Study for Uniform Convergence and Power Series

Abstract: It has been preliminary researched that function series and power series in mathematical analysis course. There are some basic properties and the basic conclusion in the courses. This article is based on the basic theory and properties, for them to make further in-depth study. First of all, as a necessary tool, it has introduced the two properties of definite integral, it is proved that the continuous function sequence limit problem under the definite integral, then it is defined the sequence of functions on subsets of real number set uniformly Cauchy's concept, basis on them several theorem is proved, it is obtained that results of a series of important properties of function terms. Using of these properties, power series of several important theorems are proved, which is about the important properties of the power series again.

Key words: mathematical analysis course; function series; power series; uniform convergence

1. Introduction

This article assumes that the reader is familiar with the basic theory of mathematical analysis course[1] and its basic results[1-7], basic on these theories and results, The properties of function series[8-11] is been further studied, it is obtained that the important properties of uniform convergence[12-15] and power series[16-18].

Our next theorem shows one can interchange integrals and uniform limits[1-7]. The adjective "uniform" here is important. We don't prove it, but admits it directly because in the mathematical analysis course[1] exist its proof.

Discussion 1. To prove Theorem 1 below we merely use some basic facts about integration which should be familiar [or believable] even if your calculus is rusty. Specifically, we use:

(a)If g and h are integrable on $[a, b]$ and if $g(x) \le h(x)$ for all $x \in [a, b]$, then

$$\int_a^b g(x)dx \le \int_a^b h(x)dx .$$

We also use the following corollary:

(b)If g is integrable on $[a, b]$, then

$$\left|\int_a^b g(x)dx\right| \le \int_a^b |g(x)|dx$$

Continuous functions on closed intervals are integrable, as noted mathematical analysis course[1].

2. The proof of Theorem 1

Now, we begin to prove Theorem 1.

Theorem 1. Let (f_n) be a sequence of continuous functions on $[a, b]$, and suppose

$f_n \to f$ uniformly on [a, b]. Then

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx \qquad (1)$$

Proof. By Theorem[1-7] f is continuous, so the functions are all integrable on [a, b]. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly on [a, b], there exists a number N such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \text{for all} \quad x \in [a, b] \text{ and } n > N.$$

Consequently $n > N$ implies

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| = \left|\int_a^b [f_n(x)dx - f(x)]dx\right|$$

$$\leq \int_a^b |f_n(x) - f(x)|\,dx \leq \int_a^b \frac{\varepsilon}{b-a}dx = \varepsilon.$$

The first $\leq$ follows from Discussion 1(b) applied to $g = f_n - f$ and the second $\leq$ follow from Discussion 1(a) applied to $g = |f_n - f|$ and $h = \dfrac{\varepsilon}{b-a}$; h happens to be a constant function, but this does no harm.

The last paragraph shows that given $\varepsilon > 0$, there exists N such that

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \leq \varepsilon \quad \text{for} \quad n > N.$$

Therefore (1) holds.

Recall one of the advantages of the notion of Cauchy sequence, A sequence (s_n) of real numbers can be shown to converge without knowing its limit by simply verifying that it is a Cauchy sequence. Clearly a similar result for sequences of functions would be valuable, since it is likely that we will not know the limit function in advance. What we need is the idea of "uniformly Cauchy."

3. A definition and its properties about the sequence of functions

Definition 1. A sequence (f_n) of functions defined on a set $S \subseteq R$ is uniformly Cauchy on S if

for each $\varepsilon > 0$ there exists a unmber N such that

$$|f_n(x) - f_m(x)| < \varepsilon \quad \text{for all} \quad x \in S \quad \text{and all} \quad m, n > N. \qquad (1)$$

Compare this definition with that of a Cauchy sequence of real numbers and that of uniform convergence. It is an easy exercise to show uniformly convergent sequences of functions are uniformly Cauchy. The interesting and useful result is the converse, just as in the case of sequences of real numbers.

Theorem 2. Let (f_n) be a sequence of functions and uniformly Cauchy on a set $S \subseteq R$. Then there exists a function f on S such that $f_n \to f$ uniformly on S.

Proof. First we have to "find" f. We begin by showing

for each $x_0 \in S$ the sequence $(f_n(x_0))$ is a

Cauchy sequence of real numbers.. (1)

For each $\varepsilon > 0$, there exists N such that $|f_n(x) - f_m(x)| < \varepsilon$ for $x \in S$ and $m, n > N$.

In particular, we have

$$|f_n(x_0) - f_m(x_0)| < \varepsilon \text{ for } m, n > N.$$

This shows $(f_n(x_0))$ is a Cauchy sequence, so(1) holds.

Now for each x in S, assertion (1) implies $\lim_{n\to\infty} f_n(x)$ exists; this is proved in Theorem[1-7] which in the end depends on the Completeness Axiom. Hence we define $f(x) = \lim_{n\to\infty} f_n(x)$. This defines a function f on S such that $f_n \to f$ uniformly on S.

Now that we have "found" f, we need to prove $f_n \to f$ uniformly on S. Let $\varepsilon > 0$. There is a number N such that

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \text{ for all } x \in S \text{ and all } m, n > N. \quad (2)$$

Consider $m > N$ and $x \in S$. Assertion (2) tells us that $f_n(x)$ lies in the open interval $\left(f_m(x) - \frac{\varepsilon}{2}, f_m(x) + \frac{\varepsilon}{2}\right)$ for all $n > N$. Therefore, as a easy fact, the $f(x) = \lim_{n\to\infty} f_n(x)$ lies in the closed interval $\left[f_m(x) - \frac{\varepsilon}{2}, f_m(x) + \frac{\varepsilon}{2}\right]$. In other words,

$$|f(x) - f_m(x)| \le \frac{\varepsilon}{2} \text{ for all } x \in S \text{ and all } m > N.$$

Then of course

$$|f(x) - f_m(x)| < \varepsilon \text{ for all } x \in S \text{ and all } m > N.$$

This shows $f_m(x) \to f$ uniformly on S, as desired.

Theorem 2 is especially useful for "series of functions." Let us recall what $\sum_{k=1}^{\infty} a_k$ signifies when the a_k 's are real numbers. This signifies $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ provided this limit exists [as a real number, +∞ or -∞]. Otherwise the symbol $\sum_{k=1}^{\infty} a_k$ has no meaning. Thus the infinite series is the limit of the sequence of partial sums $\sum_{k=1}^{n} a_k$. Similar remarks apply to series of functions. A series of functions is an expression $\sum_{k=0}^{\infty} g_k$ or $\sum_{k=0}^{\infty} g_k(x)$ which makes sense provided the sequence of partial sums converges, or diverges to -∞ or +∞ pointwise. If the sequence of partial sums $\sum_{k=0}^{\infty} g_k$ converges uniformly on a set S to $\sum_{k=0}^{\infty} g_k$, then we say the series is uniformly convergent on S.

4. Application and examples

Example 1. Any power series is a series of functions, since $\sum_{k=0}^{\infty} a_k x^k$ has the form $\sum_{k=0}^{\infty} g_k$ where $g_k(x) = a_k x^k$ for all x.

Example 2. $\sum_{k=0}^{\infty} \frac{x^k}{1+x^k} g_k$ is a series of functions, but is not a power series, at least not in its present form. This is a series $\sum_{k=0}^{\infty} g_k$ where $g_0(x) = \frac{1}{2}$ for all x, $g_1(x) = \frac{x}{1+x}$ for all x, $g_2(x) = \frac{x^2}{1+x^2}$ for all x, etc.

Example 3. Let g be the function drawn in Fig. 1,

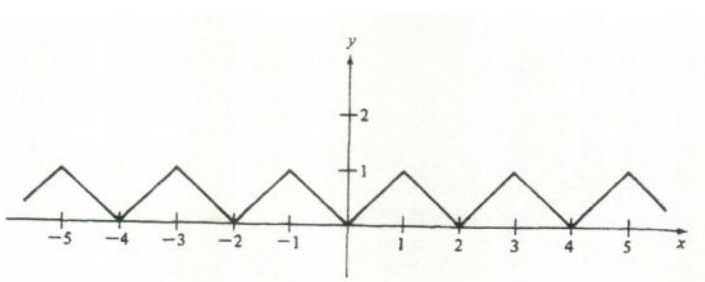

Fig. 1

and let $g_n(x) = g(4^n x)$ for all $x \in R$. Then $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n g_n(x)$ is a series of functions. The limit function f is continuous on R, but has the amazing property that it is not differentiable at any point! The proof of the non-differentiability of f is somewhat delicate[1-7].

Theorems for sequences of functions translate easily into theorems for series of functions. Here is an example.

Theorem 3. Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq R$. Suppose each g_k is continuous on *S* and the series converges uniformly on *S*. Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on *S*.

Proof. Each partial sum $f_n = \sum_{k=1}^{n} g_k$ is continuous and the sequence (f_n) converges uniformly on *S*. Hence the limit function is continuous by Theorem[1-7].

Recall the Cauchy criterion for series $\sum_{k=1}^{\infty} a_k$ given in paper[1-7] :

For each $\varepsilon > 0$ there exists a number *N* such that

$$n \geq m > N \text{ implies } \left|\sum_{k=m}^{n} a_k\right| < \varepsilon . \qquad (*)$$

The analogue for series of functions is also useful. The sequence of partial sums of a series $\sum_{k=0}^{\infty} g_k$ of functions is uniformly Cauchy on a set *S* if and only if the series satisfies the Cauchy criterion [uniformly on *S*]:

For each $\varepsilon > 0$ there exists a number *N* such that

$$n \geq m > N \text{ implies } \left|\sum_{k=m}^{n} g_k(x)\right| < \varepsilon . \text{ for all } x \in S \qquad (**)$$

Theorem 4. If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set *S*, then the series converges uniformly on *S* by Theorem 2.

Here is a useful corollary.

Theorem 5 (M-test). Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all *x* in a set *S*, then $\sum g_k$ converges uniformly on *S*.

Proof. To verify the Cauchy criterion on *S*, let $\varepsilon > 0$. Since the series $\sum M_k$ converges, it satisfies the Cauchy criterion in Definition[1-7]. So there exists a number *N* such that

$$n \geq m > N \text{ implies } \sum_{k=m}^{n} M_k < \varepsilon .$$

Hence if $n \geq m > N$ and *x* is in *S*, then

$$\left|\sum_{k=m}^{n} g_k(x)\right| \le \sum_{k=m}^{n} |g_k(x)| \le \sum_{k=m}^{n} M_k < \varepsilon$$

Thus the series $\sum_{k=0}^{\infty} g_k$ satisfies the Cauchy criterion uniformly on *S*, and Theorem 4 shows it converges uniformly on *S*.

Example 4. Show $\sum_{n=1}^{\infty} 2^{-n} x^n$ represents a continuous function f on (-2, 2), but the convergence is not uniform.

Solution. This is a power series with radius of convergence 2. Clearly the series does not converge at x=2 or at x=-2, so its interval of convergence is (-2, 2).

Consider $0 < a < 2$ and note

$$\sum_{n=1}^{\infty} 2^{-n} a^n = \sum_{n=1}^{\infty} \left(\frac{a}{2}\right)^n$$

converges. Since

$$\left|2^{-n} x^n\right| \le 2^{-n} a^n = \left(\frac{a}{2}\right)^n \text{ for } x \in [-a, a],$$

the Theorem 5 **(M-test)** shows the series converges uniformly to a function on [-a, a]. By Theorem 3 the limit function f is continuous at each point of the set [-a, a]. Since a can be any number less than 2, we conclude f represents a continuous function on (-2, 2).

Since we have $\sup\{|2^{-n} x^n| \mid x \in (-2, 2)\} = 1$ for each n, the convergence of the series cannot be uniform on (-2, 2) in view of the next example.

Example 5. Show that if the series $\sum g_n$ converges uniformly on a set *S*, then

$$\lim_{n\to\infty} \sup\left\{|g_n(x)| \mid x \in S\right\} = 0. \qquad (1)$$

Solution. Let $\varepsilon > 0$. Since the series $\sum g_n$ satisfies the Cauchy criterion, there exists N such that

$$n \ge m > N \text{ implies } \left|\sum_{k=m}^{n} g_k(x)\right| < \varepsilon \text{ for all } x \in S.$$

In particular,

$$n > N \text{ implies } |g_n(x)| < \varepsilon \text{ for all } x \in S.$$

Therefore

$$n > N \text{ implies } \sup\left\{|g_n(x)| \mid x \in S\right\} \le \varepsilon.$$

This establishes (1).

5. Properties of power series

Now we begin to study the properties of the power series.

Theorem 6. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R>0$ [possibly $R=+\infty$]. If $0<R_1<R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.

Proof. Consider $0<R_1<R$. A glance at Theorem[1-7] shows the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} |a_n| x^n$ have the same radius of convergence, since β and R are defined in terms of $|a_n|$. Since $|R_1|<R$, we have $\sum_{n=0}^{\infty} |a_n| R_1^n < \infty$. Clearly we have $|a_n x^n| \le |a_n| R_1^n$ for all x in $[-R_1, R_1]$, so the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R_1, R_1]$ by the Theorem 5 **(M-test)**. The limit function is continuous at each point of $[-R_1, R_1]$ by Theorem 3.

Corollary 7. The power series $\sum a_n x^n$ converges to a continuous function on the open interval $(-R_1, R_1)$.

Proof. If $x_0 \in (-R, R)$ then $x_0 \in (-R_1, R_1)$ for some $R_1 < R$. The theorem shows the limit of the series is continuous at x_0.

We emphasize that a power series need not converge uniformly on its interval of convergence though it might.

We are going to differentiate and integrate power series term-by-term, so clearly it would be useful to know where the new series converge. The next lemma tells us.

Lemma 8. If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence *R*, then the power series

$$\sum_{n=0}^{\infty} n a_n x^{n-1} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence *R*.

Proof. First observe the series $\sum_{n=0}^{\infty} na_n x^{n-1}$ and $\sum_{n=0}^{\infty} na_n x^n$ have the same radius of convergence: since the second series is x times the first series, they converge for exactly the same values of x. Likewise $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^n$ have the same radius of convergence.

Next recall $R = \frac{1}{\beta}$ where $\beta = \limsup |a_n|^{1/n}$. For the series $\sum_{n=0}^{\infty} na_n x^n$, we consider

$$\limsup (n|a_n|)^{1/n} = \limsup n^{1/n} |a_n|^{1/n}.$$

By Theorem[1-7], we have $\lim n^{1/n} = 1$ so $\limsup (n|a_n|)^{1/n} = \beta$ by Theorem[1-7]. Hence the series $\sum_{n=0}^{\infty} na_n x^n$ has radius of convergence R.

For the series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^n$, we consider $\limsup \left(\frac{|a_n|}{n+1} \right)^{1/n}$. It is easy to show $\lim (n+1)^{1/n} = 1$; therefore $\lim \left(\frac{1}{n+1} \right)^{1/n} = 1$. Hence by Theorem[1-7] we have $\limsup \left(\frac{|a_n|}{n+1} \right)^{1/n} = \beta$, so the series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^n$ has radius of convergence R.

Theorem 9. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \text{for } |x| < R. \qquad (1)$$

Proof. We fix x and assume $x < 0$; the case $x > 0$ is similar. On the interval $[x, 0]$, the sequence of partial sums $\sum_{k=0}^{n} a_k t^k$ converges uniformly to $f(t)$ by Theorem 6. Consequently, by Theorem 1 we have

$$\int_x^0 f(t)dt = \lim_{n\to\infty} \int_x^0 \left(\sum_{k=0}^{n} a_k t^k \right) dt$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \int_x^0 t^k dt = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \left[\frac{0^{k+1} - x^{k+1}}{k+1} \right]$$

$$= -\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} \qquad (2)$$

The second equality is valid because we can interchange integrals and finite sums; this is a basic property of integrals[1-7]. Since $\int_0^x f(t)dt = -\int_x^0 f(t)dt$. Eq. (2) implies Eq.(1).

The theorem just proved shows that a power series can be integrated term-by-term inside its interval of convergence. Term-by-term differentiation is also legal.

Theorem 10. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{for } |x| < R. \qquad (1)$$

The proof of Theorem 9 was a straightforward application of Theorem 1 but the direct analogue of Theorem 1 for derivatives is not true[1-7]. So we give a devious indirect proof of the theorem.

Proof. We begin with series $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and observe this series converges for $|x| < R$ by Lemma 8. Theorem 9 shows that we can integrate g term-by-term:

$$\int_0^x g(t)dt = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0 \quad \text{for } |x| < R.$$

Thus if $0 < R_1 < R$, then

$$f(x) = \int_{-R_1}^{x} g(t)dt + k \quad \text{for } |x| < R_1,$$

where k is a constant; in fact,

$$k = a_0 - \int_{-R_1}^{x} g(t)dt.$$

Since g is continuous, one of the versions of the Fundamental Theorem of Calculus[1-7] shows f is differentiable and $f'(x) = g(x)$. Thus

$$f'(x) = g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{for } |x| < R.$$

Example 6. Recall

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x| < 1. \qquad (1)$$

Differentiating term-by-term, we obtain

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2} \quad \text{for } |x|<1.$$

Integrating (1) term-by-term, we get

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \int_0^x \frac{1}{1-t} dt = -\log_e(1-x)$$

$$\log_e(1-x) = -\sum_{n-1}^{\infty} \frac{1}{n} x^n \quad \text{for } |x|<1. \qquad (2)$$

Replacing x by-x, we find

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } |x|<1. \qquad (3)$$

It turns out that this equality is also valid for x=1[see Example 7], so we have the interesting identity

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \qquad (4)$$

In Eq. (2) set $x = \frac{m-1}{m}$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{m-1}{m}\right)^n = -\log_e\left(1-\frac{m-1}{m}\right) = -\log_e\left(\frac{1}{m}\right) = \log_e m$$

Hence we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{m-1}{m}\right)^n = \log_e m \quad \text{for all } m.$$

Here is yet another proof that $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$.

To establish (4) we need a relatively difficult theorem about convergence of a power series at the endpoints of its interval of convergence.

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence R. If the series converges at x=R, then f is continuous at x=R. If the series converges at x=R, then f is continuous at x=-R.

Example 7. As promised, we return to (3) in Example 1:

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } |x|<1.$$

For x=1 the series converges by the Alternating Series Theorem[1-7]. Thus the series represents a function f on (-1, 1] that is continuous at x=1 by Abel's theorem. The function $\log_e(1+x)$

is also continuous at x=1 so the functions agree at x=1. [In detail, if (x_n) is a sequence in (-1, 1) converging to 1, then $f(1)=\lim_{n\to\infty} f(x_n) \; =\lim_{n\to\infty}\log_e\left(1+x_n\right)=\log_e 2$.] Therefore we have

$$\log_e 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots.$$

Example 8. Recall $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$. Note that at x=-1 the function $\frac{1}{1-x}$ is continuous and takes the value $\frac{1}{2}$. However, the series does not converge for x=-1, so Abel's theorem does not apply.

The point of view in our extremely brief introduction to power series has been: For a given power series $\sum a_n x^n$, what can one say about the function $f(x)=\sum a_n x^n$? This point of view was misleading. Often, in real life, one begins with a function f and seeks a power series that represents the function for some or all values of x. This is because power series, being limits of polynomials, are in some sense basic objects.

If we have

$$f(x)=\sum_{n=0}^{\infty} a_n x^n \quad \text{for } |x|<R,$$

then we can differentiate f term-by-term forever. At each step, we may calculate the kth derivative of f at 0, written $f^{(k)}(0)$. It is easy to show $f^{(k)}(0)=k!a_k$ for $k\ge 0$. This tells us that if f can be represented by a power series, then that power series must be $\sum_{n=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$. This is the Taylor series for f about 0. Frequently, but not always, the Taylor series will agree with f on the interval of convergence.

6. Conclusions

From the above, we have seen that the properties of the power series are very perfect, it is an extremely rare class of function series; in addition, Cauchy criterion has played important role. Using Cauchy criterion as a tool, not only can derive many properties of number series, but it can also be derived a lot of properties of function series in the deep. In addition, the limit thought is never less important tool in our study.

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