

1 Linear Continuous Operators and Generalized Complex Numbers In The 2 Generalized Function Spaces $L_\alpha(S_\alpha^\alpha(R))$

3

4 **Abstract** . In this paper in order to define ordinary differential equations ODE
5 and partial differential equations PDE in the new generalized functions space
6 $L_\alpha(X)$ we will define generalized complex numbers corresponding to this
7 space, also we will define the generalized linear operators in the space $L_\alpha(X)$.

8 **Key Words** : Gelfand – Shilov spaces, Generalized Functions.

9 1. Introduction

10 Now, let X be a separated complete locally convex algebra [1] with topology
11 defined by the family of semi norms $P_{i \in I}$ such that for each $i \in I$ there exist $j \in I$
12 and a constant $C_i > 0$ for which

$$13 \quad p_i(xy) \leq C_i p_j(x) p_j(y) \quad \forall x, y \in X \quad (*)$$

14 if we denote by $G(x)$ the set of all possible sequences (x_k) in X , then $G(X)$
15 is an algebra with operations of coordinate wise multiplication.

16 Let $\alpha > 1$ be positive real number, define the following sets [1] :

$$17 \quad G_\alpha(X) = \{x = (x_k) \in G(x) : \exists m \quad \forall i \in I \exists C_i > 0, P_i(x_k) \leq C_i \exp\left(m k^{\frac{1}{\alpha}}\right) \forall k\}.$$

$$18 \quad N_\alpha(X) = \{x = (x_k) \in G(x) : \forall m, \forall i \in I \exists C_i > 0 : P_i(x_k) \leq C_i \exp(-m k^{\frac{1}{\alpha}}) \forall k\}.$$

19 **Theorem 1.2** The space $G_\alpha(X)$ is a Sub algebra of the algebra $G(x)$, and
20 $N_\alpha(X)$ is an ideal of $G_\alpha(X)$.

21 Define the space $L_\alpha(X)$ as a factor space in the following way

$$22 \quad L_\alpha(X) = G_\alpha(X) / N_\alpha(X).$$

23 Now if α and β be a nonnegative real numbers and $k, q \in \mathbb{N}$. Define the
24 following sets [4] :

25 $S_{\alpha}^{\beta} = \{ f(x) \in C^{\infty}(R) : \exists A > 0, \exists B > 0 \forall k \forall q \exists C > 0 : |x^k f^{(q)}(x)| \leq CA^k B^q K^{k\alpha} q^{\beta} \}$

26 If $\alpha > 0$ and $\beta = \alpha$, then the space $S_{\alpha}^{\alpha}(R)$ is said to be Rome – Helfand – Shilov Space

27 The topology in $S_{\alpha}^{\alpha}(R)$ is defined by the system of semi norms in the following

28 way : $p_{n,l} = \sup_{\substack{k \leq n \\ m \leq l}} q_{k,m}(t)$ where $q_{m,k} = \sup_{x \in R} \frac{x^k f^{(m)}(x)}{A^k B^m K^{k\alpha} m^{\alpha m}}$.

29 The following theorem is true see [1,2 , 5-8] .

30 **Theorem 1.1.** If $f(x), g(x) \in S_{\alpha}^{\alpha}(R)$, then for each n, l there is a constant

31 $C_{n,l} > 0$ such that $p_{n,l}(fg) \leq C_{n,l} p_{n,l}(f) p_{n,l}(g) \quad \forall f, g \in S_{\alpha}^{\alpha}(R)$

32 Since, $S_{\alpha}^{\alpha}(R)$ is a complete separated locally convex Algebra and $P_{n,l}(g)$

33 satisfy (*) the new generalized functions space defined in the following way[1]:

$$L_{\alpha}(S_{\alpha}^{\alpha}(R)) = G_{\alpha}(S_{\alpha}^{\alpha}(R)) / N_{\alpha}(S_{\alpha}^{\alpha}(R))$$

34

35 The embeddings of spaces $S_{\alpha}^{\alpha}(R)$ and $[S_{\alpha}^{\alpha}(R)]'$ in to the Algebra $L_{\alpha}(S_{\alpha}^{\alpha}(R))$

36 have been defined [6]. Therefore, we can write $S_{\alpha}^{\alpha}(R) \subset L_{\alpha}(S_{\alpha}^{\alpha}(R))$,

37 $[S_{\alpha}^{\alpha}(R)]' \subset L_{\alpha}(S_{\alpha}^{\alpha}(R))$.

38 2.Generalized Complex Numbers Corresponding to The Space $L_{\alpha}(S_{\alpha}^{\alpha}(R))$

39 Now our aim is to construct tools in the space $L_{\alpha}(S_{\alpha}^{\alpha}(R))$. For example we

40 need set of complex generalized numbers C^* to study mathematical models as

41 Cauchy's one:

$$42 \quad \left\{ \begin{array}{l} Df = fg \\ f(0) = Z^* \\ u, v \in L_{\alpha}(S_{\alpha}^{\alpha}(R)) \end{array} \right. \quad (I) \quad , \quad \left\{ \begin{array}{l} Df = \delta^n f \\ f(a) = b \\ f \in L_{\alpha}(S_{\alpha}^{\alpha}(R)) \end{array} \right. \quad a, b \in C^* \quad (II)$$

43 We define the generalized complex numbers corresponding to the space of the
44 ultra generalized functions space $L_{\alpha}(S_{\alpha}^{\alpha}(R))$ in the following way :

45 Let $K(C)$ be the set of all sequences of complex numbers . Define $K^*(C)$ as

46 the set of all sequences $(z_k) \in K(C)$ such that there is a natural number $m \in \mathbb{N}$

47 and a constant $C > 0$, such that $|z_k| \leq C \exp(-mk^{\frac{1}{\alpha}})$, for each k . Define $I^*(C)$ as
 48 the set of all sequences $(\eta_k) \in K(C)$ such that for each $m \in \mathbb{N}$ and for each k
 49 there is a constant $d > 0$ such that $|\eta_k| \leq d \exp(-mk^{\frac{1}{\alpha}})$, for each k in the domain
 50 of the sequences (η_k) .

51 **Theorem 2.1 . a)** The set $K^*(C)$ is an algebra,

52 b) The set $I^*(C)$ be an ideal in $K^*(C)$.

53 **Proof . a)** Suppose $z_1 = (z_k)$, $z_2 = (z_k^l)$ are elements in $K^*(C)$, then there are
 54 natural numbers m_1, m_2 and a constants $C_1 > 0, C_2 > 0$ such that

$$55 \quad |z_k| \leq C_1 \exp(-m_1 k^{\frac{1}{\alpha}}), \quad |z_k^l| \leq C_2 \exp(-m_2 k^{\frac{1}{\alpha}})$$

56 Consider $|z_k z_k^l| \leq C_1 C_2 \exp(-(m_1 + m_2)k^{\frac{1}{\alpha}})$, that is $z_1 z_2 \in K^*(C)$.

57 b) Now suppose that $z = (z_k) \in K^*(C)$, then there is a natural number m_1 and a
 58 constant $C > 0$ such that $|z_k| \leq C \exp(-m_1 k^{\frac{1}{\alpha}})$, for each k . If $\eta = (\eta_k) \in I^*(C)$ then
 59 for each $m \in \mathbb{N}$ there is a constant $d > 0$ such that

$$60 \quad |\eta_k| = |\eta_k| \leq d \exp(-mk^{\frac{1}{\alpha}}) \text{ for each } k.$$

61 Consider $|z_k \eta_k| \leq C d \exp(-(m_1 + m)k^{\frac{1}{\alpha}})$, that is $z \eta \in I^*(C)$.

62 **Theorem 2 . 2 a)** If $h = ((h_k)) \in G_{\mathbb{X}}(S_{\mathbb{X}}^{\infty}(\mathbb{R}))$ and $\mu_0 \in \mathbb{R}$, then $h(\mu_0) =$
 63 $((h_k(\mu_0))) \in K^*(C)$

64 b) If $\eta = (\eta_k) \in N_{\mathbb{X}}(S_{\mathbb{X}}^{\infty}(\mathbb{R}))$ and $\mu_0 \in \mathbb{R}$, then $\eta(\mu_0) \in I^*(C)$.

65 **Proof .** The Theorem is proved by using definitions of
 66 $G_{\mathbb{X}}(S_{\mathbb{X}}^{\infty}(\mathbb{R})), N_{\mathbb{X}}(S_{\mathbb{X}}^{\infty}(\mathbb{R})), K^*(C)$ and $I^*(C)$.

67 **Definition.** The space of generalized complex numbers is defined as a factor
 68 algebras

$$C_x^* = K^*(C) / I^*(C).$$

The definition of C_x^* and Theorem 2.2 play important role when we study the models like Cauchy's in the space of new generalized functions $L_x(S_x^*(R))$.

Moreover, we define embeddings of the set of all real numbers R and the set of all complex numbers C into the space of generalized complex numbers C_x^* by the following mappings :

$$j_1 : x \in R \rightarrow (x_k + 0i) \in C_x^*, \text{ where } x_k = x \quad \forall k$$

$$j_2 : z \in C \rightarrow (z_k) \in C_x^*, \text{ where } z_k = z \quad \forall k$$

3. Linear Continuous Operators in $L_x(S_x^*(R))$

Let $A : S_x^*(R) \rightarrow S_x^*(R)$ be a linear continuous operator, then [3] for each $i \in I$ there exist j and a constant $C_i > 0$ such that

$$P_i(A(\varphi(x))) \leq C_i P_j(\varphi(x)) \quad \forall \quad \varphi(x) \in S_x^*(R).$$

The operator A is lifted coordinate wise to a map which we denote by

$$A^* : G(S_x^*(R)) \rightarrow G(S_x^*(R)).$$

Theorem 3.1. a) $A^*[G_\alpha(S_\alpha^\alpha(R))] \subset G_\alpha(S_\alpha^\alpha(R))$, b) $A^*[N_\alpha(S_\alpha^\alpha(R))] \subset N_\alpha(S_\alpha^\alpha(R))$

Proof. The proof of this theorem follows from the definition of sets $G_x(S_x^*(R))$, $N_x(S_x^*(R))$, and by using the continuity of the operator A .

Now, the operator A can be lifted to a map which we will denote by

$$A_x^* : L_x(S_x^*(R)) \rightarrow L_x(S_x^*(R))$$

88

Theorem 3.2. a) The operator A_x^* is independent on a representative,

b) If $A : S_x^*(R) \rightarrow S_x^*(R)$ is isomorphism of $S_x^*(R)$, then

$A_x^* : L_x(S_x^*(R)) \rightarrow L_x(S_x^*(R))$ is an isomorphism of $L_\alpha(S_\alpha^\alpha(R))$.

92 Proof. a) Let $f \in L_\infty(S_\alpha^\alpha(\mathbb{R}))$ and let (f_k) and (g_k) are two representative of f
93 , then $(f_k - g_k) \in N_\alpha(S_\alpha^\alpha(\mathbb{R}))$. Since A^* is continuous , then

$$\begin{aligned} \mathbf{94} \quad P_i [A_\alpha^* (f_k) - A_\alpha^* (g_k)] &= P_i (A_\alpha (f_k - g_k)) \leq C_i P_i (f_k - g_k) \leq \\ \mathbf{95} \quad C_i C_j \exp(-mk^{\frac{1}{\alpha}}) \forall k, &\text{ that is } [A_\alpha^* (f_k) - A_\alpha^* (g_k)] \in N_\alpha(S_\alpha^\alpha(\mathbb{R})). \end{aligned}$$

96 b) The proof follows immediately from the definition of A_α^* and by the fact

97 $A : S_\alpha^\alpha(\mathbb{R}) \rightarrow S_\alpha^\alpha(\mathbb{R})$ is an isomorphism of $S_\alpha^\alpha(\mathbb{R})$.

98 Now, if h be linear continuous functional $h : S_\alpha^\alpha(\mathbb{R}) \rightarrow \mathbb{C}$, then it is lifted
99 coordinate wise to $h : G(S_\alpha^\alpha(\mathbb{R})) \rightarrow \mathbb{C}$ and $h_\alpha^* : L_\infty S_\alpha^\alpha(\mathbb{R}) \rightarrow \mathbb{C}_\alpha^*$, and the
100 functional h_α^* well defined by virtue of the following results :

101 Corollary. a) $h_\alpha^* [C_\alpha^*(S_\alpha^\alpha(\mathbb{R}))] \subset K^*(\mathbb{C})$, **b)** $h_\alpha^* [N_\alpha^*(S_\alpha^\alpha(\mathbb{R}))] \subset I^*(\mathbb{C})$,

102 c) The functional h_α^* is independent on a representative .

103

104 Now we can define the Lebege's Integral on $L_\alpha(S_\alpha^\alpha(\mathbb{R}))$ as a linear continuous
105 functional by the following way .

106 Let $K \subset \mathbb{R}$ be any compact set and $\varphi \in L_\alpha(S_\alpha^\alpha(\mathbb{R}))$, then define

$$\mathbf{107} \quad \left(\int_K \varphi(x) d\mu \right) = \int_K \varphi_k(x) d\mu, \text{ where } \varphi_k(x) \text{ - be any representative of } \varphi .$$

108 Definition . The generalized complex number Z^* with representative

109 $\left(\int_K \varphi_k(x) d\mu \right)$ is called the generalized integral of new generalized function

110 $\varphi \in L_\infty(S_\alpha^\alpha(\mathbb{R}))$ over the compact K ,that is $Z^* = \int_K \varphi(x) d\mu = \int_K \varphi_k(x) d\mu$

111 where $\varphi_k(x)$ - be any representative of φ .

112 The generalized integral defined above preserve many properties of usual

113 Labegue's integral defined in $S_\alpha^\alpha(\mathbb{R})$, for example, the following properties are
114 preserved :

$$\mathbf{115} \quad 1) \quad \int_K [\lambda(x) \pm \eta(x)] d\mu = \int_K \lambda(x) d\mu \pm \int_K \eta(x) d\mu ,$$

$$2) \int_K a \lambda(x) d\mu = a \int_K \lambda(x) d\mu \quad \forall a \in C_\alpha^*$$

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