Original Research Article

A NEW AND SIMPLE MATRIX INVERSION METHOD USING DODGSON'S CONDENSATION

Abstract

This article furnishes a new and simple matrix inversion method which makes full use of the condensation technique of the author of *Alice in Wonderland*, Charles Dodgson. A special feature of this article is the adoption of Bhaskara's law of impending operation on zero in overcoming the problem of division by zero whenever zero appears as a divisor in the condensation technique of Dodgson.

keywords: Matrix, Determinant, Division by zero, Bhaskara's law of impending operation on zero, Inverse of Matrices, Dodgson Condensation, Cofactor matrix

1 Introduction

Given any matrix , for instance, the 3rd order matrix,

$$\mathbf{D} = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix},$$

one knows, by the standard method of finding the minor of each element, how to compute the cofactor matrix containing the minors with their prescribed signs, and hence the inverse of the original matrix, obtained by dividing the transpose of the cofactor matrix by the determinant of the original matrix. This process, in the above instance, would run thus:

The cofactor matrix of $\mathbf{D}$ is

$$\begin{bmatrix} \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -5 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 4 & -5 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -5 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & -5 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} \end{bmatrix}$$

which, being evaluated, gives

$$\begin{bmatrix} 13 & 11 & -7 \\ 1 & 31 & 7 \\ 16 & 6 & 14 \end{bmatrix}$$

where

$$\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix}$$

is the minor of the element 2 in row 1 and column 1 of the original matrix, obtained by deleting all elements in row 1 and column 1;

$$\begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix}$$

is the minor of the element 3 in row 1 and column 2 of the original matrix, obtained by deleting all elements in row 1 and column 2; the other minors are found in a similar fashion. The transpose of the cofactor matrix of $\mathbf{D}$ is

$$\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix},$$

and thus the inverse of $\mathbf{D}$ is

$$\begin{aligned} \mathbf{D^{-1}} &= \frac{1}{|\mathbf{D}|}\,(\text{Transpose of the cofactor matrix of } \mathbf{D}) \\ &= \frac{1}{98}\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}. \end{aligned}$$

This method, practicable only for the second and third orders, becomes tedious and painful when it is adopted in computing the inverses of higher–order matrices.

Another method, more efficient, of computing the inverses of matrices is that due to Jordan, often called Gauss–Jordan method. This method involves setting up the $n \times 2n$ matrix $\begin{bmatrix} \mathbf{D} & \mathbf{I} \end{bmatrix}$ and applying elementary row operations to this matrix to convert the left half to the identity matrix $\mathbf{I}$. Clearly, in doing this, the right half will be converted to a matrix; that is, the inverse matrix $\mathbf{D^{-1}}$ will automatically be constructed in the right half as the left half is converted to the identity.

The cardinal aim of this paper is to introduce a novel method of computing the inverses of matrices. This approach makes use of the well-known condensation method of Charles Dodgson.

The rest of this paper is structured into two sections. Because some understanding of the theory of Dodgson's condensation of determinants is required to compute the inverses of matrices to which this paper is mainly devoted, we will discuss Dodgson's condensation first, and Section 2 is set up for this purpose. Section 3 deals with the use of Dodgson's condensation in computing the inverses of matrices. It is assumed that the reader is familiar with the elementary theorems of matrices and determinants.

2 Dodgson's Condensation

Dodgson's condensation consists of the following steps or rules:

1. Employ the elementary row and column operations to rearrange, if necessary, the given nth order matrix such that there are no zeros in its interior. The interior of a matrix is the minor formed after the first and last rows and columns of the matrix have been deleted.

2. Evaluate every 2nd order determinant formed by four adjacent elements. The values of the determinants form the $(n-1)$st order matrix.

3. Condense the $(n-1)$st order matrix in the same manner, dividing each entry by the corresponding element in the interior of the nth order matrix.

4. Repeat the condensation process until a single number is obtained. This number is the value of the determinant of the nth order matrix.

To make the method clear, we consider the matrix in Section 1:

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & 4 \\ 3 & 5 & -1 \end{bmatrix}.$$

We compute its determinant by condensing it, applying rule 2, to

$$\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 5 & -1 \end{vmatrix} \end{bmatrix}$$

which when evaluated gives

$$\begin{bmatrix} -5 & 14 \\ 7 & -21 \end{bmatrix}.$$

This in turn, by rule 3, is condensed to give the value, 7. Dividing this value by the interior, 1, of the 3rd order matrix, we get 7 which is the value of the determinant of our original 3rd order matrix.

Again, we want to compute the determinant of the 4th order matrix

$$\begin{bmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{bmatrix}$$

using Dodgson's condensation technique. By rule 2 this is condensed into

$$\begin{bmatrix} \begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 0 & -4 \\ 5 & 1 \end{vmatrix} & \begin{vmatrix} -4 & 6 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 6 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ -3 & 8 \end{vmatrix} & \begin{vmatrix} 2 & 6 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 6 & -1 \\ 9 & 1 \end{vmatrix} \end{bmatrix}$$

which, when evaluated, gives

$$\begin{bmatrix} 10 & 20 & -6 \\ 8 & 28 & -1 \\ 6 & -30 & 15 \end{bmatrix}.$$

This in turn, by rule 3, is condensed into

$$\begin{bmatrix} \begin{vmatrix} 10 & 20 \\ 8 & 28 \end{vmatrix} & \begin{vmatrix} 20 & -6 \\ 28 & -1 \end{vmatrix} \\ \begin{vmatrix} 8 & 28 \\ 6 & -30 \end{vmatrix} & \begin{vmatrix} 28 & -1 \\ -30 & 15 \end{vmatrix} \end{bmatrix}$$

which, being evaluated, furnishes

$$\begin{bmatrix} 120 & 148 \\ -408 & 390 \end{bmatrix}.$$

We divide each element of the above 2×2 matrix by the corresponding element of the interior matrix of the 4th order matrix,

$$\begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix},$$

and have

$$\begin{bmatrix} \frac{120}{5} & \frac{148}{1} \\ \frac{-408}{2} & \frac{390}{6} \end{bmatrix}$$

which gives

$$\begin{bmatrix} 24 & 148 \\ -204 & 65 \end{bmatrix}$$

which, when evaluated, gives the value of 31752. Dividing this value by the interior, 28, of the 3rd order matrix, we get 1134 which is the value of our original 4th order matrix.

The simplest way of presenting the workings appears to be to arrange the series of matrices one under another, as it is displayed below; it will then be found very easy to pick out the divisors (in the interior matrices) required in rules 3 and 4:

$$\begin{bmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 20 & -6 \\ 8 & 28 & -1 \\ 6 & -30 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 24 & 148 \\ -204 & 65 \end{bmatrix}$$

$$1134.$$

Dodgson's condensation method, being interesting and excellently suited to hand–computations, is in the first place remarkable for its exceedingly great briefness, lucidity and accuracy. It is also noteworthy as it involves the evaluation of only 2nd order determinants, the elements of which are adjacent to one another.

However, it is evident that, when zeros (which Dodgson called *ciphers* in his paper [6]) appear in the interior of the original matrix or any one of the derived matrices, the process cannot be continued because of the emergence of division by zero[6]. A solution to this problem, as Dodgson suggests, is to recommence the operation by first rearranging the original matrix by transferring the top row to the bottom or the bottom row to the top so that the zero, when it occurs, is now found in an exterior row[6]. The merit of this solution is that "there is only *one* new row to be computed; the other rows are simply copied from the work already done"[6].

Suppose now we want to find the value of the determinant of the matrix

$$\begin{bmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \end{bmatrix}.$$

We compute as follows:

$$\begin{bmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -5 & -3 & -1 \\ -3 & -3 & -3 & 3 \\ 3 & 3 & 3 & -1 \\ -5 & -3 & -1 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -15 & 6 & 12 \\ 0 & 0 & 6 \\ 6 & -6 & 8 \end{bmatrix}.$$

We cannot continue the operation because of the zero which occurs in the interior of the derived 3rd order matrix. Division by zero will occur. So we rearrange the original 5th order matrix by moving the top row to the bottom and moving all the other rows up once, and recommence the operation:

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \\ 2 & -1 & 2 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 & -3 & 3 \\ 3 & 3 & 3 & -1 \\ -5 & -3 & -1 & -5 \\ 3 & -5 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 6 \\ 6 & -6 & 8 \\ -17 & 8 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 12 \\ 18 & 40 \end{bmatrix}$$

$$36.$$

There is another means of overcoming this problem of zero divisor without recommencing the condensation process. It is the use of Bhaskara's law of impending operation on zero which is discussed in the papers. This is done by first adding the zero $\mathbf{0}$ to one of the elements of the 2×2 matrix whose evaluation gives rise to the zero in the interior matrix. Thus we write

$$\begin{bmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -5 & -3 & -1 \\ -3 & -3 & -3 & 3 \\ 3 & 3 & 3+\mathbf{0} & -1 \\ -5 & -3 & -1 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -15 & 6 & 12 \\ \mathbf{0} & -\mathbf{0} & 6+\mathbf{0} \\ 6 & -6-\mathbf{0} & 8+\frac{5}{2}\cdot\mathbf{0} \end{bmatrix}.$$

$$\begin{bmatrix} -5\cdot\mathbf{0} & -12-10\cdot\mathbf{0} \\ 2\cdot\mathbf{0} & \dfrac{72+8\cdot\mathbf{0}-3\cdot\mathbf{0}^2}{2(3+\mathbf{0})} \end{bmatrix}.$$

The determinant of the matrix is therefore

$$\frac{1}{-\mathbf{0}}\left[-5\cdot\mathbf{0}\left(\frac{72+8\cdot\mathbf{0}-3\cdot\mathbf{0}^2}{2(3+\mathbf{0})}\right)+2\cdot\mathbf{0}(12+10\cdot\mathbf{0})\right]$$

which becomes

$$5\left(\frac{72+8\cdot\mathbf{0}-3\cdot\mathbf{0}^2}{2(3+\mathbf{0})}\right)-2(12+10\cdot\mathbf{0}).$$

Omitting **0** as it merely represents absolute nothing gives

$$5\left(\frac{72}{2(3)}\right)-2(12)$$

which is equal to 36.

3 Inverses of Matrices

In this section, I shall teach how to calculate, by means of a new method, the inverses of matrices of not only the second and third orders, but also the fourth and fifth orders. This method uses Dodgson's condensation and computes the inverse of any $n \times n$ matrix $\mathbf{D}$ as demonstrated in the following rules:

1. Form the four-quadrant matrix by putting $\mathbf{D}$ in four quadrants:

$$\begin{bmatrix} \mathbf{D} & \mathbf{D} \\ \mathbf{D} & \mathbf{D} \end{bmatrix}.$$

2. Form the interior matrix of the four-quadrant matrix by deleting the first and last rows and columns of the four-quadrant matrix.

3. Apply Dodgson's condensation by condensing the interior matrix of the four-quadrant matrix to matrix of the next lower order and continue the process until a matrix of the same order as $\mathbf{D}$ is obtained, that is an $n \times n$ matrix.

If n is odd, the final $n \times n$ matrix formed is the cofactor matrix of $\mathbf{D}$. If n is even and we give each element of the final $n \times n$ matrix formed its prescribed sign, the resulting matrix is the cofactor matrix of $\mathbf{D}$.

It is interesting to note that the determinant of $\mathbf{D}$ can be easily obtained from the above process. Apply the condensation one more time. We get an $n-1 \times n-1$ matrix. If n is odd, then the $n-1 \times n-1$ matrix formed is a matrix consisting of only the determinant of $\mathbf{D}$ as its elements. If n is even and we give each element of the $n-1 \times n-1$ matrix formed its prescribed sign, the resulting matrix consists of only the determinant of $\mathbf{D}$ as its elements.

To illustrate the ease with which this method is used to obtain the inverse of matrices, we begin with the simplest case, the 2nd order matrix

$$\mathbf{D} = \begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}.$$

By rule 1 we write the four-quadrant matrix as

$$\begin{bmatrix} 2 & 4 & 2 & 4 \\ -1 & 7 & -1 & 7 \\ 2 & 4 & 2 & 4 \\ -1 & 7 & -1 & 7 \end{bmatrix}$$

and by rule 2 we get the interior matrix of the four-quadrant matrix as

$$\begin{bmatrix} 7 & -1 \\ 4 & 2 \end{bmatrix}.$$

We give the elements of this matrix their prescribed signs since the original matrix is of even order, and we obtain the cofactor matrix as

$$\begin{bmatrix} 7 & -(-1) \\ -4 & 2 \end{bmatrix}$$

which gives

$$\begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix}$$

which, being transposed, becomes

$$\begin{bmatrix} 7 & -4 \\ 1 & 2 \end{bmatrix}.$$

The inverse of the original matrix is, therefore,

$$\mathbf{D^{-1}} = \frac{1}{18} \begin{bmatrix} 7 & -4 \\ 1 & 2 \end{bmatrix}$$

where the value 18 is the determinant of the original matrix.

We now compute by means of this new technique the inverse of the 3×3 matrix:

$$\mathbf{D} = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}.$$

By rule 1 we write the four-quadrant matrix as

$$\begin{bmatrix} 4 & 1 & -5 & 4 & 1 & -5 \\ -2 & 3 & 1 & -2 & 3 & 1 \\ 3 & -1 & 4 & 3 & -1 & 4 \\ 4 & 1 & -5 & 4 & 1 & -5 \\ -2 & 3 & 1 & -2 & 3 & 1 \\ 3 & -1 & 4 & 3 & -1 & 4 \end{bmatrix},$$

and by applying rule 2 we get the 4×4 interior matrix of the four-quadrant matrix as

$$\begin{bmatrix} 3 & 1 & -2 & 3 \\ -1 & 4 & 3 & -1 \\ 1 & -5 & 4 & 1 \\ 3 & 1 & -2 & 3 \end{bmatrix}$$

which, by rule 3, becomes the cofactor matrix

$$\begin{bmatrix} 13 & 11 & -7 \\ 1 & 31 & 7 \\ 16 & 6 & 14 \end{bmatrix}$$

which, after famously undergoing transposition, becomes

$$\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}.$$

Here we must not give signs to the elements since the original matrix is of odd order, and we must stop condensation at this point since the derived matrix is of the same order as the original matrix $\mathbf{D}$. We now compute the determinant of $\mathbf{D}$ as follows:

$$|\mathbf{D}| = \frac{1}{4} \begin{vmatrix} 13 & 11 \\ 1 & 31 \end{vmatrix} = 98.$$

Notice also that

$$|\mathbf{D}| = \frac{1}{3} \begin{vmatrix} 11 & -7 \\ 31 & 7 \end{vmatrix} = 98,$$

$$|\mathbf{D}| = \frac{1}{-5} \begin{vmatrix} 1 & 31 \\ 16 & 6 \end{vmatrix} = 98,$$

$$|\mathbf{D}| = \frac{1}{4} \begin{vmatrix} 31 & 7 \\ 6 & 14 \end{vmatrix} = 98.$$

Thus the inverse of $\mathbf{D}$ is

$$\mathbf{D^{-1}} = \frac{1}{98}\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}.$$

Clearly, the computation of the inverse of 2×2 and 3×3 matrices by employing Dodgson's condensation is simple and systematic and does not involve the liability of dividing by elements in the interiors of the matrices, thereby escaping the cipher problem of division by zero. This new method, therefore, deserves utmost consideration and absolute attention of all as the 2×2 and 3×3 matrices are the most common matrices employed in texts and by students and teachers of mathematics, science and engineering. Thus, the Author strongly recommend this new method for hand–computation of inverses of matrices worldwide.

Let us now compute the inverse of the 3×3 matrix,

$$\mathbf{D} = \begin{bmatrix} 0 & 3 & 0 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}.$$

By rule 1 we write the four-quadrant matrix as

$$\begin{bmatrix} 0 & 3 & 0 & 0 & 3 & 0 \\ 4 & 1 & 6 & 4 & 1 & 6 \\ 1 & 4 & 0 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 & 3 & 0 \\ 4 & 1 & 6 & 4 & 1 & 6 \\ 1 & 4 & 0 & 1 & 4 & 0 \end{bmatrix},$$

and by applying rule 2 we get the 4×4 interior matrix of the four-quadrant matrix as

$$\begin{bmatrix} 1 & 6 & 4 & 1 \\ 4 & 0 & 1 & 4 \\ 3 & 0 & 0 & 3 \\ 1 & 6 & 4 & 1 \end{bmatrix},$$

which, by rule 3, becomes the cofactor matrix ,

$$\begin{bmatrix} -24 & 6 & 15 \\ 0 & 0 & 3 \\ 18 & 0 & -12 \end{bmatrix}$$

which in its turn, after being transposed, becomes

$$\begin{bmatrix} -24 & 0 & 18 \\ 6 & 0 & 0 \\ 15 & 3 & -12 \end{bmatrix}.$$

We stop the condensation process and start the computation of the determinant of $\mathbf{D}$. Since, of the four elements in the interior of the 4×4 interior matrix of the four-quadrant matrix, only one element is non–zero, there is, therefore, only one way of finding the determinant of $\mathbf{D}$ from the cofactor matrix, namely, the evaluation in which the divisor is the element, 1. Thus the determinant of $\mathbf{D}$ is

$$|\mathbf{D}| = \frac{1}{1}\begin{vmatrix} 6 & 15 \\ 0 & 3 \end{vmatrix} = 18.$$

and its inverse is

$$\mathbf{D^{-1}} = \frac{1}{18}\begin{bmatrix} -24 & 0 & 18 \\ 6 & 0 & 0 \\ 15 & 3 & -12 \end{bmatrix}.$$

We now turn to the computation of the inverse of the 4×4 matrix,

$$\begin{bmatrix} -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 \end{bmatrix}.$$

We write the four–quadrant matrix as

$$\begin{bmatrix} -2 & 3 & 4 & 7 & -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 & 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 & 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 & 2 & 1 & 4 & 2 \\ -2 & 3 & 4 & 7 & -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 & 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 & 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 & 2 & 1 & 4 & 2 \end{bmatrix}$$

and obtain its interior matrix as

$$\begin{bmatrix} 2 & 4 & 4 & 6 & 2 & 4 \\ -3 & 6 & 3 & 3 & -3 & 6 \\ 1 & 4 & 2 & 2 & 1 & 4 \\ 3 & 4 & 7 & -2 & 3 & 4 \\ 2 & 4 & 4 & 6 & 2 & 4 \\ -3 & 6 & 3 & 3 & -3 & 6 \end{bmatrix}.$$

We employ Dodgson's condensation and obtain the following:

$$\begin{bmatrix} 24 & -12 & -6 & -24 & 24 \\ -18 & 0 & 0 & 9 & -18 \\ -8 & 20 & -18 & 8 & -8 \\ 4 & -12 & 50 & -22 & 4 \\ 24 & -12 & 50 & -22 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -36 & 0 & -18 & -72 \\ -90 & 0 & 81 & 72 \\ 4 & 112 & 2 & -48 \\ 60 & 168 & -222 & -216 \end{bmatrix}.$$

We give the elements of this matrix their prescribed signs since the original matrix is of even order, and we obtain the cofactor matrix as

$$\begin{bmatrix} -36 & 0 & -18 & 72 \\ 90 & 0 & -81 & 72 \\ 4 & -112 & 2 & 48 \\ -60 & 168 & 222 & -216 \end{bmatrix}.$$

The determinant of $\mathbf{D}$ is

$$|\mathbf{D}| = \frac{1}{9}\begin{vmatrix} -18 & -72 \\ 81 & 72 \end{vmatrix} = 504.$$

and its inverse is

$$\mathbf{D^{-1}} = \frac{1}{504}\begin{bmatrix} -36 & 90 & 4 & -60 \\ 0 & 0 & -112 & 168 \\ -18 & -81 & 2 & 222 \\ 72 & 72 & 48 & -216 \end{bmatrix}.$$

This method is exceedingly simple and lucid, but it may be rendered even more palpable to the eye by arranging the series of matrices one under another, as it is displayed below; it will then be found very easy to culled the divisors from the matrices:

$$\begin{bmatrix} -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & 4 & 7 & -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 & 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 & 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 & 2 & 1 & 4 & 2 \\ -2 & 3 & 4 & 7 & -2 & 3 & 4 & 7 \\ 6 & 2 & 4 & 4 & 6 & 2 & 4 & 4 \\ 3 & -3 & 6 & 3 & 3 & -3 & 6 & 3 \\ 2 & 1 & 4 & 2 & 2 & 1 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 4 & 6 & 2 & 4 \\ -3 & 6 & 3 & 3 & -3 & 6 \\ 1 & 4 & 2 & 2 & 1 & 4 \\ 3 & 4 & 7 & -2 & 3 & 4 \\ 2 & 4 & 4 & 6 & 2 & 4 \\ -3 & 6 & 3 & 3 & -3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 24 & -12 & -6 & -24 & 24 \\ -18 & 0 & 0 & 9 & -18 \\ -8 & 20 & -18 & 8 & -8 \\ 4 & -12 & 50 & -22 & 4 \\ 24 & -12 & 50 & -22 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -36 & 0 & -18 & -72 \\ -96 & 0 & 81 & 72 \\ 4 & 112 & 2 & -48 \\ 60 & 168 & -222 & -216 \end{bmatrix}.$$

We take the computation of the determinant of another 4×4 matrix. This is to teach us how we may handle cases wherein zero appears in the interior of the matrix derived directly from the four-quadrant matrix. Suppose the matrix

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 5 \end{bmatrix}.$$

We find the determinant as follows:

$$\begin{bmatrix} 1 & 2 & 3 & -1 & 1 & 2 & 3 & -1 \\ 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 & 2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 5 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & -1 & 1 & 2 & 3 & -1 \\ 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 & 2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 5 & 1 & 2 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 2 & 1 & -1 \\ 2 & 3 & 5 & 1 & 2 & 3 \\ 2 & 3 & -1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 2 & 1 & -1 \end{bmatrix}.$$

We add, based on Bhaskara's law of zero, the zero $\mathbf{0}$ (this notation is used to differentiate it from the non-interior 0) to the interior zero and apply condensation as usual:

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 2 & 1 & -1 \\ 2 & 3 & 5 & 1 & 2 & 3 \\ 2 & 3 & -1 & 1 & 2 & 3 \\ 1 & \mathbf{0} & 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 2 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 \\ 5 & -8 & -9 & 3 & 5 \\ 0 & -18 & 6 & 0 & 0 \\ 2 \cdot \mathbf{0} - 3 & 3 + \mathbf{0} & -3 & -3 & -3 \\ -1 - \mathbf{0} & \mathbf{0} + 1 & 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -9 & 0 & 3 \\ -30 & -42 & -18 & 0 \\ 12\cdot\mathbf{0}-18 & 6\cdot\mathbf{0}-36 & -18 & 0 \\ 2\cdot\mathbf{0}+3 & 3\cdot\mathbf{0}+3 & 0 & 3 \end{bmatrix}.$$

We omit $\mathbf{0}$ since it represents nothing. Thus, we have the matrix

$$\begin{bmatrix} -3 & -9 & 0 & 3 \\ -30 & -42 & -18 & 0 \\ -18 & -36 & -18 & 0 \\ 3 & 3 & 0 & 3 \end{bmatrix}$$

which, giving its elements their prescribed signs, becomes

$$\begin{bmatrix} -3 & 9 & 0 & -3 \\ 30 & -42 & 18 & 0 \\ -18 & 36 & -18 & 0 \\ -3 & 3 & 0 & 3 \end{bmatrix}.$$

Thus the inverse is

$$\frac{1}{18}\begin{bmatrix} -3 & 30 & -18 & -3 \\ 9 & -42 & 36 & 3 \\ 0 & 18 & -18 & 0 \\ -3 & 0 & 0 & 3 \end{bmatrix}.$$

3.1 Proof of the Validity of the New Approach

We now proceed to give a proof of the validity of this new method. In doing so, we shall take the following steps in computing the cofactor matrix of the $n \times n$ matrix $\mathbf{D}$ by means of the new method:

1. Form the cofactor matrix consisting of the cofactors or minors in determinant form.

2. Rearrange the elements in the determinants, the minors, such that

   (a) for every row, from left to right, the 2nd, 3rd, 4th, ..., $(n-1)$st columns of each determinant are the respective 1st, 2nd, 3rd, ..., $(n-2)$nd columns of the next determinant.

   (b) for every column, from top to bottom, the 2nd, 3rd, 4th, ..., $(n-1)$st rows of each determinant are the respective 1st, 2nd, 3rd, ..., $(n-2)$nd rows of the next determinant.

3. Write all the columns of the elements of all the determinants without repeating any column.

If these steps are carefully taken, it will be found that the new matrix formed is the interior of a matrix formed by putting $\mathbf{D}$ in four quadrants, adjacent to one another.

3.1.1 Derivation for 2×2 Matrix

Let us first take the simplest case, the 2×2 matrix:

$$\mathbf{D} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

If we compute the cofactors of the elements of this matrix by the method of finding the complementary minor of each element, we obtain the cofactor matrix

$$\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

which, except for the prescribed signs of the cofactors, is the interior of the matrix,

$$\begin{bmatrix} a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \\ a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \end{bmatrix},$$

formed by putting $\mathbf{D}$ in four quadrants, adjacent to one another.

3.1.2 Derivation for 3×3 Matrix

Secondly, let us take the 3×3 matrix:

$$\mathbf{D} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

We find the cofactor matrix by method of minors and get

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}.$$

We rearrange the elements of each determinant in the cofactor matrix above, such that for every row, from left to right, the 2nd column of each determinant is the 1st column of the next determinant. Thus we have the cofactor matrix of $\mathbf{D}$ rearranged as

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{13} & a_{11} \\ a_{33} & a_{31} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}.$$

Again, we rearrange the elements of each determinant, such that for every column, from top to bottom, the 2nd row of each determinant is the 1st row of the next determinant. So we have the cofactor matrix rewritten as

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}.$$

This new arrangement of the cofactor matrix may be considered as a derived matrix obtained by employing Dodgson's condensation to the matrix,

$$\begin{bmatrix} a_{22} & a_{23} & a_{21} & a_{22} \\ a_{32} & a_{33} & a_{31} & a_{32} \\ a_{12} & a_{13} & a_{11} & a_{12} \\ a_{22} & a_{23} & a_{21} & a_{22} \end{bmatrix},$$

which is clearly the interior of the matrix,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & a_{33} \end{bmatrix},$$

formed by putting **D** in four quadrants, adjacent to one another. This proves the method for a 3×3 matrix; and similar proofs might be given for larger matrices.

3.1.3 Derivation for 4×4 Matrix

Lastly, let us take the 4×4 matrix:

$$\mathbf{D} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

We find the cofactor matrix by method of minors and get

$$\begin{bmatrix}
\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\
-\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\
\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\
-\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{bmatrix}.$$

We rearrange the elements of each determinant in the cofactor matrix above, such that for every row, from left to right, the 2nd and 3rd columns of each determinant are respectively the 1st and 2nd columns of the next determinant. Thus we have the cofactor matrix of **D** rearranged as

$$\begin{bmatrix}
\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{23} & a_{24} & a_{21} \\ a_{33} & a_{34} & a_{31} \\ a_{43} & a_{44} & a_{41} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{21} & a_{22} \\ a_{24} & a_{31} & a_{32} \\ a_{24} & a_{41} & a_{42} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\
-\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} & a_{11} \\ a_{33} & a_{34} & a_{31} \\ a_{43} & a_{44} & a_{41} \end{vmatrix} & -\begin{vmatrix} a_{14} & a_{11} & a_{12} \\ a_{34} & a_{31} & a_{32} \\ a_{44} & a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\
\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{13} & a_{14} & a_{11} \\ a_{23} & a_{24} & a_{21} \\ a_{43} & a_{44} & a_{41} \end{vmatrix} & \begin{vmatrix} a_{14} & a_{11} & a_{12} \\ a_{24} & a_{21} & a_{22} \\ a_{44} & a_{41} & a_{42} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\
-\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} & a_{11} \\ a_{23} & a_{24} & a_{21} \\ a_{33} & a_{34} & a_{31} \end{vmatrix} & -\begin{vmatrix} a_{14} & a_{11} & a_{12} \\ a_{24} & a_{21} & a_{12} \\ a_{34} & a_{31} & a_{12} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{bmatrix}.$$

Again, we rearrange the elements of each determinant, such that for every column, from top to bottom, the 2nd and 3rd rows of each determinant are respectively the 1st and 2nd rows of the next determinant. So we have the cofactor matrix rewritten as

$$\begin{bmatrix}
\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} & -\begin{vmatrix} a_{23} & a_{24} & a_{21} \\ a_{33} & a_{34} & a_{31} \\ a_{43} & a_{44} & a_{41} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{21} & a_{22} \\ a_{34} & a_{31} & a_{32} \\ a_{44} & a_{41} & a_{42} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\
-\begin{vmatrix} a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \\ a_{12} & a_{13} & a_{14} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} & a_{31} \\ a_{43} & a_{44} & a_{41} \\ a_{13} & a_{14} & a_{11} \end{vmatrix} & -\begin{vmatrix} a_{34} & a_{31} & a_{32} \\ a_{44} & a_{41} & a_{42} \\ a_{14} & a_{11} & a_{12} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} \\
\begin{vmatrix} a_{42} & a_{43} & a_{44} \\ a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \end{vmatrix} & -\begin{vmatrix} a_{43} & a_{44} & a_{41} \\ a_{13} & a_{14} & a_{11} \\ a_{23} & a_{24} & a_{21} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{41} & a_{42} \\ a_{14} & a_{11} & a_{12} \\ a_{24} & a_{21} & a_{22} \end{vmatrix} & -\begin{vmatrix} a_{41} & a_{42} & a_{43} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} \\
-\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} & a_{11} \\ a_{23} & a_{24} & a_{21} \\ a_{33} & a_{34} & a_{31} \end{vmatrix} & -\begin{vmatrix} a_{14} & a_{11} & a_{12} \\ a_{24} & a_{21} & a_{22} \\ a_{34} & a_{31} & a_{12} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{bmatrix}.$$

This new arrangement of the cofactor matrix may be considered, removing the prescribed signs, as a derived matrix obtained by employing Dodgson's condensation to the matrix,

$$\begin{bmatrix}
a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} \\
a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} \\
a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{43} \\
a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} \\
a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} \\
a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33}
\end{bmatrix},$$

which is clearly the interior of the matrix,

$$\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{43} & a_{44} \\
a_{11} & a_{12} & a_{13} & a_{14} & a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} & a_{34} \\
a_{11} & a_{12} & a_{13} & a_{44} & a_{11} & a_{12} & a_{13} & a_{44}
\end{bmatrix},$$

formed by putting $\mathbf{D}$ in four quadrants, adjacent to one another. This proves the method for a 4×4 matrix; and similar proofs might be given for larger matrices.

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