

On the inverting of a general heptadiagonal matrix

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Abstract

In this paper, the author presents a new algorithm computing the inverse of any nonsingular heptadiagonal matrix. The computational cost of our algorithms is $O(n^2)$ operations in $\mathbb{C}$. The algorithms are suitable for implementation using computer algebra systems such as MAPLE, MATLAB and MATHEMATICA. Examples are given to illustrate the efficiency of the algorithms.

Keywords: Heptadiagonal matrices; LU factorization; Determinants; Computer algebra systems(CAS).

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1 Introduction

The $n \times n$ general heptadiagonal matrices take the form:

$$H = \begin{pmatrix} d_1 & e_1 & f_1 & g_1 & & & & \\ c_2 & d_2 & e_2 & f_2 & g_2 & & & \\ b_3 & c_3 & d_3 & e_3 & f_3 & g_3 & & 0 \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 & g_4 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & 0 & a_{n-3} & b_{n-3} & c_{n-3} & d_{n-3} & e_{n-3} & f_{n-3} & g_{n-3} \\ & & & a_{n-2} & b_{n-2} & c_{n-2} & d_{n-2} & e_{n-2} & f_{n-2} & \\ & & & & a_{n-1} & b_{n-1} & c_{n-1} & d_{n-1} & e_{n-1} & \\ & & & & & a_n & b_n & c_n & d_n & \end{pmatrix}, n > 4. \quad (1.1)$$

where $\{a_i\}_{4 \leq i \leq n}$, $\{b_i\}_{3 \leq i \leq n}$, $\{c_i\}_{2 \leq i \leq n}$, $\{d_i\}_{1 \leq i \leq n}$, $\{e_i\}_{1 \leq i \leq n}$, $\{f_i\}_{1 \leq i \leq n}$, and $\{g_i\}_{1 \leq i \leq n}$ are sequences of numbers such that $g_i \neq 0$, $g_{n-2} = g_{n-1} = g_n = 1$ and $f_{n-1} = f_n = e_n = 0$.

Heptadiagonal matrices frequently arise from boundary value problems. The heptadiagonal systems emanate in many numerical models, for example the traditional discretization of the implicit method to resolve partial differential equations on 3-D problems with regular grids. Also, these kinds of matrices appear in many areas of science and engineering[1-9]. So a good technique for computing the inverse of such matrices is required. To the best of our knowledge, the inversion of a general heptadiagonal matrix of the form (1.1) has not been considered.

In [10], Karawia described a reliable symbolic computational algorithm for inverting general cyclic heptadiagonal matrices by using parallel computing along with recursion. An explicit formula for the determinant of a heptadiagonal symmetric matrix is given in [6]. Many researchers studied special cases of heptadiagonal matrix. In [11], the authors presented a symbolic algorithm for finding the inverse of any general nonsingular tridiagonal matrix. A new efficient computational algorithm to find the inverse of a general tridiagonal

matrix is presented in [12] based on the Doolittle LU factorization. In [13], the authors introduced a computationally efficient algorithm to obtain the inverse of a tridiagonal matrix and a pentadiagonal matrix and they assumed few conditions to avoid failure in their own algorithm. The motivation of the current paper is to establish efficient algorithms for inverting heptadiagonal matrix. I generalized the algorithm[13] to find the inverse of a general invertible heptadiagonal matrix and we presented an efficient symbolic algorithm to find the inverse of such matrices. The development of a symbolic algorithm is considered in order to remove all cases where the numeric algorithm fails when at least one of g_i at lease equals to 0, $i = 1, 2, \dots, n - 3$. If, for every i such that $g_i = 0$, then I put $g_i = t$ for a small $t \neq 0$. The current work will be in the field of complex numbers $\mathbb{C}$.

The paper is organized as follows: In Section 2, the main result is presented. New numeric and symbolic algorithms are given in Section 3. In Section 4, illustrative examples are presented. Conclusions of the work are given in Section 5.

2 Main Result

In this section, we present recurrence formulas for the columns of the inverse of a heptadiagonal matrix H . When the matrix H is nonsingular, its inversion is computed as follows.

Let

$$H^{-1} = [S_{ij}]_{1 \leq i, j \leq n} = [\text{Col}_1, \text{Col}_2, \dots, \text{Col}_n]$$

where Col_k is the k th column of the inverse matrix H^{-1} .

By using the fact $HH^{-1} = I_n$, where I_n is the identity matrix, the first $(n - 3)$ columns can be obtained by relations

$$\left. \begin{aligned} \text{Col}_{n-3} &= \frac{1}{g_{n-3}}(E_n - d_n \text{Col}_n - e_{n-1} \text{Col}_{n-1} - f_{n-2} \text{Col}_{n-2}), \\ \text{Col}_{n-4} &= \frac{1}{g_{n-4}}(E_{n-1} - c_n \text{Col}_n - d_{n-1} \text{Col}_{n-1} - e_{n-2} \text{Col}_{n-2} - f_{n-3} \text{Col}_{n-3}), \\ \text{Col}_{n-5} &= \frac{1}{g_{n-5}}(E_{n-2} - b_n \text{Col}_n - c_{n-1} \text{Col}_{n-1} - d_{n-2} \text{Col}_{n-2} - e_{n-3} \text{Col}_{n-3} - f_{n-4} \text{Col}_{n-4}), \\ \text{Col}_j &= \frac{1}{g_j}(E_{j+3} - a_{j+6} \text{Col}_{j+6} - b_{j+5} \text{Col}_{j+5} - c_{j+4} \text{Col}_{j+4} - d_{j+3} \text{Col}_{j+3} - e_{j+2} \text{Col}_{j+2} - f_{j+1} \text{Col}_{j+1}), \end{aligned} \right\} \quad j = n - 6, n - 7, \dots, 1, \quad (2.1)$$

where E_k is the k th unit vector.

From (2.1), we note that if we know the last three columns Col_n , Col_{n-1} , and Col_{n-2} then we can recursively compute the remaining $(n - 3)$ columns Col_{n-3} , Col_{n-4} , ..., Col_1 .

At this point it is convenient to give recurrence formulas for computing Col_n , Col_{n-1} , and Col_{n-2} .

Consider the sequence of numbers $\{A_i\}_{1 \leq i \leq n+3}$, $\{B_i\}_{1 \leq i \leq n+3}$, and $\{C_i\}_{1 \leq i \leq n+3}$ characterized by a term recurrence relations

$$\left. \begin{aligned} A_1 &= 0, \\ A_2 &= 0, \\ A_3 &= 1, \\ d_1 A_1 + e_1 A_2 + f_1 A_3 + g_1 A_4 &= 0, \\ c_2 A_1 + d_2 A_2 + e_2 A_3 + f_2 A_4 + g_2 A_5 &= 0, \\ b_3 A_1 + c_3 A_2 + d_3 A_3 + e_3 A_4 + f_3 A_5 + g_3 A_6 &= 0, \\ a_i A_{i-3} + b_i A_{i-2} + c_i A_{i-1} + d_i A_i + e_i A_{i+1} + f_i A_{i+2} + g_i A_{i+3} &= 0, \quad i \geq 4, \end{aligned} \right\} \quad (2.2)$$

$$\left. \begin{aligned} B_1 &= 0, \\ B_2 &= 1, \\ B_3 &= 0, \\ d_1 B_1 + e_1 B_2 + f_1 B_3 + g_1 B_4 &= 0, \\ c_2 B_1 + d_2 B_2 + e_2 B_3 + f_2 B_4 + g_2 B_5 &= 0, \\ b_3 B_1 + c_3 B_2 + d_3 B_3 + e_3 B_4 + f_3 B_5 + g_3 B_6 &= 0, \\ a_i B_{i-3} + b_i B_{i-2} + c_i B_{i-1} + d_i B_i + e_i B_{i+1} + f_i B_{i+2} + g_i B_{i+3} &= 0, \quad i \geq 4, \end{aligned} \right\} \quad (2.3)$$

and

$$\left. \begin{aligned} C_1 &= 1, \\ C_2 &= 0, \\ C_3 &= 0, \\ d_1 C_1 + e_1 C_2 + f_1 C_3 + g_1 C_4 &= 0, \\ c_2 C_1 + d_2 C_2 + e_2 C_3 + f_2 C_4 + g_2 C_5 &= 0, \\ b_3 C_1 + c_3 C_2 + d_3 C_3 + e_3 C_4 + f_3 C_5 + g_3 C_6 &= 0, \\ a_i C_{i-3} + b_i C_{i-2} + c_i C_{i-1} + d_i C_i + e_i C_{i+1} + f_i C_{i+2} + g_i C_{i+3} &= 0, \quad i \geq 4. \end{aligned} \right\} \quad (2.4)$$

Now, we can give matrix forms for term recurrences (2.2), (2.3) and (2.4)

$$\mathbf{HA} = -A_{n+1}E_{n-2} - A_{n+2}E_{n-1} - A_{n+3}E_n, \quad (2.5)$$

$$\mathbf{HB} = -B_{n+1}E_{n-2} - B_{n+2}E_{n-1} - B_{n+3}E_n, \quad (2.6)$$

$$\mathbf{HC} = -C_{n+1}E_{n-2} - C_{n+2}E_{n-1} - C_{n+3}E_n, \quad (2.7)$$

where $\mathbf{A} = [A_1, A_2, \dots, A_n]^t$, $\mathbf{B} = [B_1, B_2, \dots, B_n]^t$, and $\mathbf{C} = [C_1, C_2, \dots, C_n]^t$.

Let's define the following determinants:

$$X_i = \begin{vmatrix} A_i & A_{n+2} & A_{n+3} \\ B_i & B_{n+2} & B_{n+3} \\ C_i & C_{n+2} & C_{n+3} \end{vmatrix}, \quad i = 1, 2, \dots, n+1, \quad (2.8)$$

$$Y_i = \begin{vmatrix} A_i & A_{n+1} & A_{n+3} \\ B_i & B_{n+1} & B_{n+3} \\ C_i & C_{n+1} & C_{n+3} \end{vmatrix}, \quad i = 1, 2, \dots, n+2, \quad (2.9)$$

$$Z_i = \begin{vmatrix} A_i & A_{n+1} & A_{n+2} \\ B_i & B_{n+1} & B_{n+2} \\ C_i & C_{n+1} & C_{n+2} \end{vmatrix}, \quad i = 1, 2, \dots, n+3. \quad (2.10)$$

By simple calculations, we have

$$\mathbf{HX} = -X_{n+1}E_{n-2}, \quad (2.11)$$

$$\mathbf{HY} = -Y_{n+2}E_{n-1}, \quad (2.12)$$

$$\mathbf{HZ} = -Z_{n+3}E_n, \quad (2.13)$$

where $\mathbf{X} = [X_1, X_2, \dots, X_n]^t$, $\mathbf{Y} = [Y_1, Y_2, \dots, Y_n]^t$, and $\mathbf{Z} = [Z_1, Z_2, \dots, Z_n]^t$.

Remark 2.1. $X_{n+1} = -Y_{n+2} = Z_{n+3}$.

Theorem 2.1. (generalization version of theorem 3.1 in [13]) $\mathbf{H}$ is invertible iff $X_{n+1} \neq 0$ and , for every $i, g_i \neq 0$.

Moreover

$$\text{Col}_n = \left[\frac{-Z_1}{\mathbf{X}_{n+1}}, \frac{-Z_2}{\mathbf{X}_{n+1}}, \dots, \frac{-Z_n}{\mathbf{X}_{n+1}} \right]^t, \quad (2.14)$$

$$\text{Col}_{n-1} = \left[\frac{Y_1}{\mathbf{X}_{n+1}}, \frac{Y_2}{\mathbf{X}_{n+1}}, \dots, \frac{Y_n}{\mathbf{X}_{n+1}} \right]^t, \quad (2.15)$$

$$\text{Col}_{n-2} = \left[\frac{-X_1}{\mathbf{X}_{n+1}}, \frac{-X_2}{\mathbf{X}_{n+1}}, \dots, \frac{-X_n}{\mathbf{X}_{n+1}} \right]^t. \quad (2.16)$$

Proof. Since $\det(\mathbf{H}) = -\left(\prod_{i=1}^{n-3} g_i\right) X_{n+1}$. So if $\mathbf{H}$ is invertible then $X_{n+1} \neq 0$ and if $X_{n+1} \neq 0$ and $g_i \neq 0$ for every i then $\det(\mathbf{H}) \neq 0$ and $\mathbf{H}$ is invertible. From (2.11), (2.12) and (2.13) we obtain Col_n , Col_{n-1} , and Col_{n-2} . The proof is completed. $\square$

Remark 2.2. If, for every i such that $g_i = 0$, we put $g_i = t$ for a small $t \neq 0$, and if $H|_{t=0}$ is invertible, then $X_{n+1}(t) \neq 0$. then $\mathbf{H}$ is invertible.

3 New numeric and symbolic algorithms for the inverse of heptadiagonal matrix

In this section, we formulate the result in the previous section . It is a numerical algorithm to compute the inverse of a general heptadiagonal matrix of the form (1.1) when it exists.

Algorithm 3.1 To find the inverse of heptadiagonal matrix (1.1).

let $f_{n-1} = f_n = e_n = 0$ and $g_{n-2} = g_{n-1} = g_n = 1$.

INPUT: Order of the matrix n and the components $a_i, b_j, c_k, d_l, e_l, f_l$, and g_l for $i = 4, 5, \dots, n$, $j = 3, 4, \dots, n$, $k = 2, 3, \dots, n$, and $l = 1, 2, \dots, n$,

OUTPUT: The inverse of heptadiagonal matrix $\mathbf{H}^{-1}$.

Step 1: Compute the sequence of numbers A_i, B_i , and C_i for $i = 1, 2, \dots, n+3$ using (2.2), (2.3) and (2.4) respectively.

Step 2: Compute X_i , $i = 1, 2, \dots, n+1$ using (2.8), Y_i , $i = 1, 2, \dots, n+2$ using (2.9) and Z_i , $i = 1, 2, \dots, n+3$ using (2.10).

Step 3: Compute the last three columns Col_n , Col_{n-1} , and Col_{n-2} using (2.14), (2.15), and (2.16) respectively.

Step 4: Compute the remaining $(n-3)$ -columns Col_j , $j = n-3, n-4, \dots, 1$ using (2.1).

Step 5: Set $\mathbf{H}^{-1} = [\text{Col}_1, \text{Col}_2, \dots, \text{Col}_n]$.

The numeric algorithm 3.1 will be referred to as **NINVHEPTA** algorithm in the sequel. The computational cost of **NINVHEPTA** algorithm is $11n^2 - 75n + 21$ operations.

As can be easily seen, it breaks down unless the conditions $g_i \neq 0$ are satisfied for all $i = 1, 2, \dots, n-3$. So the following symbolic algorithm is developed in order to remove the cases where the numeric algorithm fails.

Algorithm 3.2 To find the inverse of heptadiagonal matrix (1.1).
let $f_{n-1} = f_n = e_n = 0$ and $g_{n-2} = g_{n-1} = g_n$.

INPUT: Order of the matrix n and the components $a_i, b_j, c_k, d_l, e_l, f_l$, and g_l for $i = 4, 5, \dots, n$,
 $j = 3, 4, \dots, n$, $k = 2, 3, \dots, n$, and $l = 1, 2, \dots, n$,

OUTPUT: The inverse of heptadiagonal matrix $\mathbf{H}^{-1}$.

Step 1: If $g_i = 0$ for any $i = 1, 2, \dots, n-3$ set $g_i = t$ (t is just a symbolic name).

Step 2: Compute the sequence of numbers A_i, B_i , and C_i for $i = 1, 2, \dots, n+3$ using (2.2), (2.3) and (2.4) respectively.

Step 3: Compute X_i , $i = 1, 2, \dots, n+1$ using (2.8), Y_i , $i = 1, 2, \dots, n+2$ using (2.9) and Z_i , $i = 1, 2, \dots, n+3$ using (2.10).

Step 4: Compute the last three columns Col_n , Col_{n-1} , and Col_{n-2} using (2.14), (2.15), and (2.16) respectively.

Step 5: Compute the remaining $(n-3)$ -columns Col_j , $j = n-3, n-4, \dots, 1$ using (2.1).

Step 6: Substitute the actual value $t = 0$ in all expressions to obtain the elements of columns Col_j ,
 $j = 1, 2, \dots, n$.

Step 7: Set $\mathbf{H}^{-1} = [\text{Col}_1, \text{Col}_2, \dots, \text{Col}_n]$.

The symbolic algorithm 3.2 will be referred to as **SINVHEPTA** algorithm in the sequel. The **SINVHEPTA** algorithm use $O(n^2)$ elementary operations. Based on **SINVHEPTA** algorithm, a MAPLE procedure [14] for inverting a general nonsingular heptadiagonal matrix $\mathbf{H}$ is listed as an Appendix.

4 ILLUSTRATIVE EXAMPLES

In this section we give many examples for the sake of illustration.

Example 4.1. (Case I: $g_i \neq 0$ for all i)

Find the inverse of following 10×10 heptadiagonal matrix

$$H_1 = \begin{bmatrix} 2 & 1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & -3 & 2 & 7 & 2 & 0 & 0 & 0 & 0 \\ 6 & 1 & 3 & 2 & 3 & -1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 2 & -3 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 4 & 4 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 3 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & 1 & 2 & 1 & 11 & 3 \\ 0 & 0 & 0 & 0 & 0 & 4 & -3 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7 & 1 & 1 & 2 \end{bmatrix} \quad (4.1)$$

Solution: By applying the **SINVHEPTA** algorithm, it yields

• **Step 1:** $A = [0, 0, 1, 4, -\frac{9}{2}, \frac{53}{4}, \frac{21}{4}, \frac{83}{4}, -\frac{93}{2}, \frac{663}{4}, -\frac{67}{4}, -198, -269]$,

$B = [0, 1, 0, 1, -\frac{3}{2}, \frac{13}{4}, \frac{19}{12}, \frac{41}{12}, -\frac{47}{6}, \frac{341}{12}, -\frac{53}{12}, -\frac{107}{3}, -\frac{124}{3}]$, and

$C = [1, 0, 0, 2, -\frac{9}{2}, \frac{53}{4}, \frac{67}{12}, \frac{269}{12}, -\frac{221}{6}, \frac{1781}{12}, -\frac{881}{12}, -\frac{578}{3}, -\frac{730}{3}]$.

• **Step 2:** $X = [\frac{4231}{3}, -\frac{10942}{3}, -\frac{2146}{3}, -3688, \frac{4687}{2}, -\frac{31741}{12}, -\frac{19873}{12}, \frac{51721}{12}, \frac{19773}{2}, -\frac{154735}{12}, -\frac{905413}{12}]$,

$$Y = \left[\frac{1983}{4}, -\frac{62693}{4}, \frac{11759}{6}, -\frac{82109}{12}, \frac{49839}{4}, -\frac{220819}{12}, -\frac{141107}{12}, -\frac{5312}{3}, \frac{160577}{12}, -\frac{563539}{12}, 0, \frac{905413}{12} \right], \text{ and}$$

$$Z = \left[\frac{3325}{12}, -\frac{33928}{3}, \frac{21211}{12}, -\frac{22109}{6}, 7763, -\frac{19327}{2}, -\frac{84955}{12}, \frac{50981}{12}, -\frac{15235}{4}, \frac{76363}{6}, 0, 0, -\frac{905413}{12} \right].$$

• **Step 3:** $\text{Col}_{10} = \left[\frac{3325}{905413}, -\frac{135712}{905413}, \frac{21211}{905413}, -\frac{44218}{905413}, \frac{93156}{905413}, -\frac{115962}{905413}, -\frac{84955}{905413}, \frac{50981}{905413}, -\frac{45705}{905413}, \frac{152726}{905413} \right]^t$,

$\text{Col}_9 = \left[-\frac{5949}{905413}, \frac{188079}{905413}, -\frac{23518}{905413}, \frac{82109}{905413}, -\frac{149517}{905413}, \frac{220819}{905413}, \frac{141107}{905413}, \frac{21248}{905413}, -\frac{160577}{905413}, \frac{563539}{905413} \right]^t$, and

$\text{Col}_8 = \left[\frac{16924}{905413}, -\frac{43768}{905413}, -\frac{8584}{905413}, -\frac{44256}{905413}, \frac{28122}{905413}, -\frac{31741}{905413}, -\frac{19873}{905413}, \frac{51721}{905413}, \frac{118638}{905413}, -\frac{154735}{905413} \right]^t$.

• **Step 4:** $\text{Col}_7 = \left[-\frac{51473}{905413}, \frac{214649}{905413}, \frac{6848}{905413}, \frac{139095}{905413}, -\frac{121161}{905413}, \frac{106328}{905413}, \frac{88422}{905413}, -\frac{278373}{905413}, -\frac{103927}{905413}, \frac{500627}{905413} \right]^t$,

$\text{Col}_6 = \left[-\frac{80594}{905413}, -\frac{217}{905413}, \frac{83035}{905413}, \frac{170735}{905413}, -\frac{10659}{905413}, \frac{31638}{905413}, -\frac{14393}{905413}, -\frac{84414}{905413}, \frac{14531}{905413}, -\frac{15434}{905413} \right]^t$,

$\text{Col}_5 = \left[-\frac{82176}{905413}, \frac{447486}{905413}, -\frac{28082}{905413}, \frac{170806}{905413}, -\frac{175068}{905413}, -\frac{6589}{905413}, \frac{102273}{905413}, \frac{9510}{905413}, -\frac{78091}{905413}, \frac{392246}{905413} \right]^t$,

$\text{Col}_4 = \left[\frac{205297}{905413}, -\frac{910556}{905413}, \frac{6935}{905413}, -\frac{472222}{905413}, \frac{410790}{905413}, -\frac{147233}{905413}, \frac{42741}{905413}, \frac{427692}{905413}, -\frac{147953}{905413}, \frac{9724}{905413} \right]^t$,

$\text{Col}_3 = \left[-\frac{2619}{905413}, -\frac{30890}{905413}, -\frac{25421}{905413}, -\frac{137812}{905413}, \frac{172515}{905413}, -\frac{19216}{905413}, -\frac{46090}{905413}, \frac{62775}{905413}, \frac{11493}{905413}, -\frac{198449}{905413} \right]^t$,

,

$\text{Col}_2 = \left[-\frac{29328}{905413}, \frac{877900}{905413}, -\frac{53389}{905413}, \frac{605688}{905413}, -\frac{491917}{905413}, \frac{172702}{905413}, -\frac{34896}{905413}, -\frac{501141}{905413}, \frac{156703}{905413}, \frac{50083}{905413} \right]^t$,

and

$\text{Col}_1 = \left[-\frac{88555}{905413}, \frac{552363}{905413}, \frac{125378}{905413}, -\frac{28648}{905413}, -\frac{88835}{905413}, \frac{19552}{905413}, -\frac{17938}{905413}, -\frac{61611}{905413}, \frac{46355}{905413}, -\frac{55155}{905413} \right]^t$.

• **Step 5:** $H_1^{-1} =$

$$\begin{bmatrix} -\frac{88555}{905413} & -\frac{29328}{905413} & -\frac{2619}{905413} & \frac{205297}{905413} & -\frac{82176}{905413} & -\frac{80594}{905413} & -\frac{51473}{905413} & \frac{16924}{905413} & -\frac{5949}{905413} & \frac{3325}{905413} \\ \frac{552363}{905413} & \frac{877900}{905413} & -\frac{30890}{905413} & -\frac{910556}{905413} & \frac{447486}{905413} & -\frac{217}{905413} & \frac{214649}{905413} & -\frac{43768}{905413} & \frac{188079}{905413} & -\frac{135712}{905413} \\ \frac{125378}{905413} & -\frac{53389}{905413} & -\frac{25421}{905413} & \frac{6935}{905413} & -\frac{28082}{905413} & \frac{83035}{905413} & \frac{6848}{905413} & -\frac{8584}{905413} & -\frac{23518}{905413} & \frac{21211}{905413} \\ -\frac{28648}{905413} & \frac{605688}{905413} & -\frac{137812}{905413} & -\frac{472222}{905413} & \frac{170806}{905413} & \frac{170735}{905413} & \frac{139095}{905413} & -\frac{44256}{905413} & \frac{82109}{905413} & -\frac{44218}{905413} \\ -\frac{88835}{905413} & -\frac{491917}{905413} & \frac{172515}{905413} & \frac{410790}{905413} & -\frac{175068}{905413} & -\frac{10659}{905413} & -\frac{121161}{905413} & \frac{28122}{905413} & -\frac{149517}{905413} & \frac{93156}{905413} \\ \frac{19552}{905413} & \frac{172702}{905413} & -\frac{19216}{905413} & -\frac{147233}{905413} & -\frac{6589}{905413} & \frac{31638}{905413} & \frac{106328}{905413} & -\frac{31741}{905413} & \frac{220819}{905413} & -\frac{115962}{905413} \\ -\frac{17938}{905413} & -\frac{34896}{905413} & -\frac{46090}{905413} & \frac{42741}{905413} & \frac{102273}{905413} & -\frac{14393}{905413} & \frac{88422}{905413} & -\frac{19873}{905413} & \frac{141107}{905413} & -\frac{84955}{905413} \\ -\frac{61611}{905413} & -\frac{501141}{905413} & \frac{62775}{905413} & \frac{427692}{905413} & \frac{9510}{905413} & -\frac{84414}{905413} & -\frac{278373}{905413} & \frac{51721}{905413} & \frac{21248}{905413} & \frac{50981}{905413} \\ \frac{46355}{905413} & \frac{156703}{905413} & \frac{11493}{905413} & -\frac{147953}{905413} & -\frac{78091}{905413} & \frac{14531}{905413} & -\frac{103927}{905413} & \frac{118638}{905413} & -\frac{160577}{905413} & -\frac{45705}{905413} \\ -\frac{55155}{905413} & \frac{50083}{905413} & -\frac{198449}{905413} & \frac{9724}{905413} & \frac{392246}{905413} & -\frac{15434}{905413} & \frac{500627}{905413} & -\frac{154735}{905413} & \frac{563539}{905413} & \frac{152726}{905413} \end{bmatrix}$$

Example 4.2.(Case II: $g_i = 0$ for at least one of i)

Find the inverse of following 5×5 heptadiagonal matrix

$$H_2 = \begin{bmatrix} 2 & 3 & 4 & 1 & 0 \\ -1 & 1 & -2 & 3 & 0 \\ 3 & 5 & 1 & -1 & 2 \\ 4 & -1 & 3 & 2 & 6 \\ 0 & 2 & 1 & 4 & -3 \end{bmatrix} \quad (4.2)$$

Solution:

- i- By applying the **NINVHEPTA** algorithm, it breaks down since $g_2 = 0$.
- ii- By applying the **SINVHEPTA** algorithm, it yields

- **Step 1:** $A = [0, 0, 1, -4, 14x^{-1}, -\frac{5x+28}{x}, \frac{5x-84}{x}, 3\frac{5x+14}{x}]$,
 $B = [0, 1, 0, -3, 8x^{-1}, -8\frac{x+2}{x}, \frac{7x-48}{x}, 2\frac{5x+12}{x}]$, and
 $C = [1, 0, 0, -2, 7x^{-1}, -\frac{5x+14}{x}, -42x^{-1}, \frac{8x+21}{x}]$.
- **Step 2:** $X = [\frac{55x+294}{x}, \frac{63+40x}{x}, -\frac{183+56x}{x}, -3\frac{2x+15}{x}, -79x^{-1}, -\frac{315x+901}{x}]$,
 $Y = [-2\frac{35x+88}{x}, 7\frac{5x+13}{x}, 2\frac{7x+18}{x}, -\frac{21x+65}{x}, -14x^{-1}, 0, \frac{315x+901}{x}]$, and
 $Z = [-\frac{5x-548}{x}, 5\frac{5x-28}{x}, -\frac{35x+194}{x}, 25\frac{3x+4}{x}, -325x^{-1}, 0, 0, -\frac{315x+901}{x}]$.
- **Step 3:** $Col_5 = [-\frac{5x-548}{315x+901}, 5\frac{5x-28}{315x+901}, -\frac{35x+194}{315x+901}, 25\frac{3x+4}{315x+901}, \frac{-325}{315x+901}]^t$,
 $Col_4 = [2\frac{35x+88}{315x+901}, -7\frac{5x+13}{315x+901}, -2\frac{7x+18}{315x+901}, \frac{21x+65}{315x+901}, \frac{14}{315x+901}]^t$, and
 $Col_3 = [\frac{55x+294}{315x+901}, \frac{63+40x}{315x+901}, -\frac{183+56x}{315x+901}, -3\frac{2x+15}{315x+901}, \frac{-79}{315x+901}]^t$.
- **Step 4:** $Col_2 = [\frac{-545}{315x+901}, \frac{205}{315x+901}, \frac{91}{315x+901}, \frac{111}{315x+901}, \frac{315}{315x+901}]^t$, and
 $Col_1 = [-5\frac{13x+123}{315x+901}, 10\frac{x+19}{315x+901}, 56\frac{2x+7}{315x+901}, -\frac{33x+7}{315x+901}, \frac{248}{315x+901}]^t$.
- **Step 5:** $H_2^{-1} = \begin{bmatrix} -5\frac{13x+123}{315x+901} & \frac{-545}{315x+901} & \frac{55x+294}{315x+901} & 2\frac{35x+88}{315x+901} & -\frac{5x-548}{315x+901} \\ 10\frac{x+19}{315x+901} & \frac{205}{315x+901} & \frac{63+40x}{315x+901} & -7\frac{5x+13}{315x+901} & 5\frac{5x-28}{315x+901} \\ 56\frac{2x+7}{315x+901} & \frac{91}{315x+901} & -\frac{183+56x}{315x+901} & -2\frac{7x+18}{315x+901} & -\frac{35x+194}{315x+901} \\ -\frac{33x+7}{315x+901} & \frac{111}{315x+901} & -3\frac{2x+15}{315x+901} & \frac{21x+65}{315x+901} & 25\frac{3x+4}{315x+901} \\ \frac{248}{315x+901} & \frac{315}{315x+901} & \frac{-79}{315x+901} & \frac{14}{315x+901} & \frac{-325}{315x+901} \end{bmatrix}_{x=0}$
- **Step 6:** $H_2^{-1} = \begin{bmatrix} -\frac{615}{901} & -\frac{545}{901} & \frac{294}{901} & \frac{176}{901} & \frac{548}{901} \\ \frac{190}{901} & \frac{205}{901} & \frac{63}{901} & -\frac{91}{901} & -\frac{140}{901} \\ \frac{392}{901} & \frac{91}{901} & -\frac{183}{901} & -\frac{36}{901} & -\frac{194}{901} \\ -\frac{7}{901} & \frac{111}{901} & -\frac{45}{901} & \frac{65}{901} & \frac{100}{901} \\ \frac{248}{901} & \frac{315}{901} & -\frac{79}{901} & \frac{14}{901} & -\frac{325}{901} \end{bmatrix}$.

Example 4.3. We consider the following $n \times n$ heptadiagonal matrix in order to demonstrate the efficiency of **SINVHEPTA** algorithm.

$$H = \begin{pmatrix} -2 & -1 & 2 & 1 & & & & \\ 3 & -2 & -1 & 2 & 1 & & & \\ 1 & 3 & -2 & -1 & 2 & 1 & 0 & \\ 2 & 1 & 3 & -2 & -1 & 2 & 1 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & 2 & 1 & 3 & -2 & -1 & 2 & 1 \\ 0 & & 2 & 1 & 3 & -2 & -1 & 2 \\ & & & 2 & 1 & 3 & -2 & -1 \\ & & & & 2 & 1 & 3 & -2 \end{pmatrix}. \quad (4.3)$$

In Table 1. we give a comparison of the mean time between **SINVHEPTA**, **CHEPTA**[10] (symbolic algorithm to find the inverse of Cyclic Heptadiagonal matrix) algorithms and **MatrixInverse** function in Maple 13.0 for different orders, over 100 trials. It was tested in an Intel(R) Core(TM) i7-4700MQ CPU@2.40GHz 2.40 GHz.

Table 1.

The mean values of the time elapsed(in seconds) over 100 trials, of the proposed algorithm, **CHEPTA**[10] algorithm and **MatrixInverse** function in Maple 13.0.

Algorithms	n	10	20	50	100	150	200	500
SINVHEPTA	Mean time(s)	0.00563	0.01326	0.08171	0.44915	1.29597	2.95462	48.74440
CHEPTA [10]	Mean time(s)	0.00233	0.01189	0.06819	0.43924	1.27376	2.95963	49.35749
MatrixInverse	Mean time(s)	0.00375	0.01512	0.17214	1.31768	4.41329	10.82438	198.68109

Example 4.4. The performance of the algorithm **NINVHEPTA** is recorded in table 2, which gives us the mean values of time elapsed(in seconds) and relative error $\|H \cdot H^{-1} - I_n\|_F / \|I_n\|_F$, over 100 trials, for the computation of the inverse of a heptadiagonal matrix **H** with random integers entries h_{ij} for $|i - j| \leq 3$ and with nonzero entries on its third superdiagonal, and $h_{ij} = 0$ otherwise.

Table 2.

The mean values of the time elapsed(in seconds)and the mean of the relative error(MRE) $\|H \cdot H^{-1} - I_n\|_F / \|I_n\|_F$, over 100 trials, of the proposed algorithm, **NINVHEPTA** algorithm to obtain more accurate information about the full inverse of a random heptadiagonal matrix **H**.

n	10	20	50	100	150	200	500
Mean time(s)	$5.60e - 3$	$1.36e - 2$	$7.68e - 2$	$3.37e - 1$	$9.24e - 1$	2.05	5.74
MRE	$4.12e - 14$	$7.42e - 13$	$1.62e - 12$	$8.09e - 12$	$5.38e - 11$	$9.66e - 11$	$3.54e - 10$

5 CONCLUSIONS

In this work new numeric and symbolic algorithms have been developed for finding the inverse of any nonsingular heptadiagonal matrix. The symbolic algorithm removes the cases where the numeric algorithms fail when at least one of $g_i = 0$, $i = 1, 2, \dots, n - 3$. It has the smallest mean time the methods in literature for large sizes

Appendix. A MAPLE procedure for inverting a general nonsingular heptadiagonal matrix

```

>restart:with(LinearAlgebra):
hepta_inv := proc(n::posint,a::vector,b::vector,c::vector,d::vector,e::vector,f::vector,g::vector)
local i,j;
global A,B,C,X,Y,Z,S,Hinv,DetH,DetH1;
A := array(1 .. n+3): B := array(1 .. n+3): C := array(1 .. n+3): X := array(1 .. n+1):
Y := array(1 .. n+2): Z := array(1 .. n+3):S:=array(1..n,1..n,sparse):
for i from 1 to n-3 do
    if g[i] = 0 then g[i] := x end if
end do:
A[1]:=0:A[2]:=0:A[3]:=1:B[1]:=0:B[2]:=1:B[3]:=0:C[1]:=1:C[2]:=0:C[3]:=0:
A[4]:=-simplify(f[1]/g[1]):A[5]:=-simplify((e[2]+A[4]*f[2])/g[2]):A[6]:=-simplify((d[3]+e[3]*A[4]+A[5]*f[3])/g[3]):
B[4]:=-simplify(e[1]/g[1]):B[5]:=-simplify((d[2]+B[4]*f[2])/g[2]):B[6]:=-simplify((c[3]+e[3]*B[4]+B[5]*f[3])/g[3]):
C[4]:=-simplify(d[1]/g[1]):C[5]:=-simplify((c[2]+C[4]*f[2])/g[2]):C[6]:=-simplify((b[3]+e[3]*C[4]+C[5]*f[3])/g[3]):
for i from 4 to n do
    A[i+3] := -simplify((a[i]*A[i-3]+b[i]*A[i-2]+c[i]*A[i-1]+d[i]*A[i]+e[i]*A[i+1]+f[i]*A[i+2])/g[i]):
    B[i+3] := -simplify((a[i]*B[i-3]+b[i]*B[i-2]+c[i]*B[i-1]+d[i]*B[i]+e[i]*B[i+1]+f[i]*B[i+2])/g[i]):
    C[i+3] := -simplify((a[i]*C[i-3]+b[i]*C[i-2]+c[i]*C[i-1]+d[i]*C[i]+e[i]*C[i+1]+f[i]*C[i+2])/g[i]):
end do:
i := 'i':
for i from 1 to n+1 do
    X[i]:=simplify(Determinant(Matrix([A[n+3], A[n+2], A[i]], [B[n+3], B[n+2], B[i]], [C[n+3], C[n+2], C[i]]))):
    Y[i]:=simplify(Determinant(Matrix([A[n+3], A[n+1], A[i]], [B[n+3], B[n+1], B[i]], [C[n+3], C[n+1], C[i]]))):
    Z[i]:=simplify(Determinant(Matrix([A[n+2], A[n+1], A[i]], [B[n+2], B[n+1], B[i]], [C[n+2], C[n+1], C[i]]))):

```

```

end do:
Y[n+2]:=simplify(Determinant(Matrix([[A[n+3], A[n+1], A[n+2]], [B[n+3], B[n+1], B[n+2]], [C[n+3], C[n+1],
C[n+2]]]]))):
Z[n+2]:=simplify(Determinant(Matrix([[A[n+2], A[n+1], A[n+2]], [B[n+2], B[n], B[n+2]], [C[n+2], C[n],
C[n+2]]]]))):
Z[n+3]:=simplify(Determinant(Matrix([[A[n+2], A[n+1], A[n+3]], [B[n+2], B[n+1], B[n+3]], [C[n+2], C[n+1],
C[n+3]]]]))):
DetH:=simplify(product(g[k],k=1..n-3)*X[n+1]):
DetH1:=eval(DetH,x=0):
if DetH=0 then
    printf("%s\n", "H is a Singular Matrix");
else
    i := 'i':
    for i from 1 to n do
        S[i,n]:=-Z[i]/Z[n+3]:
        S[i,n-1]:=-Y[i]/Y[n+2]:
        S[i,n-2]:=-X[i]/X[n+1]:
    end do:
    i := 'i':
    for i to n do
        S[i, n-3] := -simplify((d[n]*S[i, n]+e[n-1]*S[i, n-1]+f[n-2]*S[i, n-2])/g[n-3]);
        if i = n then
            S[i, n-3] := simplify(1/g[n-3]+S[i, n-3]):
        end if
    end do:
    if n=5 then
        i := 'i':
        for i to n do
            S[i, n-4] := -simplify((c[n]*S[i,n]+d[n-1]*S[i, n-1]+e[n-2]*S[i, n-2]+
f[n-3]*S[i, n-3])/g[n-4]);
            if i = n-1 then
                S[i, n-4] := simplify(1/g[n-4]+S[i, n-4])
            end if
        end do
    elif n=6 then
        i := 'i':
        for i to n do
            S[i, n-5] := -simplify((b[n]*S[i,n]+c[n-1]*S[i,n-1]+d[n-2]*S[i, n-2]+e[n-3]*S[i, n-3]+
f[n-3]f[n-4]*S[i, n-4])/g[n-5]);
            if i = n-2 then
                S[i, n-5] := simplify(1/g[n-5]+S[i, n-5]):
            end if
        end do:
    else
        i := 'i':
        for i to n do
            S[i, n-4] := -simplify((c[n]*S[i,n]+d[n-1]*S[i, n-1]+e[n-2]*S[i, n-2]+f[n-3]*S[i, n-3])/g[n-4]);
            if i = n-1 then
                S[i, n-4] := simplify(1/g[n-4]+S[i, n-4])
            end if
        end do:
        i := 'i':
        for i to n do
            S[i, n-5] := -simplify((b[n]*S[i,n]+c[n-1]*S[i,n-1]+d[n-2]*S[i, n-2]+e[n-3]*S[i, n-3]+f[n-4]*
S[i, n-4])/g [n-5]);
            if i = n-2 then
                S[i, n-5] := simplify(1/g[n-5]+S[i, n-5])
            end if
        end do:
    end if
end if

```

```

        end if
    end do:
    i:='i':
    for j from n-6 by -1 to 1 do
        for i to n do
            S[i, j] := -simplify((a[j+6]*S[i,j+6]+b[j+5]*S[i,j+5]+c[j+4]*S[i,j+4]+d[j+3]*S[i,j+3]+
                                f[n-3]e[j+2]*S[i,j+2]+f[j+1]*S[i, j+1])/g[j]);
            if i = j+3 then
                S[i,j] := simplify(1 / g[j] + S[i,j])
            fi:
        od:
    od:
fi:
Hinv:=evalm(S):
eval(evalm(Hinv),x=0):
fi:
end proc:

```

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