k_2 may be finite or infinite.

Then, the Kamal transform denoted by the operator $K(.)$ for function of exponential order and belonging to set B is defined (by [10]) by the integral equation:

$$G(u) = K[f(t)] = \int_0^{\infty} e^{-\frac{t}{u}} f(t) dt, \quad u \in (k_1, k_2) \quad (1)$$

2.1.1 Kamal Transform for Derivatives

For any given function $f(t)$ defined over the set B , Kamal transform of derivatives are given as follows:

$$K[f'(t)] = u^{-1}G(u) - f(0),$$

$$K[f''(t)] = u^{-2}G(u) - u^{-1}f(0) - f'(0),$$

$$K[f^{(n)}(t)] = u^{(-n)}G(u) - \sum_{k=0}^{n-1} u^{k-n+1} f^{(k)}(0), \quad \text{for } n = 0, 1, 2, \dots$$

2.2 Mahgoub Transforms

The newly introduced integral transform by Mahgoub [11] is an integral transform that has the same setting with Kamal transform [10]. Thus, the Mahgoub integral transform denoted by the operator $M(.)$ is defined by:

$$H(u) = M[f(t)] = u \int_0^{\infty} e^{-ut} f(t) dt, \quad u \in (k_1, k_2) \quad (2)$$

2.2.1 Mahgoub Transform for Derivatives

For any given function $f(t)$ defined over the set B , Mahgoub transform of derivatives are given as follows:

$$M[f'(t)] = uH(u) - u f(0),$$

$$M[f''(t)] = u^2H(u) - u f'(0) - u^2 f(0),$$

$$M[f^{(n)}(t)] = u^{(n)}H(u) - \sum_{k=0}^{n-1} u^{n-k} f^{(k)}(0), \quad \text{for } n = 0, 1, 2, \dots$$

2.3 Connection Between Kamal and Mahgoub Transforms

To examine the connection between Kamal and Mahgoub transforms, we give the following theorem:

Theorem 2.3.1 : Let $f(t) \in B$ with Kamal transform $G(u)$ and Mahgoub transform $H(u)$, then

$$G(u) = u H\left(\frac{1}{u}\right) \quad (3)$$

and

$$H(u) = u G\left(\frac{1}{u}\right) \quad (4)$$

Proof: To prove Eq. (3), hence $f(t) \in B$, then from Eq. (1)

$$G(u) = \int_0^{\infty} e^{-\frac{t}{u}} f(t) dt = u \left(\frac{1}{u} \int_0^{\infty} e^{-\frac{t}{u}} f(t) dt \right)$$

Thus, from Eq. (2) $G(u) = u H\left(\frac{1}{u}\right)$.

To prove Eq. (4), from the Eq. (2), we have

$$H(u) = u \int_0^{\infty} e^{-ut} f(t) dt = u \left(\int_0^{\infty} e^{\frac{-t}{u}} f(t) dt \right).$$

Thus, from Eq. (1), we get $H(u) = uG\left(\frac{1}{u}\right)$.

Example 2.3.2

Consider the function $f(t) = t^2$. Kamal and Mahgoub transforms are obtained respectively as follows: $G(u) = 2u^3$ and $H(u) = \frac{2}{u^2}$, and their graph is given in figure 1:

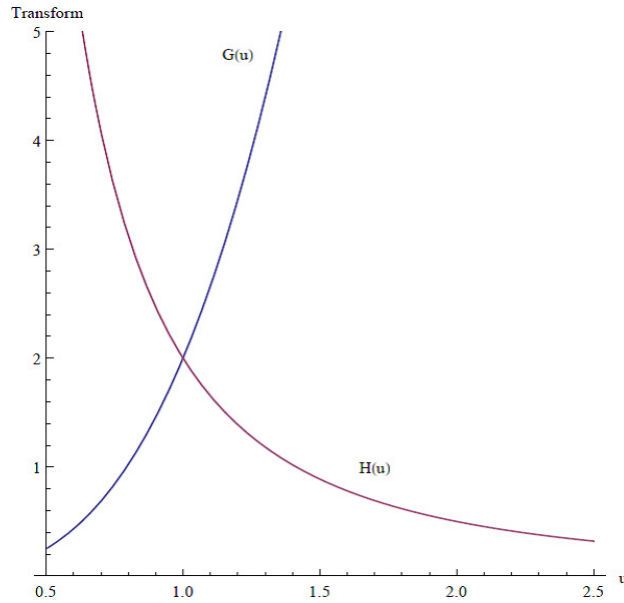


Figure 1: Plots for Kamal and Mahgoub Transforms of $f(t) = t^2$

3 Kamal Transform Dualities

In this section, we present the Kamal transform dualities starting with the Laplace, Sumudu, Elzaki and Aboodh transforms.

3.1 Kamal -Laplace Duality

Laplace transform of $f(t)$ for $t \geq 0$, denoted by $F(u)$, is defined by the equation

$$F(u) = L[f(t)] = \int_0^{\infty} e^{-ut} f(t) dt \quad (5)$$

Theorem 3.1.1: Let $f(t) \in B$ with Kamal transform $G(u)$ and Laplace transform $F(u)$, then

$$G(u) = F\left(\frac{1}{u}\right) \quad (6)$$

and

$$F(u) = G\left(\frac{1}{u}\right) \quad (7)$$

Proof: To prove Eq. (6), let $f(t) \in B$, then from Eq. (1)

$$G(u) = \int_0^{\infty} e^{-\frac{t}{u}} f(t) dt = \int_0^{\infty} e^{-\left(\frac{1}{u}\right)t} f(t) dt.$$

87 Thus, from Eq. (5) $G(u) = F\left(\frac{1}{u}\right)$.

88 To prove Eq. (7), from the Eq. (5), we have

$$F(u) = \int_0^{\infty} e^{-ut} f(t) dt = \int_0^{\infty} e^{-\frac{t}{\left(\frac{1}{u}\right)}} f(t) dt.$$

89 Thus, from Eq. (1), we get $F(u) = G\left(\frac{1}{u}\right)$.

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91 3.2 Kamal-Sumudu Duality

92 Sumudu transform of $f(t)$ for $t \geq 0$ denoted by $N(u)$, is defined by the equation

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$$94 N(u) = S[f(t)] = \int_0^{\infty} e^{-t} f(ut) dt, \quad k_1, k_2 > 0. (8)$$

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96 **Theorem 3.2.1:** Let $f(t) \in B$ with Kamal transform $G(u)$ and Sumudu transform $N(u)$, then

97

$$98 G(u) = uN(u) \quad (9)$$

99 and

$$100 N(u) = \frac{1}{u} G(u) \quad (10)$$

101 **Proof:** To prove Eq. (9), hence $f(t) \in B$, then from Eq. (1)

102

$$G(u) = \int_0^{\infty} e^{-\frac{t}{u}} f(t) dt.$$

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104 Now let, $w = \frac{t}{u} \Rightarrow t = uw$ and $dt = u dw$. Then,

$$G(u) = \int_0^{\infty} e^{-w} f(uw) u dw = u \left(\int_0^{\infty} e^{-w} f(uw) dw \right).$$

105

106 Hence, on comparing with Eq. (8), we thus obtain

$$G(u) = uN(u).$$

107 To prove Eq. (10), from Eq. (8),

$$N(u) = \int_0^{\infty} e^{-t} f(ut) dt.$$

108 Let, $w = ut \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$. Then,

$$N(u) = \int_0^{\infty} e^{-\frac{w}{u}} f(w) \frac{dw}{u} = \frac{1}{u} \left(\int_0^{\infty} e^{-\frac{w}{u}} f(w) dw \right).$$

109 Hence, on comparing with Eq. (1), we thus obtain

$$N(u) = \frac{1}{u} G(u).$$

3.3 Kamal-Elzaki Duality

Elzaki transform of $f(t)$ for $t \geq 0$, denoted by $T(u)$, is defined by the equation

$$T(u) = E[f(t)] = u^2 \int_0^\infty e^{-t} f(ut) dt, \quad u \in (k_1, k_2). \quad (11)$$

Theorem 3.3.1 : Let $f(t) \in B$ with Kamal transform $G(u)$ and Elzaki transform $T(u)$, then

$$G(u) = \frac{1}{u} T(u) \quad (12)$$

and

$$T(u) = uG(u). \quad (13)$$

Proof: To prove Eq. (12), hence $f(t) \in B$, then from Eq. (1)

$$G(u) = \int_0^\infty e^{-\frac{t}{u}} f(t) dt.$$

Let, $w = \frac{t}{u} \Rightarrow t = uw$ and $dt = u dw$. Then,

$$G(u) = \int_0^\infty e^{-w} f(uw) u dw = u \left(\int_0^\infty e^{-w} f(uw) dw \right) = \frac{1}{u} \left(u^2 \int_0^\infty e^{-w} f(uw) dw \right).$$

Hence, on comparing with Eq. (11), we thus obtain

$$G(u) = \frac{1}{u} T(u).$$

To prove Eq. (13), from Eq. (11),

$$T(u) = u^2 \int_0^\infty e^{-t} f(ut) dt.$$

Let, $t = \frac{w}{u} \Rightarrow dt = \frac{dw}{u}$. Then,

$$T(u) = u^2 \int_0^\infty e^{-\frac{w}{u}} f(w) \frac{dw}{u} = u \left(\int_0^\infty e^{-\frac{w}{u}} f(w) dw \right).$$

Hence, on comparing with Eq. (1), we thus obtain

$$T(u) = uG(u).$$

3.4 Kamal-Aboodh Duality

Aboodh transform of $f(t)$ for $t \geq 0$, denoted by $K(u)$, is defined by the equation

$$K(u) = A[f(t)] = \frac{1}{u} \int_0^\infty e^{-ut} f(t) dt, \quad u \in (k_1, k_2). \quad (14)$$

Theorem 3.4.1 : Let $f(t) \in B$ with Kamal transform $G(u)$ and Aboodh transform $K(u)$, then

$$G(u) = \frac{1}{u} K\left(\frac{1}{u}\right) \quad (15)$$

and

$$K(u) = \frac{1}{u} G\left(\frac{1}{u}\right). \quad (16)$$

Proof: To prove Eq. (15), hence $f(t) \in B$, then from Eq. (1)

$$G(u) = \int_0^\infty e^{-\frac{t}{u}} f(t) dt = \frac{1}{u} \left(\frac{1}{\frac{1}{u}} \int_0^\infty e^{-\left(\frac{1}{u}\right)t} f(t) dt \right).$$

Thus, from Eq. (14)

$$G(u) = \frac{1}{u} K\left(\frac{1}{u}\right).$$

136 To prove Eq. (16), from the Eq. (14), we have

$$K(u) = \frac{1}{u} \int_0^\infty e^{-ut} f(t) dt = \frac{1}{u} \left(\int_0^\infty e^{\frac{-t}{\frac{1}{u}}} f(t) dt \right)$$

137 Thus, from Eq. (1), we get:

$$K(u) = \frac{1}{u} G\left(\frac{1}{u}\right).$$

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140 **Example 3.5**

141 Consider the function $f(t) = e^{-2t}$. We obtain the following transforms:

$$142 \quad F(u) = \frac{1}{u+2}, N(u) = \frac{1}{1+2u}, T(u) = \frac{u^2}{1+2u}, K(u) = \frac{1}{u^2+2u} \text{ and } G(u) = \frac{u}{1+2u}$$

143 Figure 2 gives the illustration of the above transforms of the function under consideration:

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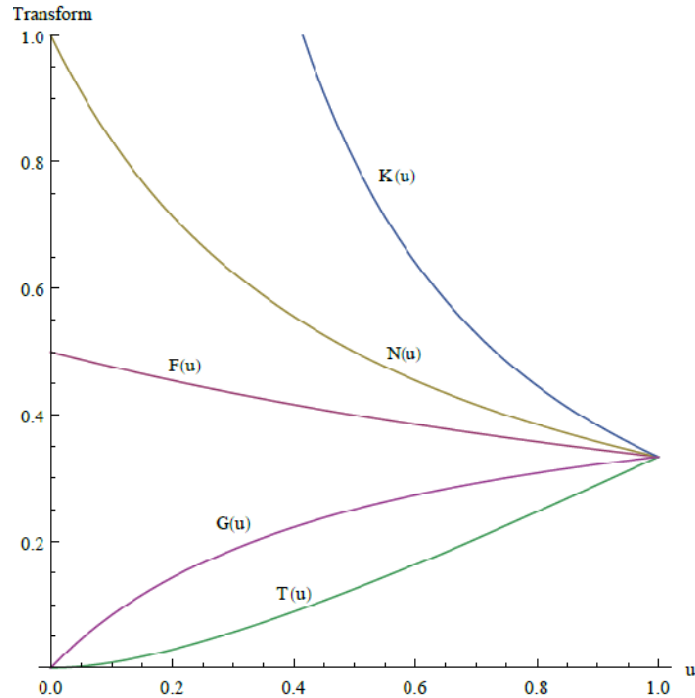


Figure 2: Plots for Laplace, Sumudu, Elzaki, Aboodh and Kamal Transforms of $f(t) = e^{-2t}$.

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148 **4 Mahgoub Transform Dualities**

149 In the same pattern, we present Mahgoub transform dualities, at the same time maintaining the
150 definitions of the integral transforms presented in section 3.

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152 **4.1 Mahgoub-Laplace Duality**

153 **Theorem 4.1.1 :** Let $f(t) \in B$ with Mahgoub transform $H(u)$ and Laplace transform $F(u)$, then

154

$$155 \quad H(u) = uF(u) \quad (17)$$

156 and

$$F(u) = \frac{1}{u} H(u). \quad (18)$$

Proof: To prove Eq. (17), hence $f(t) \in B$, then from Eq. (2)

$$H(u) = u \int_0^\infty e^{-ut} f(t) dt = u \left(\int_0^\infty e^{-ut} f(t) dt \right).$$

Thus, from Eq. (5):

$$H(u) = uF(u).$$

To prove Eq. (18), from the Eq. (5), we have

$$F(u) = \int_0^\infty e^{-ut} f(t) dt = \frac{1}{u} \left(u \int_0^\infty e^{-ut} f(t) dt \right).$$

Thus, from Eq. (2), we get

$$F(u) = \frac{1}{u} H(u).$$

4.2 Mahgoub-Sumudu Duality

Theorem 4.2.1 : Let $f(t) \in B$ with Mahgoub transform $H(u)$ and Sumudu transform $N(u)$, then

$$H(u) = N\left(\frac{1}{u}\right) \quad (19)$$

and

$$N(u) = H\left(\frac{1}{u}\right). \quad (20)$$

Proof: To prove Eq. (19), let $f(t) \in B$, then from Eq. (2)

$$H(u) = u \int_0^\infty e^{-ut} f(t) dt.$$

Let, $w = ut \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$. Then,

$$H(u) = \int_0^\infty e^{-w} f\left(\frac{w}{u}\right) dw = \int_0^\infty e^{-w} f\left(\frac{1}{u} w\right) dw.$$

Hence, on comparing with Eq. (8), we thus obtain:

$$H(u) = N\left(\frac{1}{u}\right).$$

To prove Eq. (20), from Eq. (8),

$$N(u) = \int_0^\infty e^{-t} f(ut) dt.$$

Let, $ut \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$. Then,

$$N(u) = \int_0^\infty e^{-\frac{w}{u}} f(w) \frac{dw}{u} = \frac{1}{u} \left(\int_0^\infty e^{-\left(\frac{1}{u}\right)w} f(w) dw \right).$$

Hence, on comparing with Eq. (2), we thus obtain:

$$N(u) = H\left(\frac{1}{u}\right).$$

4.3 Mahgoub-Elzaki Duality

Theorem 4.3.1: Let $f(t) \in B$ with Mahgoub transform $H(u)$, and Elzaki transform $T(u)$, then

$$H(u) = u^2 T(u) \quad (21)$$

and

$$T(u) = u^2 H\left(\frac{1}{u}\right). \quad (22)$$

Proof: To prove Eq. (21), hence $f(t) \in B$, then from Eq. (2)

$$H(u) = u \int_0^\infty e^{-ut} f(t) dt.$$

186

187 Let, $w = ut \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$. Then,

188

$$H(u) = \int_0^\infty e^{-w} f\left(\frac{w}{u}\right) dw = u^2 \left(\left(\frac{1}{u}\right)^2 \int_0^\infty e^{-w} f\left(\frac{1}{u}w\right) dw \right).$$

189 Hence, on comparing with Eq. (11), we thus obtain:

$$H(u) = u^2 T\left(\frac{1}{u}\right).$$

190 To prove Eq. (22), from Eq. (11)

$$T(u) = u^2 \int_0^\infty e^{-t} f(ut) dt.$$

191 Let, $w = ut \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$. Then,

$$T(u) = u \int_0^\infty e^{-\frac{w}{u}} f(w) dw = u^2 \left(\frac{1}{u} \int_0^\infty e^{-\left(\frac{1}{u}\right)w} f(w) dw \right).$$

192 Hence, on comparing with Eq. (2), we thus obtain

$$T(u) = u^2 H\left(\frac{1}{u}\right).$$

193 4.4 Mahgoub-Aboodh Duality

194 **Theorem 4.4.1:** Let $f(t) \in B$ with Mahgoub transform $H(u)$ and Aboodh transform $K(u)$, then

195

$$H(u) = u^2 K(u) \quad (23)$$

197 and

$$K(u) = \frac{1}{u^2} H(u). \quad (24)$$

199 **Proof:** To prove Eq. (23), hence $f(t) \in B$, then from Eq. (2)

$$H(u) = u \int_0^\infty e^{-ut} f(t) dt = u^2 \left(\frac{1}{u} \int_0^\infty e^{-ut} f(t) dt \right).$$

200 And, we obtain from Eq. (14) :

$$H(u) = u^2 K(u).$$

201 To prove Eq. (24), from the Eq. (14), we have

$$K(u) = \frac{1}{u} \int_0^\infty e^{-ut} f(t) dt = \frac{1}{u^2} \left(u \int_0^\infty e^{-ut} f(t) dt \right).$$

202 Thus, from Eq. (2), we get

$$K(u) = \frac{1}{u^2} H(u).$$

203 Example 4.5

204 Consider the function $f(t) = t^2$. We obtain the following transforms:

$$205 \quad F(u) = \frac{2}{u^3}, N(u) = 2u^2, \quad T(u) = 2u^4, \quad K(u) = \frac{2}{u^4} \text{ and } H(u) = \frac{2}{u^2}.$$

206 Figure 3 gives the illustrations of the connections between the transforms under consideration as
207 follows:
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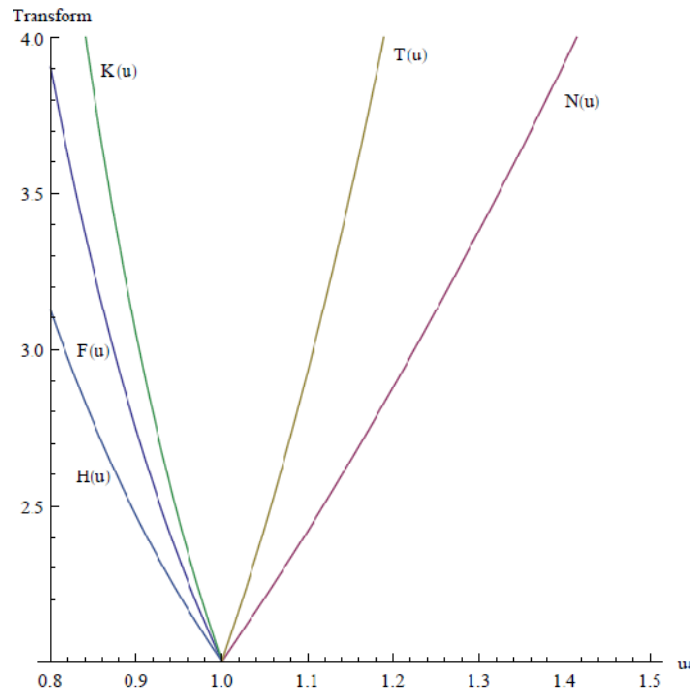


Figure 3: Plots for Laplace, Sumudu, Elzaki, Aboodh and Mahgoub Transforms of $f(t) = t^2$.

5. Conclusion

In conclusion, the duality relations existing between the Kamal and Mahgoub integral transforms with the Laplace, Sumudu, Elzaki and Aboodh integral transforms are established together with their converses. The dualities are presented in forms of theorems and are later proved in a friendly manner. Some test functions are considered as examples and illustrated graphically with the help of Mathematica software. Thus, we therefore recommend the use of these new integral transforms in solving various differential and integral equations that are found in mathematical physics and engineering problems; having established their duality relations with some famous integral transforms.

References

- [1] Watugula G.K., Sumudu transform: a new integral transform to solve differential equations and control engineering problems, International Journal of Mathematical Education in Science and Technology, 24(1) (1993) 35-43
- [2] Khan Z.H., and Khan W.A., Natural transform-properties and applications, NUST Journal of Engineering Sciences, 1(2008) 127-133
- [3] Elzaki T.M., The new integral transform "Elzaki transform", Global Journal of Pure and Applied Mathematics, 7(1) (2011) 57-64
- [4] Aboodh K.S., The new integral transform "Aboodh transform" Global Journal of Pure and Applied Mathematics, 9(1) (2013) 35-43
- [5] Kashuri A. and Fundo A., "A new integral transform," Advances in Theoretical and Applied Mathematics, 8 (1) (2013) 27-43
- [6] Zafar Z., ZZ transform method, International Journal of Advanced Engineering and Global Technology, 04 (1) 2016
- [7] Elzaki T.M. and Elzaki S.M., On the connections between Laplace and Elzaki transforms, Advances in Theoretical and Applied Mathematics, 6(1) (2011) 1-10
- [8] Khalaf R.F. and Belgacem F.B.M., Extraction of the Laplace, Fourier, and Mellin Transforms from the Sumudu transform, AIP Proceedings, 1637, 1426 (2014)

- 239 [9] Shah K., Junaid M. and Ali N., Extraction of Laplace, Sumudu, Fourier and Mellin Transform
240 from the Natural transform, J. Appl. Environ. Biol. Sci., 5(9) (2015) 108-115
- 241 [10] Abdelilah Kamal H.S., The new integral transform "Kamal transform", Advances in Theoretical
242 and Applied Mathematics, 11(4) (2016) 451-458.
- 243 [11] Mohnad Mahgoub A.M, The new integral transform "Mahgoub transform", Advances in
244 Theoretical and Applied Mathematics, 11(4) (2016) 391-398.
- 245 [12] Debnath L. and Bhatta D., Integral Transforms and their Applications, second edition, C.R.C.
246 Press, London, (2007)
- 247 [13] Abdelilah Kamal H.S., The use of Kamal transform for solving partial differential equations,
248 Advances in Theoretical and Applied Mathematics, 12(1) (2017) 7-13
- 249 [14] Mohnad Mahgoub A.M, An application of new transform "Mahgoub transform" to partial
250 differential equations, Mathematical Theory and Modeling, 7(1) (2017) 2224-5804