Symmetric q-Gamma Function

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Review Article

Abstract

In this work we are interested by giving new characterizations of the symmetric q-Gamma function and show that there are intimately related. For that, some special q-calculus technics are used.

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1 Introduction

In literature the characterizations of the well known Gamma function are studied by many authors [1, 2] and [3]. As same as the Gamma function, the characterization of the q-Gamma function was studied by Elmonser, Brahim and Fitouhi in [4], they proved the following results:

Theorem 1.1. *The q-Gamma function is the unique function* $f(x) > 0$ *on* $]0, +\infty[$ *that satisfies the following properties:*
a) $f(1) = 1$
b)$f(x+1) = [x]_q f(x)$
c)$f(x+n) = (1-q)^{[x]_q - x} f(n)[n]_q^{[x]_q} t_n(x)$, *where* $t_n(x) \to 1$ *as* $n \to \infty$.

The second theorem gives the relationship between three different characterizations of the q-Gamma function:

Theorem 1.2. *For a q-PG function* f*, the following properties are equivalent:*
(C) $\ln f$ *is convex on* $]0, +\infty[$,
(L)$L(n+x) = ([x]_q - x)\ln(1-q) + L(n) + x\ln(n+1) + r_n(x)$,
where $L(x) = \ln f(x+1)$ *and* $r_n(x) \to 0$ *as* $n \to \infty$,
(P) $f(x+n) = (1-q)^{[x]_q - x} f(n)[n]_q^{[x]_q} t_n(x)$,
where $t_n(x) \to 1$ *as* $n \to \infty$.
A q-PG function f *satisfying these properties is equal to* $c\Gamma_q(x)$*, for some constant c.*

where the a q-PG function (pre-q-gamma function) is a positive function f on $]0, +\infty[$ satisfying the functional equation $f(x+1) = [x]_q f(x)$.

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A generalization of the q-gamma function, called symmetric q-Gamma function, was introduced and studied by K. Brahim and Yosr Sidomou in [6].

In the present paper, we continue the study of this function by giving some new characterizations and prove that they are intimately related.

2 Notations and Preliminaries

We recall some usual notions and notation used in the q-theory [7, 8, 9] and [10]. Throughout this paper, we assume $q \in]0,1[$.

For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) = (1-a)(1-aq)....(1-aq^{n-1}), \qquad n = 1,2,..... \tag{2.1}$$

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i). \tag{2.2}$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbf{C}, \tag{2.3}$$

$$\widetilde{[x]}_q = \frac{q^x - q^{-x}}{q-q^{-1}} =, \quad x \in \mathbf{C}, \tag{2.4}$$

$$[n]_q! = \prod_{k=1}^{n}[k]_q = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbf{N}. \tag{2.5}$$

and

$$\widetilde{[n]}_q! = \prod_{k=1}^{n}\widetilde{[k]}_q, \quad n \in \mathbf{N}. \tag{2.6}$$

One can see that

$$\widetilde{[x]}_q = q^{-(x-1)}[x]_{q^2}. \tag{2.7}$$

3 The symmetric q-Gamma function:

The q-Gamma function $\Gamma_q(x)$, a q-analogue of Euler's gamma function, was introduced by Thomae [11] and later by Jackson [12] as the infinite product:

$$\Gamma_q(x) = \frac{(q;q)_\infty(1-q)^{1-x}}{(q^x;q)_\infty} \quad ,x>0, \tag{3.1}$$

where q is a fixed real number $0<q<1$.

Recently, K. Brahim and Yosr Sidomou [6] introduced the symmetric q-Gamma function as follows:

$$\widetilde{\Gamma}_q(z) = q^{-\frac{(z-1)(z-2)}{2}}\Gamma_{q^2}(z), \quad ,z>0, q>0, q\neq 1, \tag{3.2}$$

where

$$\Gamma_q(z) = \begin{cases} \frac{(q,q)_\infty}{(q^x,q)_\infty}(1-q)^{1-x}, & \text{if } 0<q<1, \\ \frac{(q^{-1},q^{-1})_\infty}{(q^{-x},q^{-1})_\infty}(1-q)^{1-x}q^{\frac{x(x-1)}{2}}, & \text{if } q>1. \end{cases} \tag{3.3}$$

They proved that it is symmetric under the interchange $q \leftrightarrow q^{-1}$ and satisfies a q-analogue of the Bohr-Mollerup theorem for $q \neq 1$:

Theorem 3.1. *Let $q > 0$, $q \neq 1$. The only function $f \in C^2((0,\infty))$ satisfying the conditions:*
(a) $f(1) = 1$.
(b) $f(x+1) = \widetilde{[x]}_q f(x)$.
(c) $\frac{d^2}{dx^2} Log f(x) \geq |Log q|$ for positive x,
is the symmetric q-Gamma function.

In [4], the author proved the following relation

$$\Gamma_q(x) = \lim_{n\to+\infty} (1-q)^{[x]_q - x} \frac{[n]_q^{[x]_q} [n]_q!}{[x]_q [x+1]_q ... [x+n]_q}, \quad x > 0. \tag{3.4}$$

Using the relation (3.2) and (3.4) , we derive the following relation:

$$\widetilde{\Gamma}_q(x) = \lim_{n\to+\infty} q^{-\frac{(x-1)(x-2)}{2}} (1-q^2)^{[x]_{q^2} - x} \frac{[n]_{q^2}^{[x]_{q^2}} [n]_{q^2}!}{[x]_{q^2} [x+1]_{q^2} ... [x+n]_{q^2}}, \quad x > 0, 0 < q < 1. \tag{3.5}$$

4 Characterization of the q-Gamma function:

As it is proved in [4] and [5] we establish new characterizations of the symmetric q-Gamma function. The first characterization is given by the following theorem:

Theorem 4.1. *The symmetric q-Gamma function $\widetilde{\Gamma}_q(x)$ is the unique function $f(x) > 0$ on $]0, +\infty[$ that satisfies the following properties:*
a) $f(1) = 1$
b)$f(x+1) = \widetilde{[x]}_q f(x)$
c)$f(x+n) = q^{-\frac{x^2+2nx-3x}{2}} (1-q^2)^{[x]_{q^2} - x} [n]_{q^2}^{[x]_{q^2}} f(n) t_n(x)$, where $t_n(x) \to 1$ as $n \to \infty$.

Proof. .
First we prove that $\widetilde{\Gamma}_q(x)$ satisfies conditions (a), (b) and (c).

From Theorem 3.1, the symmetric q-Gamma function satisfies the condition (a) $\widetilde{\Gamma}_q(1) = 1$, and the condition (b) $\widetilde{\Gamma}_q(x+1) = \widetilde{[x]}_q \widetilde{\Gamma}_q(x)$.

As a consequence of the two properties, we get $\widetilde{\Gamma}_q(n) = \widetilde{[n-1]}_q!$

(c) Let $s_n(x) = \frac{\widetilde{\Gamma}_q(x)}{q^{-\frac{(x-1)(x-2)}{2}} (1-q^2)^{[x]_{q^2} - x} \widetilde{\Gamma}_{n,q}(x)}$,

where $\widetilde{\Gamma}_{n,q}(x) = \frac{[n]_{q^2}^{[x]_{q^2}} [n]_{q^2}!}{[x]_{q^2} [x+1]_{q^2} ... [x+n]_{q^2}} = \frac{[n]_{q^2}^{[x]_{q^2}} \widetilde{[n]}_q!}{q^{nx+x-1} \widetilde{[x]}_q \widetilde{[x+1]}_q ... \widetilde{[x+n]}_q}$,

then $\widetilde{\Gamma}_q(x) = s_n(x) q^{-\frac{(x-1)(x-2)}{2}} (1-q^2)^{[x]_{q^2} - x} \widetilde{\Gamma}_{n,q}(x)$ and $\lim_{n\to+\infty} s_n(x) = 1$.

For $n \in \mathbf{N}$ and $x > 0$, we apply (b) n times to get

$$\begin{aligned}\widetilde{\Gamma}_q(x+n) &= \widetilde{[x+n-1]}_q...\widetilde{[x+1]}_q\widetilde{[x]}_q\widetilde{\Gamma}_q(x)\\ &= \frac{\widetilde{[x+n]}_q...\widetilde{[x+1]}_q\widetilde{[x]}_q}{\widetilde{[x+n]}_q}.q^{-\frac{(x-1)(x-2)}{2}}(1-q^2)^{[x]_{q^2}-x}\frac{[n]_{q^2}^{[x]_{q^2}}\widetilde{[n]}_q!}{q^{nx+x-1}\widetilde{[x]}_q\widetilde{[x+1]}_q...\widetilde{[x+n]}_q}.s_n(x)\\ &= q^{-\frac{x^2+2nx-3x}{2}}(1-q^2)^{[x]_{q^2}-x}[n]_{q^2}^{[x]_{q^2}}\widetilde{\Gamma}_q(n)t_n(x).\end{aligned}$$

Where $t_n(x) = q^{-x}\frac{\widetilde{[n]}_q}{\widetilde{[x+n]}_q}.s_n(x)$. Thus, $\widetilde{\Gamma}_q(x+n) = q^{-\frac{x^2+2nx-3x}{2}}(1-q^2)^{[x]_{q^2}-x}[n]_{q^2}^{[x]_{q^2}}\widetilde{\Gamma}_q(n)t_n(x)$ and $t_n(x) \to 1$ as $n \to +\infty$.

To show uniqueness, we assume $f(x)$ is a function that satisfies (a), (b) and (c). From properties (a) and (b), we have

$$f(n) = \widetilde{[n-1]}_q!. \tag{4.1}$$

$$f(x+n) = \widetilde{[x+n-1]}_q\widetilde{[x+n-2]}_q...\widetilde{[x+1]}_q\widetilde{[x]}_q f(x). \tag{4.2}$$

Combining (4.1),(4.2) and (c) together, we have

$$\begin{aligned}f(x) &= q^{-\frac{x^2+2nx-3x}{2}}(1-q^2)^{[x]_{q^2}-x}\frac{[n]_{q^2}^{[x]_{q^2}}\widetilde{[n-1]}_q!}{\widetilde{[x+n-1]}_q\widetilde{[x+n-2]}_q...\widetilde{[x+1]}_q\widetilde{[x]}_q}t_n(x)\\ &= q^{-\frac{(x-1)(x-2)}{2}}(1-q^2)^{[x]_{q^2}-x}\widetilde{\Gamma}_{n,q}(x).s_n(x),\end{aligned}$$

where $s_n(x) = q^x\frac{\widetilde{[x+n]}_q}{\widetilde{[n]}_q}t_n(x) \to 1$ as $n \to +\infty$. Therefore $f(x) = \widetilde{\Gamma}_q(x)$ and hence f is uniquely determined. This completes the proof.

5 Relationship between Characterizations

In what follows, we will adopt the terminology of the following definition.

Definition 5.1. A function f is said to be a qs-PG function (pre-symmetric-q-gamma function), if f is positive on $]0,+\infty[$ and satisfies the functional equation

$f(x+1) = \widetilde{[x]}_q f(x)$.

In the previous section we showed that the property

$$f(x+n) = q^{-\frac{x^2+2nx-3x}{2}}(1-q^2)^{[x]_{q^2}-x}[n]_{q^2}^{[x]_{q^2}}f(n)t_n(x)$$

characterizes the q-gamma function. In this section we will give three properties which are equivalent to one another for a qs-PG function and characterize the symmetric q-gamma function.

Theorem 5.1. *For a q-PG function f, the following properties are equivalent:*
(C) $\ln f$ is convex on $]0,+\infty[$,
(L)$L(n+x) = -\frac{x^2+2nx-3x}{2}\ln q + ([x]_{q^2}-x)\ln(1-q^2) + L(n) + [x]_{q^2}\ln[n+1]_{q^2} + r_n(x)$,
where $L(x) = \ln f(x+1)$ and $r_n(x) \to 0$ as $n \to \infty$,
(P) $f(x+n) = q^{-\frac{x^2+2nx-3x}{2}}(1-q^2)^{[x]_{q^2}-x}[n]_{q^2}^{[x]_{q^2}}f(n)t_n(x)$,
where $t_n(x) \to 1$ as $n \to \infty$.
A qs-PG function f satisfying these properties is equal to $c\widetilde{\Gamma}_q(x)$,for some constant c.

Proof. .
(a) $(P) \Leftrightarrow (L)$. We have

$$
\begin{aligned}
(P) \quad \Leftrightarrow \quad & f(x+(n+1)) = q^{-\frac{x^2+2nx-3x}{2}}(1-q^2)^{[x]_{q^2}-x} f(n+1)[n+1]_{q^2}^{[x]_{q^2}} t_{n+1}(x), \\
& t_{n+1}(x) \to 1 \\
\Leftrightarrow \quad & \ln f(x+(n+1)) = -\frac{x^2+2nx-3x}{2}\ln q + ([x]_{q^2}-x)\ln(1-q^2) + \ln f(n+1) \\
& +[x]_{q^2}\ln[n+1]_{q^2} + \ln t_{n+1}(x), t_{n+1}(x) \to 1 \\
\Leftrightarrow \quad & L(x+n) = -\frac{x^2+2nx-3x}{2}\ln q + ([x]_{q^2}-x)\ln(1-q^2) + L(n) \\
& +[x]_{q^2}\ln[n+1]_{q^2} + r_n(x), r_n(x) \to 0 \\
\Leftrightarrow \quad & (L).
\end{aligned}
$$

(b) $(C) \Longrightarrow (P)$. Let $m < x \le m+1$, where $m = 0,1,2,...$ For any natural n, $n+m-1 < n+m < n+x \le n+m+1$. The convexity of $\ln f$ gives us (we write $L_m = \ln f(n+m)$)

$$
\begin{aligned}
\frac{L_m - L_{m-1}}{n+m-(n+m-1)} \le \frac{\ln f(n+x) - \ln f(n+m)}{(n+x)-(n+m)} \le \frac{L_{m+1}-L_m}{(n+m+1)-(n+m)} \\
\Leftrightarrow (x-m)\ln \widetilde{[n+m-1]}_q \le \ln \frac{f(n+x)}{f(n+m)} \le (x-m)\ln \widetilde{[n+m]}_q \\
\Leftrightarrow \widetilde{[n+m-1]}_q^{\,x-m} \le \frac{f(n+x)}{\widetilde{[n+m-1]}_q \widetilde{[n+m-2]}_q ... \widetilde{[n]}_q f(n)} \le \widetilde{[n+m]}_q^{\,x-m} \\
\Leftrightarrow \widetilde{[n+m-1]}_q^{\,x} T_m \le \frac{f(n+x)}{f(n)} \le \widetilde{[n+m]}_q^{\,x} T_m \frac{\widetilde{[n+m-1]}_q^{\,m}}{\widetilde{[n+m]}_q^{\,m}},
\end{aligned}
$$

where $T_m = \frac{\widetilde{[n+m-1]}_q \widetilde{[n+m-2]}_q ... \widetilde{[n]}_q}{\widetilde{[n+m-1]}_q^{\,m}} = q^{\frac{m(m-1)}{2}} \frac{[n+m-1]_{q^2}[n+m-2]_{q^2}...[n]_{q^2}}{[n+m-1]_{q^2}^m}$.
Therefore, we have

$$
\lim_{n\to+\infty} q^{nx}\frac{f(n+x)}{f(n)} = \frac{q^{-\frac{x^2-3x}{2}}}{(1-q^2)^x},
$$

by the squeezing theorem. If we let

$$
t_n(x) = \frac{q^{\frac{x^2+2nx-3x}{2}} f(n+x)}{(1-q^2)^{[x]_{q^2}-x} f(n)[n]_{q^2}^{[x]_{q^2}}},
$$

then

$$
f(n+x) = q^{-\frac{x^2+2nx-3x}{2}}(1-q^2)^{[x]_{q^2}-x} f(n)[n]_{q^2}^{[x]_{q^2}} t_n(x),
$$

where $t_n(x) \to 1$ as $n \to \infty$. This proves that f satisfies (P).

(c) $(P) \Longrightarrow (C)$. From the uniqueness part of the proof of the Theorem 1.1 we have
$f(x) = f(1)\lim_{n\to+\infty} q^{-\frac{(x-1)(x-2)}{2}}(1-q^2)^{[x]_{q^2}-x}\Gamma_{n,q}(x)$.

Using the fact that the limit function of a convergent sequence of convex functions is convex, it suffices to show that $\ln\left(q^{-\frac{(x-1)(x-2)}{2}}(1-q^2)^{[x]_{q^2}-x}\Gamma_{n,q}(x)\right)$ is convex.
Now

$$
\begin{aligned}
\ln\left(q^{-\frac{(x-1)(x-2)}{2}}(1-q^2)^{[x]_{q^2}-x}\Gamma_{n,q}(x)\right) \quad = \quad & -\frac{(x-1)(x-2)}{2}\ln q + ([x]_{q^2}-x)\ln(1-q^2) \\
& +[x]_{q^2}\ln[n]_{q^2} + \ln([n]_{q^2}!) - \ln[x]_{q^2} - ... - \ln[x+n]_{q^2}.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}\left(\ln\left(q^{-\frac{(x-1)(x-2)}{2}}(1-q^2)^{[x]_{q^2}-x}\Gamma_{n,q}(x)\right)\right)' &= (-x+\frac{3}{2})\ln q+\left(-2\frac{\ln q}{1-q^2}q^{2x}-1\right)\ln(1-q^2)\\ &+\left(-2\frac{\ln q}{1-q^2}q^{2x}\ln[n]_{q^2}\right)+\frac{2\ln q}{1-q^2}\frac{q^{2x}}{[x]_{q^2}}+...\\ &+\frac{2\ln q}{1-q^2}\frac{q^{2(x+n)}}{[x+n]_{q^2}}.\end{aligned}$$

And so

$$\begin{aligned}\left(\ln\left(q^{-\frac{(x-1)(x-2)}{2}}(1-q^2)^{[x]_{q^2}-x}\Gamma_{n,q}(x)\right)\right)'' &= -\ln q-4\frac{(\ln q)^2}{1-q^2}q^{2x}(\ln(1-q^2)+\ln\frac{1-q^{2n}}{1-q^2})\\ &+4\frac{(\ln q)^2}{1-q^2}[\frac{q^{2x}[x]_{q^2}+\frac{q^{4x}}{1-q^2}}{[x]_{q^2}^2}+...\\ &+\frac{q^{2(x+n)}[x+n]_{q^2}+\frac{q^{4(x+n)}}{1-q^2}}{[x+n]_{q^2}^2}]\\ &= -\ln q-4\frac{(\ln q)^2}{1-q^2}q^{2x}(\ln(1-q^{2n}))\\ &+4\frac{(\ln q)^2}{1-q^2}[\frac{q^{2x}[x]_{q^2}+\frac{q^{4x}}{1-q^2}}{[x]_{q^2}^2}+...\\ &+\frac{q^{2(x+n)}[x+n]_{q^2}+\frac{q^{4(x+n)}}{1-q^2}}{[x+n]_{q^2}^2}].\end{aligned}$$

Then
$\left(\ln\left((1-q)^{[x]_q-x}\Gamma_{n,q}(x)\right)\right)''>0.$

This completes the proof.

Competing Interests

Author has declared that no competing interests exist.

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