A NovelModel for Removing the Mixture Noise and Blur of the Image

ABSTRACT

In order to deal with the problem of the mixture noise removing and deblurring, a new model based on partial differential equations is proposed in this paper. We analyzesome basic properties of the model, such as the existence and uniqueness of the solution of the model. Furthermore, we use an alternating minimization algorithm to find out the minimum of the proposed model. The experimental results showthat the proposed model can restrain the stair-case and edge blurring effectivelywhile removing the mixture noise and blur of the image.

Keywords: mixture noise; image deblurring; image denoising.

1. INTRODUCTION

The image is interfered by any unexpected blur or noise signal in the process of the emerged, transmission and storage. For example, the radar images, remote sensing images and medical ultrasonic images have irrelevant trails, the scientific research and medical diagnosis will be disturbed, so the results is distorted. In view of these facts, the technology of image deblurrring and denoising plays an important role in the domain of image processing, which continue to attract more and more attention of the researchers.

In this paper, we use a mathematical descriptionto show how the noise and blur image is formed[1].Suppose that the degraded image f is formedas

$$f = (Hw+b)v,$$

where$f$is the observed image, w is the original image,$b$is the additional noise, and$v$is the multiplicative noise. H is the blurred operatorwhich follows the corresponding distributionsuch as the motion blur, Gaussian blur or average blur. If let $H=Id$, Iddenotes the identity matrix,the observed image $f$is only disturbed by the noise.In the past few years, there are so many efficient methods which have been studied todeal with the imagedenoising problem[2,3]. Initially, the researchers removed the noise signal by the spatial filter or timing filter, such as the Fourier transmission filter, average filter, etc.[4,5]. In the following years, with the development of the mathematical theory, especially the partial differential equation(PDE) and total variation(TV) theory, the image denoisingbased on the PDEs and TVtheory draw more attention of the researchers.

In fact, there are two important types of the noise, so we must take different measures to restore the image.Under the additive noise scheme, the most typical method is the ROF model, which was based on the total variationproposed by Rudin , Osher , Fatemiin 1992[2], and the ROF model is defined as:

$$\min_{u\in BV(\Omega)} E(u) := \min_{u\in BV(\Omega)} \left\{ \int_\Omega |Du| + \lambda \int_\Omega (f-u)^2 \right\}. \qquad (1.1)$$

Here$BV(\Omega)$ is the bounded variation space defined in the compact support domain Ω of the image u, $f$is the degradation image.The first itemin $E(u)$ is called the regularization term, which serves to penalize high noisesolutions and restrict the solution space to the desired class of functions.The second item measures the fidelity to the data, which ensures that the denoisedimage u reserves main characteristic of the observed image f. The parameterλ is a positive weighting constant, which coordinates the regularization termand fitting term.

ROF model has been defined in the BV space,the biggest advantage of it is that it allows the energy function to bediscontinued, so it can preserve more edges information whenremoving the noise. However, the model for retaining image texture features is not perfect, which causes some virtual edges in the smooth area of image, brings the piecewise constant solution and the "stair-case" phenomenon.Later, a lot of improved models for this problem have been put forward[6-11], and these models are used to remove the multiplicative noise. In addition, great deals of denoisingapproaches were proposedto removethe impulse noise[12,13] and Possionnoise[8,14]. The most important method is RLO modelproposed by Rudin, Lions and Osher[15]:

$$\min_{u\in BV(\Omega)} E(u) := \min_{u\in BV(\Omega)} \{\int_\Omega |Du| + \lambda_1 \int_\Omega \frac{f}{u} + \lambda_2 \int_\Omega (\frac{f}{u} - 1)^2\} \cdot \tag{1.2}$$

Thenbased on this model, Aubert and Aujor proposed their modelas follows[16]

$$\min_{u\in BV(\Omega)} E(u) := \min_{u\in BV(\Omega)} \{\int_\Omega |Du| + \lambda \int_\Omega (\frac{f}{u} + \log u)\} \cdot \tag{1.3}$$

In [16], the Gamma noise with mean 1 is considered. Although their function is not convex, they still proved that their model have a unique solution by the numerical examples. Later, Huanget al.alsoimprovedthis model(called HNW model)[17]:

$$\min_{u,w} E(u,w) = \min_{u,w} \left\{\int_\Omega (u + fe^{-u}) + \alpha_2 \int_\Omega |Dw| + \alpha_1 \|u - w\|_2^2\right\} \cdot \tag{1.4}$$

Numerical results have shown that this model can provide the denoisedimages with better visual qualities. From then on, many fast and efficient algorithms and models were presented to solve above problem[9-11,16,17]. In fact, a lot of methods focus on the improvement of the fidelity term[16,17]. However, instead of the fidelity term, the main damage to the texture features of the image is the regularization term, so we need to pay more attention to improvethe regularization term. Then a series of models have been proposedto solve this problem[9-11]. One of the typical improved models is as follows

$$\min_{u\in BV(\Omega)} E(u) := \min_{u\in BV(\Omega)} \{\int_\Omega |Dw| \ln(1 + Dw) + B\} \cdot \tag{1.5}$$

Here$B$ is the traditional fidelity term such as $\lambda_1 \int_\Omega \frac{f}{u} + \lambda_2 \int_\Omega (\frac{f}{u} - 1)^2$, $\lambda \int_\Omega (\log u + \frac{f}{u})$, $\lambda \int_\Omega (u + fe^{-u})$, etc. Compared with HNW and AA model, these models have better restored results and retain more texture features than HNW and AA model.

Howeverthe image is not only disturbed by the noise practically, but also interfered by the atmospheric attenuation or lens/geometric distortion[18], etc. Somany researchers begin to restore the blurred image.Aubert and Aujor havealso extended their model to handle the image deblurring problem[16].Because of the non-convexity, Huang et al.improved this methodand proposed a new model [1]

$$\min_{u>0.w} E(u,w) := \min_{u>0.w} \left\{\frac{\|Hw - u\|^2}{2\sigma^2} + \int_\Omega (\log u + \frac{f}{u}) + \lambda \int_\Omega |Dw|\right\} , \tag{1.6}$$

where$u$is an intermediate image and$u=Hw+v$. In [1], they use the convex relaxation technique to compensatethe shortcoming of non-convexity. Except for analysing the basic properties of themodel, the paper also provide some experiments, which show that model(1.6) has a goodeffect.

To remove mixture noise and blur effectively, and retain more edges and texture features inthe restored imagein this paper, inspired by[9,11], we propose a novel model as follows:

$$\min_{u>0,w} E(u,w) := \min \left\{\frac{\|Hw - u\|^2}{2\sigma^2} + \lambda \int_\Omega (u + fe^{-u}) + \int_\Omega |Dw| \ln(1 + |Dw|)\right\} . \tag{1.7}$$

Here$u$is an intermediate image and$u=Hw+b$, b is Gaussian whiter noise with standard varianceσ.Moreover, we analyse some basic properties of the proposed model and list some numerical results, the experiments results demonstrate that our model have a good visual effect and preserve more texture features in the restored image.

The rest of this paper is organized as follows. Some properties about the proposed model are obtained in the next section. In section 3, an alternating minimization algorithm of our model is developed. Some numerical experiments are listed in section 4 to illustrate the performance of the proposed algorithm. Finally, a conclusion is given in section 5.

2 SOME BASIC PROPERTIES OF THE PROPOSED MODEL

In this section, we shall discuss some basic properties of the proposed model to prove the existence, uniqueness, and comparing principle ofthe model. For the convenience of the analysis and the discretization,themodel(1.7) can be rewritten as:

$$\min_{u>0,w} E(u,w) := \min\left\{ \frac{\|H[w]_i - [u]_i\|^2}{2\sigma^2} + \lambda\sum_{i=0}^{mn}([u]_i + [f]_i e^{-[u]_i}) + \sum_{i=0}^{mn}|D[w]_i|\ln(1+|D[w]_i|) \right\}. \quad (2.1)$$

Theorem 1. Assume that $f>0$ and$\lambda>0$.Then $E(u,w)$ in problem (2.1) isconvex.

Proof.This is a standard result. It is based on $f>0$,$\lambda>0$, and $\frac{\partial^2 E}{\partial u^2} = \frac{1}{\sigma^2} + \lambda[f]_i e^{-[u]_i} > 0$.

Remark 1. By theorem 1, we deduce that the model (2.1) hasat mostone solution.

Theorem 2. Let(u, w) be the solution of problem (1.7). Suppose that $H=Id\geq 0$ and $H1=1$, where 1 is constant image with entries 1,Then

1) $\min(\inf w, \inf(\ln f)) \leq u \leq \max(\sup w, \sup(\ln f))$.
2) $\inf w \geq \inf u \geq \inf f$, $\sup w \leq \sup u \leq \sup f$.

Proof.1) If $\inf f > 0$, for any i, as the function

$$t \to t + fe^{-t} + \frac{\|Hw - t\|^2}{2\sigma^2}$$

is monotone decreasing when $t \in (0, \min(\ln[f]_i, [Hw]_i))$. We know that

$$[u]_i > \min(\ln[f]_i, [Hw]_i),$$

whichmeans

$$u > \min(\inf(\ln f), \inf Hw).$$

Using the fact that $H \geq 0$ and for each $i$fixed , $\sum_{i,j}[H]_{i,j} = 1$,we have

$$[Hw]_i = \sum_j [H]_{i,j} w_j \geq \sum_j [H]_{i,j}(\inf w) = \inf w.$$

Thatis

$$\inf Hw \geq \inf w.$$

We get

$$u > \min(\inf(\ln f), \inf w)$$

So we obtain the left side of the first assertion.Similarly, we have the right side.

2) Now suppose that $H=Id$and consider

$$\min \frac{\|w-u\|^2}{2\sigma^2} + |Dw|\ln(1+|Dw|). \quad (2.2)$$

With u fixed,denote $\beta = \inf w$ and let $w_0=\max(w, \beta)$, by[10], we know that $|Dw|\ln(1+|Dw|)$ is convergence and regularization, and it has a minimum

$$|Dw_0|\ln(1+|Dw_0|). \quad (2.3)$$

Moreover, by the proposition 3.2 in[1], for each i fixed, by the definition of w_0,we have either

$$[w_0]_i = [w]_i \text{ or } [w_0]_i = \beta \geq [w]_i.$$

As $[u]_i \geq \beta$, both cases lead to

$$\left([w_0]_i - [w]_i\right)\left([w_0]_i + [w]_i - 2[u]_i\right) \leq 0 \cdot$$

The vector form gives:

$$\|w_0 - u\|^2 \leq \|w - u\|^2 \cdot$$

Together with (2.3), we can see that the replacing of $w$withw_0 will decrease the objective function value of (2.2), this implies that

$$\inf w \geq \beta = \inf u \ .$$

Combining with the fact that

$$\inf u \geq \min(\inf w, \inf f) \ .$$

We immediately have

$$\inf u \geq \inf f \ .$$

Then we get

$$\inf w \geq \inf u \geq \inf f \ .$$

This finishes the proof of the first part of the second assertion, with the same argument, we can get that the rest part is also true.

Remark 2. Note that the condition $H \geq 0$ and for each i, $H1=1$ is classical in the domain of image processing, we know that the above theoremalso holds in the continuous settings.

Theorem 3. If $w \in L^\infty(\Omega)$ with $\inf_\Omega w > 0$, then the model(1.7) exists a unique solution.

Proof. Denote by $\alpha = \inf_\Omega w$and $\beta = \sup_\Omega w$.Since $w \in L^\infty(\Omega)$ with $\inf_\Omega w > 0$,we can choose a sequence $\{w_n\} \in C^\infty(\Omega)$ such that $w_n \to w$ in $L^1(\Omega)$ and a.e. in Ω as $n \to \infty$, and

$$\inf_\Omega w \leq w_n \leq \sup_\Omega w \ . \qquad (2.4)$$

We let $h(s) = \frac{\|Hw - s\|^2}{2\sigma^2}$, then replace $w$with w_n, we can get that $h(s)$ is decreasing when $s \in (0, w_n)$ and increasing when $s \in (w_n, \infty)$ for $n \in N$, therefore, if $A \geq w_n$, we know

$$\frac{\|Hw - \min(s, A)\|^2}{2\sigma^2} \leq \frac{\|Hw - s\|^2}{2\sigma^2} \ .$$

Note that $\beta = \sup_\Omega w \geq w_n$, let $A=\beta$, we have

$$\frac{\|Hw_n - \min(u, A)\|^2}{2\sigma^2} \leq \frac{\|Hw_n - u\|^2}{2\sigma^2} \ .$$

Letting $n \to \infty$ in the above inequality, using Lebesgue Convergence Theorem and (2.4), we deduce

$$\frac{\|Hw - \min(u, A)\|^2}{2\sigma^2} \leq \frac{\|Hw - u\|^2}{2\sigma^2} \ . \qquad (2.5)$$

Thencombine (2.5) with the results of [10]and[19], we obtain

$$E(\min(u, w)) \leq E(u, w) \ .$$

In the same way, we have

$$E(\max(u, w)) \leq E(u, w) \ .$$

Hence, we get that there exists a constant C such that

$$\int h(u_n) + \int \phi(w_n) + \frac{\|Hw_n - u_n\|^2}{2\sigma^2} \leq C \cdot$$

Here $h(u_n) = u_n + fe^{-u_n}$, $\phi(w_n) = |Dw_n| \ln(1 + |Dw_n|)$. We know that (u_n, w_n) is bounded in $BV(\Omega)$, then exist $(u, w) \in BV(\Omega)$ such that (u,w) is the solution of problem(1.7).

Remark 3. Since $h,\varphi$is strictly convex as $f>0$,the uniqueness of the minimum follows the strict convexity of the energy function. We deduce that the (u,w) is the unique solution of the problem(1.7).

3. ALTERNATING MINIMIZATION METHOD

In this section, we propose using an alternatingminimization algorithm to solve theproblem(1.7).Starting from an initial guess $w^{(0)}$, the method computes a sequence of iterationssuch that

$$\begin{cases} u(\mathrm{k}+1)=\operatorname{argmin}_w\left\{\lambda\int(u+fe^{-u})+\dfrac{\left\|Hw^{(k)}-u\right\|^2}{2\sigma^2}\right\} & (3.1)\\ w(\mathrm{k}+1)=\operatorname{argmin}_u\left\{\dfrac{\left\|Hw-u^{(\mathrm{k}+1)}\right\|^2}{2\sigma^2}+|Dw|\ln(1+|Dw|)\right\}. & (3.2)\end{cases}$$

Step 1. Consider the problem(3.1), let

$$g(u)=\frac{\left\|Hw^{(k)}-u\right\|^2}{2\sigma^2}+\lambda\int(u+fe^{-u}),$$

then the discrete form of$g(u)$ is

$$g(u(i,j))=\frac{\left\|Hw^{(k)}(i,j)-u(i,j)\right\|^2}{2\sigma^2}+\lambda\int(u(i,j)+f(i,j)e^{-u(i,j)})\cdot$$

Because $\mathrm{g}(u)$ is continuously differentiable in the domain of $u(i, j)$, the minimizer of the problem (3.2) is equivalent to solve the following n^2decoupled nonlinear equations

$$g'(u(i,j))=\lambda(1-f(i,j)e^{-u(i,j)})+\frac{\left\|Hw^{(k)}(i,j)-u(i,j)\right\|}{\sigma^2}=0,\qquad i=1,2,\cdots,n^2.\quad(3.3)$$

As the original objective function $g(u)$ is strictly convex, the corresponding nonlinear equation has a unique solution.We can use the Newton method to solve the problem (3.3) as

$$u^{(m)}(u(i,j))=u^{(m-1)}(i,j)-\frac{g'(u^{(m-1)}(i,j))}{g''(u^{(m)}(i,j))}\cdot$$

Step2. Consider the variation problem (3.2), let

$$G(x,y,w,w_x,w_y)=\frac{\left\|Hw-u\right\|^2}{2\sigma^2}+|Dw|\ln(1+|Dw|)\cdot$$

Using the variation theory, we know the solution of G is determined by the corresponding Euler-Lagrange equation, which is

$$J(w)=\frac{H\left\|Hw-u\right\|}{\sigma^2}+div(\frac{|Dw|+(1+|Dw|)\ln(1+|Dw|)}{|Dw|\ln(1+|Dw|)})\cdot$$

When $J(w)=0$, we can get the solution of the problem G, the equation is as follows

$$\frac{H\left\|Hw-u\right\|}{\sigma^2}+div(\frac{|Dw|+(1+|Dw|)\ln(1+|Dw|)}{|Dw|\ln(1+|Dw|)})=0\cdot\quad(3.4)$$

Usingthe numerical discrete method of [9], the optimum numerical solution can be obtained by

$$w\_t=\sum_{p\in\Lambda}t(\left|Dw_p\right|_\varepsilon)(w_p-w(i,j))+\frac{H\left\|Hw-u^{(k)}\right\|}{\sigma^2}\cdot$$

The iterative scheme is

$$w^{(m)}=w^{(m-1)}+dt\times w\_t\,.$$

Here $t(x)=\dfrac{x+(1+x)\ln(1+x)}{x(1+x)}$ and $\left|Dw_p\right|_\varepsilon=\sqrt{\varepsilon+(w_p)_x^2+(w_p)_y^2}$, $\mathrm{p}\in\Lambda$, Λ is the four adjacent pixels of the pixel $u(i,j)$, and $\Lambda=((i-1,j),(i+1,j),(i,j-1),(i,j+1))$.

The ending condition of the iterative of the proposed method is that the relative difference between the successive iterates of the restored images should satisfy the following inequality

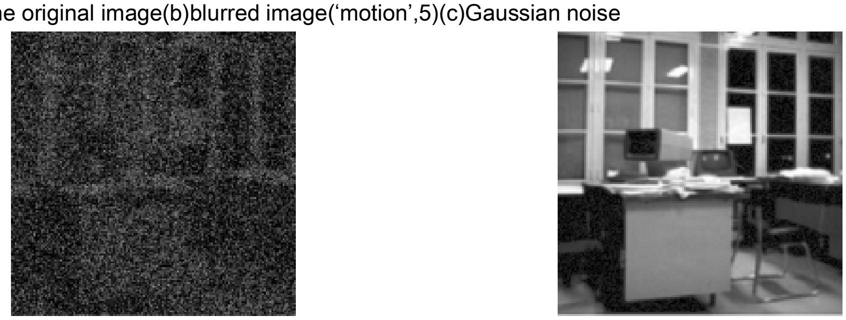
$$\frac{\left\|\tilde{w}^{(k+1)}-\tilde{w}^{(k)}\right\|}{\tilde{w}^{(k+1)}} \le 10^{-3}.$$

4. NUMERICAL RESULTS

In this section, numerical simulations are performed to illustrate the effectiveness of the proposed model for deblurring and denoisingmixturenoise simultaneously. All experiments are done by the MATALB2009a on the same machine. In order to make the experimental data powerful, we take the average value of 10 times results of our experiment.

First, we select the image 'office', on the basis of the emergence mechanism of the noise.Figure(a) is the original image.Figure(b) is the image whichis blurred by motion blur as *fspecial*('*motion*',5).Figure(c) is addedtheGaussian noise with standard variance 1.Figure(d) is the final noised image corrupted bythe Gamma noise with standard variance 0.01.Figure(e) is the result by the proposed model.

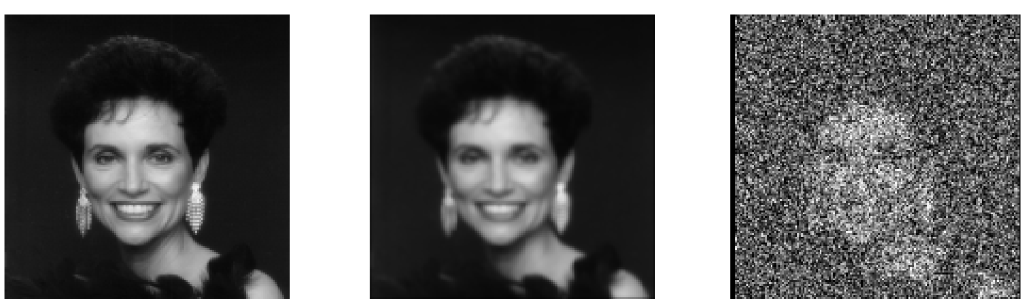
(a) the original image(b)blurred image('motion',5)(c)Gaussian noise



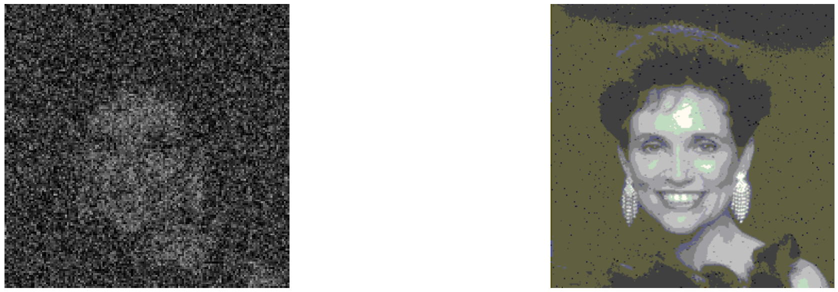
(d)Gamma noise(e)the denoised and deblurred image

Fig.1. the deblurred and denoised results of the "office" image

Next, we select the image 'women', on the basis of the emergence mechanism of the noise, Figure(a) is the original image,Figure(b) is the image which is blurred by Gaussian blur as*fspecial*('*Gaussian*',5,2), Figure(c) is addedtheGaussian noise with standard variance 1, Figure(d) is the final noised image by Gamma noise with standard variance 0.01, Figure(e) is the deblurred and denoised image by the proposed model



(a) the original image(b)blurred image('Gaussian',5,2)(c)Gaussian noise

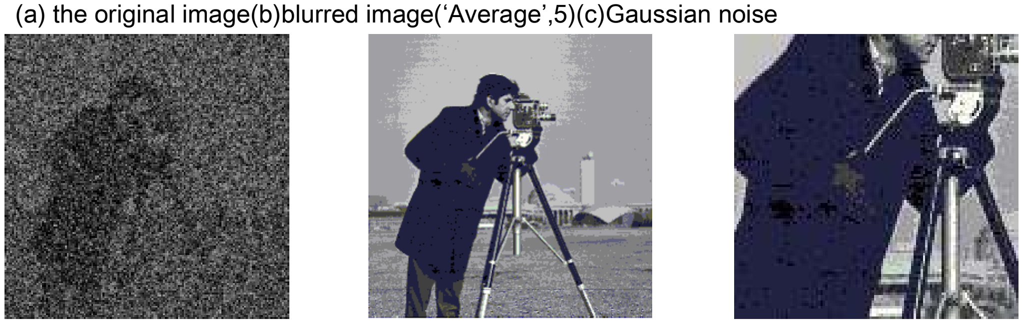


(d)Gamma noise(e)the denoised and deblurred image

Fig.2. the deblurred and denoised results of the "woman" image

Finally, we select the image 'cameraman', on the basis of the emergence mechanism of the noise.Figure(a) is the original image, Figure(b) is the image which is blurred by average blur as*fspecial*('*Average*',5), Figure(c) is added the Gaussian noise with standard variance 1, Figure(d) is the final noised image by Gamma noise with standard variance 0.01, Figure(e) is the deblurred and denoised image by the proposed model. For observing the effect conveniently, especially, we enlarge the local of the figure(e) as Figure(f).

(a) the original image(b)blurred image('Average',5)(c)Gaussian noise



(d)Gamma noise(e)the denoised and deblurred image(f) the detail view

Fig.3. the deblurred and denoised results of the "cameraman" image

The above experiments have the same standard variance of the Gaussian white noise and the multiplicative Gamma noise.The unique difference is the value of the blur operator.

Now, we introduce table 1 by two objective evaluation index to show the restoring quality of the proposed model. Suppose that the image size is *m*-by-*n*, The peak signal noise ratio (*PSNR*) and the relative error (*ReErr*) value[11]are defined as follows:

$$PSNR = 10\ln\left(\frac{V^2}{\tilde{w}(i,j) - w(i,j)}\right),$$

$$\mathrm{Re}\,Err = \frac{\left\|\tilde{w}(i,j) - w(i,j)\right\|^2}{\left\|w(\mathrm{i},\mathrm{j})\right\|^2}.$$

Where$w(i, j)$ is the original image, $\tilde{w}(i, j)$ is the restored image, and$V = \max_{i,j}((w(i, j), \tilde{w}(i, j))$. By the definition of *PSNR* and *ReErr*, we know a fact thatthe greater*PSRN*is, the better denoised effect will be reached, and the smaller*ReErr*is.

Table 1. The related evaluation index of the denoised image

Image	Cpu time	*PSNR*	*ReErr*	Iteration
office	35.0222	21.0136	0.1875	275
women	87.1734	12.9429	0.5272	624
cameraman	16.1619	28.8948	0.0668	112

5. CONCLUSION

In this paper, we put forward an improved model in the $BV(\Omega)$ space, and discuss the existence, uniqueness and comparing principle of the solution of the model.We also provide three experiments results including the images and the parameter index in the paper.The results demonstratethat the proposed model can restrain the stair-case effectively when removing the mixture noise and blur.

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