

ISAC'S CONES

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Abstract. This is a very short research work representing an homage to the regretted Professor George Isac, Department of Mathematics and Computer Science, Royal Military College of Canada, P.O. 17000, Kingston, Ontario, Canada, K7K 7B4. Professor Isac introduced the notion of “nuclear cone” in 1981, published in 1983 and called later as “supernormal cone” since it appears stronger than the usual concept of “normal cone”. For the first time, we named these convex cones as “Isac’s Cones” in 2009, after the acceptance on professor Isac’s part. This study is devoted to Isac’s cones, including significant examples, comments and several pertinent references, with the remark that this notion has its real place in Hausdorff locally convex spaces not in the normed linear spaces, having strong implications and applications in the efficiency and optimization. Isac’s cones represent the largest class of convex cones discovered till now in separated locally convex spaces ensuring the existence and important properties for the efficient points under completeness instead of compactness.

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1. Introduction.

This is a very short research work representing an homage to Professor George Isac, Department of Mathematics and Computer Science, Royal Military College of Canada, P.O. 17000, Kingston, Ontario, Canada, K7K 7B4. Professor Isac introduced the notion of “nuclear cone” in [4], published in [5] and called later on “supernormal cone” since it appears stronger than the usual concept of “normal cone”. For the first time, we named

these convex cones as “Isac’s Cones” in [25], after the acceptance on professor Isac’s part. This study is devoted to Isac’s cones, including significant examples, comments and several pertinent references, with the remark that this notion is more interesting in Hausdorff locally convex spaces as in normed linear spaces, having strong implications and applications in efficiency and optimization. A generalization of Isac’s cones in general vector spaces was given by us in [26].

2. Isac's (nuclear or supernormal) cones

Throughout the research works devoted to nuclear (supernormal) cones professor Isac considered any locally convex space in the sense of the next definition.

Definition 1. (Treves, F., 1967). *A locally convex space is any couple $(X, Spec(X))$ which is composed of a real linear space X and a family $Spec(X)$ of seminorms on X such that:*

- (i) $\chi p \in Spec(X), \forall \chi \in R_+, p \in Spec(X)$;
- (ii) if $p \in Spec(X)$ and q is an arbitrary seminorm on X such that $q \leq p$, then $q \in Spec(X)$;
- (iii) $\sup(p_1, p_2) \in Spec(X), \forall p_1, p_2 \in Spec(X)$ where

$$\sup(p_1, p_2)(x) = \sup(p_1(x), p_2(x)), \forall x \in X.$$

It is well known [28] that whenever such a family as this $Spec(X)$ is given on a real vector space X , there exists a locally convex topology τ on X such that (X, τ) is a topological linear space and a seminorm p on X is τ -continuous iff $p \in Spec(X)$. A non-empty subset B of $Spec(X)$ is a *base* for it if for every $p \in Spec(X)$ there exist $\chi > 0$ and $q \in B$ such that $p \leq \chi q$ and (X, τ) is a Hausdorff locally convex space iff $Spec(X)$ has a base B , named *Hausdorff base*, with the property that $\{x \in X : p(x) = 0, \forall p \in B\} = \{\theta\}$ where θ is the null vector in X . In this research paper we will suppose that the space (X, τ) sometimes denoted by X is a Hausdorff locally convex space. Every non-empty subset K of X satisfying the following properties: $K + K \subseteq K$ and $\chi K \subseteq K, \forall \chi \in R_+$ is named *convex cone*. If, in addition, $K \cap K = \{\theta\}$, then K is called *pointed*. Clearly, any pointed convex cone K in X generates an ordering on X defined by $x \leq y (x, y \in X)$ iff $y - x \in K$. If X^* is the dual of X , then the *dual cone* of K is defined by $K^* = \{x^* \in X^* : x^*(x) \geq 0, \forall x \in K\}$ and its corresponding *polar* is $K^0 = -K$. We recall that a pointed convex cone $K \subset (X, Spec(X))$ is *normal* with respect to the topology defined by $Spec(X)$ if it fulfils one of the next equivalent assertions:

- (i) *there exists a base Ω of neighborhoods for the origin θ in X such that $V = (V + K) \cap (V - K), \forall V \in \Omega$;*
- (ii) *there exists a base B of $Spec(X)$ with $p(x) \leq p(y), \forall x, y \in K, x \leq y, \forall p \in B$;*

(iii) for any two nets $\{x_i\}_{i \in I}, \{y_i\}_{i \in I} \subset K$ with $\theta \leq x_i \leq y_i, \forall i \in I$ and $\lim y_i = \theta$ it follows that $\lim x_i = \theta$. In particular, a convex cone K is normal in a normed linear space $(E, \|\cdot\|)$ iff there exists $t \in (0, \infty)$ such that $x, y \in E$ and $y - x \in K$ implies that $\|x\| \leq t \|y\|$.

It is well known that the concept of *normal cone* is the most important notion in the theory and applications of convex cones in topological ordered vector spaces. Thus, for example, for every separated locally convex space $(X, \text{Spec}(X))$ and any closed normal cone

$K \subset (X, \text{Spec}(X))$ we have $X^* = K^* - K^*$ (see, for instance, [2], [3]). Each pointed convex cone $K \subset (X, \text{Spec}(X))$ for which there exists a non-empty, convex bounded set

$T \subseteq X$ such that $0 \notin \overline{T}$ and $K = \bigcup_{\lambda \geq 0} \lambda T$ is called *well-based*. A cone $K \subset (X, \text{Spec}(X))$ is *well-based* iff there exists a base $B = \{p_i\}_{i \in I}$ of $\text{Spec}(X)$ and a linear continuous functional $f \in K^*$ such that for every $p_i \in B$ there exists $c_i > 0$ with $c_i p_i(x) \leq f(x), \forall x \in K$ ([2], [3]).

Clearly, every well-based cone is a normal cone, but, in general, the converse is not true, as we can see in the examples below, starting from the next basic notion.

Definition 2. (Isac, G., 1981, 1983) In a Hausdorff locally convex space $(X, \text{Spec}(X))$ a pointed convex cone $K \subset X$ is *nuclear (supernormal)* with respect to the topology induced by $\text{Spec}(X)$ if there exists a base $B = \{p_i\}_{i \in I}$ of $\text{Spec}(X)$ such that for every $p_i \in B$ there exists $f_i \in X^*$ with $p_i(x) \leq f_i(x), \forall x \in K$.

Remark 1. For the first time, we called any such as this cone “Isac’s cone” in [25], taking into account that the above definition of locally convex spaces is equivalent with the following: Let X be a real or complex linear space and $P = \{p_\alpha : \alpha \in A\}$ a family of seminorms defined on X . For every $x \in X, \varepsilon > 0$ and $n \in \mathbb{N}^*$ let

$V(x; p_1, p_2, \dots, p_n; \varepsilon) = \{y \in X : p_\alpha(y - x) < \varepsilon, \forall \alpha = \overline{1, n}\}$, then the family

$\mathcal{V}_0(x) = \{V(x; p_1, p_2, \dots, p_n; \varepsilon) : n \in \mathbb{N}^*, p_\alpha \in P, \alpha = \overline{1, n}, \varepsilon > 0\}$

has the properties :

(V₁) $x \in V, \forall V \in \mathcal{V}_0(x)$;

(V₂) $\forall V_1, V_2 \in \mathcal{V}_0(x), \exists V_3 \in \mathcal{V}_0(x) : V_3 \subseteq V_1 \cap V_2$;

(V₃) $\forall V \in \mathcal{V}_0(x), \exists U \in \mathcal{V}_0(x), U \subseteq V$ such that $\forall y \in U, \exists W \in \mathcal{V}_0(y)$ with $W \subseteq V$.

Therefore, $\mathcal{V}_0(x)$ is a base of neighbourhoods for x and taking $\mathcal{V}(x) = \{V \subseteq X : \exists U \in \mathcal{V}_0(x) \text{ with } U \subseteq V\}$, the set

$\tau = \{D \subseteq X : D \in \mathcal{V}(x), \forall x \in D\} \cup \{\emptyset\}$ is the locally convex topology generated by the family P . Obviously, the usual operations which induce the structure of linear space on X are continuous with respect to this topology. The corresponding topological space (X, τ) is a Hausdorff locally convex space iff the family P is sufficient, that is, $\forall$

$x_0 \in X \setminus \{\theta\}, \exists p_\alpha \in P$ with $p_\alpha(x_0) \neq 0$. In this context, a convex cone $K \subset X$ is an Isac's cone iff $\forall p_\alpha \in P, \exists f_\alpha \in X^* : p_\alpha(x) \leq f_\alpha(x), \forall x \in K$. The best special, refined and non-trivial Isac's cones class associated to normal cones in Hausdorff locally convex spaces was introduced and studied in [11] as *the full nuclear cones* family defined as follows: if $(X, Spec(X))$ is an arbitrary locally convex space $B \subset Spec(X)$ is a Hausdorff base of $Spec(X)$ and $K \subset X$ is a normal cone, then for any mapping $\varphi : B \rightarrow K^* \setminus \{0\}$ one says that the set

$K_\varphi = \{x \in X : p(x) \leq \varphi(p)(x), \forall p \in B\}$ is a *full nuclear cone* associated to K whenever $K_\varphi \neq \{\theta\}$. Taking into account that in a real normed linear space $(E, \|\cdot\|)$ a non-empty set $T \subset E$ is called a *Bishop-Phelps cone* if there exists y^* in the usual dual space E^* of E and $\alpha \in (0, 1]$ such that $T = \{y \in E : \alpha \|y\| \leq y^*(y)\}$ and the applications of such as these cones in Nonlinear Analysis and in Pareto type optimization for vector-valued mappings, we conclude that, for the Hausdorff locally convex spaces, *the full nuclear cones are the best generalizations of Bishop-Phelps cones* in normed vector spaces.

In the next considerations we offer *significant examples* and *adequate remarks* on the *supernormal cones*. The existence of the *efficient points* and *important properties* of the *efficient points sets* are ensured in *separated locally convex spaces ordered by (weak) supernormal cones* named by us “Isac's cones”, through the agency of the (weak) *completeness* instead of *compactness* (the reader is referred to, Isac, G., 1981, 1983, 1985, 1994, 1998, Isac, G., Postolică, V., 1993, Isac, G., Hyers, D. H., Rassias, T., M., 1997, Postolică, V., 1993, 1994, 1995, 1997, 1999, 2001, 2002, 2009, Sonntag, Z., Zălinescu, C., 2000, Truong, X. D. H., 1994 and so on).

Theorem 1. (Bahya, A. O., 1989) *A convex and normal cone K in a Hausdorff locally convex space is supernormal if and only if every net of K weakly convergent to zero converges to zero in the locally convex topology.*

Let us consider some pertinent examples

1. Any convex, closed and pointed cone in an arbitrary usual Euclidean space R^k ($k \in N^*$) is supernormal.
2. In every locally convex space any well-based convex cone is an Isac's cone.
3. A convex cone is an Isac's cone in a normed linear space if and only if it is well-based.
4. Let $n \in N^*$ be arbitrary fixed and let Y be the space of all real symmetric (n, n) matrices ordered by the pointed, convex cone $C = \{A \in Y : x^T A x \geq 0, \forall x \in R^n\}$. Then, Y is a real Hilbert space with respect to the scalar product defined by $\langle A, B \rangle = \text{trace}(A \cdot B)$ for all $A, B \in Y$ and C is well-based by $B = \{A \in C : \langle A, I \rangle = 1\}$ where I denotes the identity matrix.
5. Every pointed, locally or weakly compact convex cone in any Hausdorff locally convex space is an Isac's cone.
6. A convex cone is an Isac's cone in a nuclear space (Pietch, A., 1972) if and only if it is a normal cone.
7. In any Hausdorff locally convex space a convex cone is an weakly Isac's cone if and only if it is weakly normal.
8. In $L^p([a, b])$, ($p \geq 1$), the convex cone $K_p = \{x \in L^p([a, b]) : x(t) \geq 0 \text{ almost everywhere}\}$ is an Isac's cone if and only if $p=1$, being well based in this case by the set $B = \{x \in K_1 :$

$\int_a^b x(t) dt = 1\}$. Indeed, if $p > 1$, then the sequence (x_n) defined by

$$x_n(t) = \begin{cases} n^{1/p}, & a \leq t \leq a+(b-a)/2n \\ 0, & a+(b-a)/2n < t \leq b \end{cases} \quad n \in N$$

converges to 0 in the weak topology but not in the usual norm topology. Therefore, by virtue of Theorem 1, K_p is not an Isac's cone. Generally, for every $p > 1$, K_p has a base $B = \{x \in K_p : \int_a^b x(t) dt = 1\}$ which is unbounded. Any convex cone generated by a closed and

bounded set $B_t = \{x \in B : \int_a^b |x(t)|^p dt \leq t\}$ with $t \geq 0$ is certainly an Isac's cone. A similar result holds for $L^p(R)$. Thus, if we consider a countable family (A_n) of disjoint sets which covers R such that $\mu(A_n) = 1$ for all n in N , where μ is the Lebesgue measure, then the sequence (y_n) given by $y_n(t) = 1$ if $t \in A_n$ and $y_n(t) = 0$ for $t \in R \setminus A_n$ converges weakly to zero while it is not convergent to zero in the norm topology. Taking into account the above theorem, it follows that the usual positive cone in $L^p(R)$ is not an Isac's cone if $p > 1$, that is, it is not well-based in all these cases. However, these cones are normal for every $p \geq 1$. The same conclusion

concerning the non-supernormality is valid for the positive orthant of the usual Orlicz spaces.

9. In l^p ($p \geq 1$) equipped with the usual norm $\|\cdot\|_p$ the positive cone $C_p = \{(x_n) \in l^p: x_n \geq 0 \text{ for all } n \in N\}$ is also normal with respect to the norm topology, but it is not an Isac's cone excepting the case $p = 1$. Indeed, for every $p > 1$, the sequence (e_n) having 1 at the n th coordinate and zeros elsewhere converges to zero in the weak topology, but not in the norm topology and by virtue of Theorem 1 it follows that C_p is not an Isac's cone. For $p = 1$, C_p is well-based by the set $B = \{x \in C_1: \|x\|_1 = 1\}$ and Proposition 5 (Isac, G., 1983) ensures that it is an Isac's cone. If we consider in this case the locally convex

topology in l^1 defined by the seminorms $p_n((x_k)) = \sum_{k=0}^n |x_k|$ for every (x_k) in l^1 and $n \in N$,

which is weaker than its usual weak topology, then the usual positive cone remains an Isac's cone with respect to this topology (now it is normal in a nuclear space and one applies Proposition 6 of (Isac, G., 1983) but it is not well based. Taking into account the concept of H -locally convex space introduced by Precupanu, T. in 1969 and defined as any Hausdorff locally convex space with the seminorms satisfying the parallelogram law and the property that every nuclear space is also a H -locally convex space with respect to an equivalent system of seminorms (Pietch, A., 1972), the above example shows that in a H -locally convex space a proper convex cone may be an Isac's cone without to be well-based. Moreover, if we consider in l^2 the H -locally convex topology induced by the seminorms

$$\tilde{p}_n((x_k)) = \left(\sum_{i \geq n} |x_i|^2 \right)^{1/2}, \quad n \in N, (x_k) \in \ell^2,$$

then the convex cone $C_2 = \{(x_k) \in l^2: x_k \geq 0 \text{ for all } k \in N\}$ is normal in the H -locally convex space $(l^2, \{\tilde{p}_n\}_{n \in N})$, but it is not a supernormal cone because the same sequence (e_k) is weakly convergent to zero while $(\tilde{p}_n(e_k))$ is convergent to 1 for each $n \in N$ and one applies again Theorem 1. Another interesting example of normal cone in a H -locally convex space which is not supernormal is the usual positive cone in the space $L^2_{loc}(R)$ of all functions from R to C which are square integrable over any finite interval of R , endowed with the system of

seminorms $\{\bar{p}_n : n \in \mathbb{N}\}$ defined by $\bar{p}_n(x) = \left(\int_{-n}^n |x(t)|^2 dt \right)^{1/2}$ for every x in $L^2_{loc}(R)$. In this case, the sequence (x_k) given by:

$$x_k(t) = \begin{cases} 0, & t \in (-\infty, 0) \cup (1/k, +\infty) \\ \sqrt{k}, & t \in [0, 1/k] \end{cases}$$

converges weakly to zero, but it is not convergent in the H -locally convex topology. The results follows by Theorem 1. It is clear that every weak topology is a H -locally convex topology and, in these cases, the supernormality of convex cones coincides with the normality thanks to the Corollary of Proposition 2 in (Isac, G., 1983).

10. In the space $C([a, b])$ of all continuous, real valued functions defined on every non-trivial, compact interval $[a, b]$ equipped with the usual supremum norm the convex cone $K = \{x \in C([a, b]) : x \text{ is concave, } x(a) = x(b) = 0 \text{ and } x(t) \geq 0 \text{ for all } t \in [a, b]\}$ is supernormal, being well based by the set $\{x \in K : x(t_0) = 1\}$ for some arbitrary $t_0 \in [a, b]$. The hypothesis that all $x \in K$ are concave is essential for the supernormality.

11. The convex cone of all nonnegative sequences in the space of all absolutely convergent sequences is the dual of the usual positive cone in the space of all convergent sequences. Consequently, it has a weak star compact base and hence it is a weak star supernormal cone.

12. In l^∞ or in c_0 equipped with the supremum norm, the convex cone consisting of all sequences having all partial sums non-negative is not normal, hence it is not supernormal.

13. In every Hausdorff locally convex space any normal cone is supernormal with respect to the weak topology.

14. In every locally convex lattice which is a (L) -space the ordering cone is supernormal (see also the Example 7 given by Isac, G. in 1994).

15. If we consider the space of all locally integrable functions on a locally compact space Y with respect to a Radon measure μ endowed with the topology induced by the family of seminorms $\{p_A\}$ where $p_A(f) = \int_A |f(x)| d\mu$ for every non-empty and compact subset A of Y and every locally integrable function f , then the convex cone $K = \{f : f(x) \geq 0, x \in Y\}$ is supernormal.

16. If Z is any locally convex lattice ordered by an arbitrary convex cone K and Z^* is its topological dual ordered by the corresponding dual cone K^* , then the cone K is supernormal with respect to the locally convex topology defined on Z by the neighbourhood base at the origin $\{[-f, f]\}_{f \in K}^{\sim}$

17 In every regular vector space (E, K) (that is, the order dual E^* separates the points of E) with the property that $E = K - K$ the convex cone K is supernormal with respect to the topology defined in the preceding example.

18. Any semicomplete cone in a Hausdorff locally convex space is supernormal (for this concept see the Example 11 of Isac, G., 1994).

Remark 2. Clearly, if a convex cone K is supernormal in a normed space, then K admits a strictly positive, linear and continuous functional, that is, there exists a linear, continuous functional f such that $f(k) > 0$ for all $k \in K \setminus \{0\}$. Generally, the converse is not true even in a Banach space as we can see in the following examples:

19. If one considers in the usual space ℓ^p ($1 \leq p \leq \infty$) the convex cone $K = \ell_+^p = \{x = (x_i) \in \ell^p : x_i \geq 0 \text{ for every } i \in \mathbb{N}\}$ of infinite vectors with non-negative components,

then the functional φ defined by $\varphi(k) = \sum_{i=1}^{\infty} k_i$ for any $k = (k_i) \in \ell^p$ is linear, continuous and strictly positive. But, as we have seen in the above considerations (Example 9), this cone is supernormal if and only if $p = 1$.

20. Let K be the usual positive cone $L_+^p = \{x \in L^p([a, b]) : x(t) \geq 0 \text{ almost everywhere}\}$ in $L^p([a, b])$ ($1 \leq p \leq \infty$). Then, the linear and continuous functional ψ on $L^p([a, b])$ given by $\psi(x) = \int_a^b x(t) dt$ for every $x \in L^p([a, b])$ is strictly positive on K while K is supernormal (see the above Example 8) if and only if $p = 1$.

Therefore, ℓ_+^1 and L_+^1 are supernormal cones with empty topological interiors and for every $p \in (1, +\infty)$ it follows that ℓ_+^p and L_+^p are normal cones with empty interiors which are not supernormal. Hence, these convex cones are not well based. A very simple example of supernormal cones having non-empty topological interior is $\mathbb{R}_+^n$ ($n \in \mathbb{N}^*$).

Remark 3. In the order complete vector lattice $B([a, b])$ of all bounded, real valued functions on a compact non-singleton interval $[a, b]$ endowed with its usual norm the standard positive cone $K = \{u \in B([a, b]) : u(t) \geq 0 \text{ for all } t \in [a, b]\}$ is normal but it has not a base, that is, it is not supernormal. However, this cone has non-empty interior. If we consider the linear space ℓ^1 endowed with the separated locally convex topology generated

by the family $\{p_n : n \in \mathbb{N}\}$ of seminorms defined by $p_n(x) = \sum_{k=0}^n |x_k|$ for every $x = (x_k) \in \ell^1$, then the convex cone $K = \{x = (x_k) \in \ell^1 : x_k \geq 0 \text{ whenever } k \in \mathbb{N}\}$ is supernormal but it is not well based.

Remark 4. *The natural context of supernormality (nuclearity) for convex cones is any separated locally convex space. Isac, G. introduced the concept of “nuclear cone” in 1981, published it in 1983 and he showed that in a normed space a convex cone is nuclear if and only if it is well based or equivalently iff it is “with plastering”, the last concept being defined by Krasnoselski, M. A. in fifties (see for example , Krasnoselski, M. A., 1964 and so on). Such a convex cone was initially called “nuclear cone” by Isac, G. (1981) because in every nuclear space (Pietch, A., 1972) any normal cone is a nuclear cone in Isac’s sense (Proposition 6 of Isac, G., 1983). Afterwards, since the nuclear cone introduced by Isac appears as a reinforcement of the normal cone, it was called supernormal. The class of supernormal cones in Hausdorff locally convex spaces was initially imposed by the theory and the applications of the efficient (Pareto minimum type) points (especially existence conditions based on completeness instead of compactness were decisive together with the main properties of the efficient points sets), the study of critical points for dynamical systems and conical support points and their importance was very well illustrated by important results, examples and comments in the specified references and in other connected papers. It is also very significant to mention again that the concept of supernormality introduced by Isac, G. (1981) is not a simple generalization of the corresponding notion defined in normed linear spaces by Krasnoselski, M. A. and his colleagues in the fifties. Thus, for example, Isac’s supernormality attached to the convex cones has his sense in every Hausdorff locally convex space identically with the well known Grothendieck’s nuclearity. By analogy with the fact that a normed space is nuclear in Grothendieck’s sense if and only if it is isomorphic with an usual Euclidean space, a convex cone is supernormal in a normed space if and only if it is well based, that is, it is generated by a convex bounded set which does not contain the origin in its closure. Beside Pareto type optimization, we also mention Isac’s significant contributions, through the agency of supernormal cones, to the convex cones in product linear spaces and Ekeland’s variational type principles(Isac.G.,2003;Isac,G.,Tammer,Chr.,2003). Therefore, the more appropriate background for Isac’s cones is any separated locally convex space.*

3. Conclusions and future research directions

The family of Isac’s cones represents the largest class of ordering cones in Hausdorff locally convex spaces ensuring the existence and the adequate properties for the efficient points sets involved in the general optimization, following different completeness types instead of compactness. Consequently, one of the main goal of the next research is to identify new applications of Isac’s cones in the efficiency projected in the best

approximation problems, the set - valued fixed point theory including the dynamical systems and the nuclearity of vector spaces.

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