Original Research Article

ON THE IMPACT OF CONSUMPTION ON AN INVESTOR'S STRATEGY UNDER STOCHASTIC INTEREST RATE AND CORRELATING BROWNIAN MOTIONS

Abstract

This work considered an investor's portfolio to comprise of two assets- a risk-free asset driven by Ornstein-Uhlenbeck Stochastic interest rate of return model and a risky stock which price process is governed by the geometric Brownian motion where consumption, taxes, transaction costs and dividends are in involved. The aim was to investigate the effect of consumption on an investor's trading strategy under correlating Brownian motions. Using maximum principle the relating Hamilton-Jacobi-Bellman (HJB) equation was obtained. The application of elimination of variable dependency gave optimal investment strategy for the investor's problem. Among the findings is that more fund should be made available for investment on the risky asset when there is consumption to keep the investor solvent.

Keywords: effect of consumption, Hamilton-Jacobi-Bellman (HJB) equation, optimal investment, Ornstein-Uhlenbeck, power utility, stochastic interest rate, maximum principle.

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1. Introduction

In the field of mathematical finance asset allocation problems in continuous time framework are among the most widely studied problems, and dates back to Merton [1, 2]. In the Merton's original work provided explicit solutions on how one's expected utility is maximized while trading on stocks and consumption taking place as the underlying assets follow the Black–Scholes–Merton model with specific utility preference. After these pioneer works, many researches have been done and more are going on in many facets of Mathematical Finance. Among them, some allow for imperfections in the financial markets, Magill and Constantinides [3]. In the case of transaction costs, Guasoni and Muhle-Karbe [4], have made contributions. For investment under drawdown constraint, contributors include, Elie and Touzi [5]. In the case of trading with price impact we have, Cuoco and Cvitani´c [6] etc.
In the area of the volatility being stochastic, contributors include Zariphopoulou [7], Chacko and Viceira [8], Fouque et al. [9] and Lorig and Sircar [10].
Empirical studies have shown that non-Markovian (dependence) structure models in long-term investment which is much related to daily data and long range dependence exhibits in both return and volatility describe the data better, (Cont [11], Chronopoulou and Viens [12]).
The introduction of transaction costs into the investment and consumption problems follow from the works of, Shreve and Soner [13], Akian et al. [14], and Janeˇcek and Shreve [15]. Investigators into optimal consumption problem with borrowing constraints include, Fleming and Zariphopoulou [16], Vila and Zariphopoulou [17], Ihedioha [18] and Yao and Zhang [19].

The mentioned models were studies under the assumption that the risky asset's price dynamics was driven by the geometric Brownian motion (GBM) and the risk-free asset with a rate of return that is assumed constant. Some authors have studied the problem under the extension of geometric Brownian motion (GBM) called the constant elasticity of variance (CEV) model which is a natural extension of the geometric Brownian motion (GBM). The constant elasticity of variance (CEV) model has an advantage that the volatility rate has correlation with the risky asset price. Cox and Ross originally proposed the use of constant elasticity of variance (CEV) model as an alternative diffusion process for pricing European option; Cox and Ross [20]. Schroder [21], Lo et al. [22], Phelim and Yisong [23], and Davydov and Linetsky [24] have applied it to analyze the option pricing formula. Further applications of the constant elasticity of variance (CEV) model, in the recent years, has been in the areas of annuity contracts and the optimal investment strategies in the utility framework using dynamic programming principle.
Detailed discussions can be found in, Xiao et al.[25], Gao [26, 27], Gu et al. [28], Lin and Li [29], Gu et al.[30], Jung and Kim [31] and Zhao and Rong [32].
This paper aims at investigating and giving a closed form solution to an investment and consumption decision problem where the risk-free asset has a rate of return that is driven by the Ornstein-Uhlenbeck Stochastic interest rate of return model. Dynamic programming principle, specifically, the maximum principle is applied to obtain the HJB equation for the value function.
The rest of this paper is organized as follows: In section 2 is the problem formulation and the model. In section 3, maximum principle is applied to obtain, the HJB equation, the optimal investment strategy and the impact of consumption investigated. Section 4 concludes the paper.

2. The problem formulation:

Two cases are considered in the work, thus;

1. When there is no consumption
2. When there is consumption

2.1. When there is no consumption

We assume that an investor trades two assets in an economy continuously-c riskless asset (bond) and a risky asset (stock), Let the price of the riskless asset be denoted by $P(t)$ with a rate of return$r(t)$ which is stochastic and driven by the Orinstein-Uhlenbeck model.That is

$$dP(t) = r(t)P(t)dt \qquad (1)$$

where

$$dr(t) = \alpha\big(\beta - r(t)\big)dt + \sigma dz_1(t) \colon r(0) = r_0 \qquad (2)$$

where α is the speed of mean reversion, β the mean level attracting the interest rate and σ the constant volatility of the interest rate. $Z_1(t)$ is a standard Brownian motion. Also, let the price of the risky asset be denoted by$P_1(t)$ with the process

$$dP_1(t) = P_1(t)[\mu dt + \lambda dZ_2(t)], \qquad (3)$$

where μ and λ are constants and μ the drift parameter while λ is the diffusion parameter.$z_2(t)$ is another standard Brownian motion.
Through this work, we assume a probability space $(\Omega, \mathcal{F}, \rho)$ and a filtration $\{\mathcal{F}_t\}$. Uncertainty in the models are generated by the Brownian motions $Z_1(t)$and $Z_2(t)$.
Let $\pi(t)$ to the amount of money the investor decides to put in the risky asset at time t,then the balance $[X(t) - \pi(t)]$ is the amount to be invested in the riskless assets, where$w(t)$ is the total amount of money available for investment.
Assumption:

We assume that transaction cost, tax and dividend are paid on the amount invested in the risky asset at constant rates, σ, θ and d respectively. Therefore for any policy π, the total wealth process of the investor follows the stochastic differential equation (SDE)

$$dX^{\pi}(t) = \pi(t)\frac{dP_1(t)}{P_1(t)} + [X(t) - \pi(t)]\frac{dP(t)}{P(t)} - (\vartheta + \theta - d)\pi(t)dt. \quad (4)$$

Applying (1) and (3) in (4) gives

$$dX^{\pi}(t) = \{[(\mu + d) - (r(t) + \vartheta + \theta)]\pi(t) + r(t)X(t)\}dt + \lambda\pi(t)dZ_2(t). \quad (5)$$

Suppose the investor has a utility function $U(.)$ which is strictly concave and continuously differentiable on $(-\infty, +\infty)$ and wishes to maximize his expected utility of terminal wealth, then his problem can therefore be written as

$$\underset{\pi}{Max}\epsilon[U(X(T))] \quad (6)$$

subject to (5).

This work assumes a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and uncertainties in the models are generated by the Brownian motions $Z_1(t)$ and $Z_2(t)$.

Case 2: When there is consumption

Further assumptions is that consumption withdrawals are made from the risk-free account, therefore for any trading strategy $(\pi(t), K(t))$ the total wealth process of the investor follows the stochastic differential equation (SDE)

$$dX(t) = \pi(t)\frac{dP_1(t)}{P_1(t)} + [X(t) - \pi(t)]\frac{dP(t)}{P(t)} - [(\vartheta + \theta - d)\pi(t) + K(t)]dt, \quad (7)$$

where $K(t)$ is the rate of consumption.

Applying (2) and (3) in (4) obtains:

$$dX(t) = \pi(t)[\mu dt + \lambda dZ_2(t)] + [X(t) - \pi(t)]r(t)dt - [(\vartheta + \theta - d)\pi(t) + K(t)]dt. \quad (8)$$

which becomes

$$dX(t) = \{[(\mu + d) - (r + \vartheta + \theta)]\pi(t) + r(t)X(t) - K(t)\}dt + \lambda\pi(t)dZ_2(t). \quad (9)$$

Definition: (admissible strategy). An investment and consumption $(\pi(t), K(t))$ strategy is said to be admissible if the following conditions are satisfied:

i. $(\pi(t), k(t))$ is $\mathcal{F}_t$ −progressively measurable and

ii. $\int_0^T \pi^2(t)dt < \infty, \int_0^T k(t)dt < \infty \ ; \ \forall\, T > 0$ (10)

iii. $E\left[\int_0^T (\lambda^2\pi^2(t))dt\right] < \infty$ (11)

iv. For $\forall$ $(\pi(t), k(t))$,the stochastic differential equation (9) has a unique solution, Chang et al. [33].

Assuming the set of all admissible investment and consumption strategies $(\pi(t), k(t))$ is denoted by $B = \left[\left(\pi(t), k(t)\right): 0 \leq t \leq T\right]$,then the investor`s problem can be stated mathematically thus:

$$\mathrm{Max}_{[\pi(t),k(t)]\in B}\, E[(U(X(T)]. \quad (12)$$

This study considers the power utility function given by

$$U\left(X(t)\right) = \frac{X^{1-\phi}}{1-\phi}; \phi \neq 1. \quad (13)$$

Using the classical tools of stochastic optimal control where consumption is involved, define the value function at time t as:

$$G(t, r(t), P_1(t), X(t)) = \underset{B}{sup}\ E\left[\int_0^T e^{-\varrho\tau}\frac{K^{1-\phi}}{1-\phi}d\tau + e^{-\varrho T}\frac{{X_T}^{1-\phi}}{1-\phi}\right];$$

$P_1(t) = p_1; X(t) = x; r(t) = r, K(t) = k; 0 < t < T$ (14)

Therefore the investor`s problem becomes

$$G(t,r,p_1,x) = \sup_{[\pi(t),k(t)] \in B} E\left[\int_0^T e^{-\varrho\tau} \frac{k^{1-\phi}}{1-\phi} d\tau + e^{-\varrho T} \frac{x^{1-\phi}}{1-\phi}\right] \quad (15)$$

subject to (9).

3. The Optimal investment strategy for the power utility function

Here we obtain the explicit strategies for the optimization problem using the maximum principle and stochastic control.

3.1. When there is no consumption

Define the value function as

$$G(t,r,p_1,x) = {}^{Max}_{\pi}[\epsilon(U(w)] = 0; U(T,W) = U(w), 0 < t < T$$
$$r(t) = r, X(t) = x, P_1(t) = p_1, \quad (16)$$

then the Hamilton-Jacobi-Bellman equation (HJB) is

$$G_t + \alpha(\beta - r)G_r + \mu p_1 G_{p_1} + \{[(\mu + d) - (r + \vartheta + \theta)]\pi + rx\}G_x + \lambda^2 p_1 \pi G_{p_1 x} + \rho\sigma p_1 G_{rp_1} +$$
$$\rho\sigma\lambda\pi G_{rx} + \frac{1}{2}\left[\sigma^2 G_{rr} + \lambda^2 {p_1}^2 G_{p_1 p_1} + \lambda^2 \pi^2 G_{xx}\right] = 0 \quad (17)$$

where the Brownian motions have correlate correlation coefficient ρ.

G_t, G_{p_1}, G_x and G_r, are first partial derivatives with respect to $t, s, wandr$ respectively. Also $G_{rp_1}, G_{rx}, G_{p_1 x}, G_{rr}, G_{p_1 p_1}$ and G_{xx} are second partial derivatives.

Differentiating (17) with respect to π gives

$$[(\mu + d) - (r + \vartheta + \theta)]G_x + \lambda^2 G_{p_1 x} + \rho\sigma\lambda G_{rx} + \lambda^2 \pi G_{xx} = 0, \quad (18)$$

and the optimal strategy

$$\pi^*_{d,\vartheta,\theta} = \frac{-[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2 G_{xx}} - \frac{p_1 G_{p_1 x}}{G_{xx}} - \frac{\rho\sigma\lambda G_{rx}}{\lambda^2 G_{xx}}, \quad (19)$$

To eliminate the dependency on x, let the solution to the HJB equation (17) be

$$G(t,r,p_1,x) = H(t,r,p_1)\frac{x^{1-\phi}}{1-\phi}, \quad (20)$$

with boundary condition

$$H(T,r,p_1) = 1, \quad (21)$$

Then we obtain from (20)

$$G_x = x^{-\phi}H,\ G_{xx} = -\phi x^{-\phi-1}H,\ \ G_{p_1 x} = x^{-\phi}H_{p_1}, G_{rx} = x^{-\phi}H_r. \quad (22)$$

Applying the equivalent of $G_x, G_{xx}, G_{p_1 x}, and\ G_{rx}$ from equation (22) to (19) gives

$$\pi^*_{d,\vartheta,\theta} = \frac{[(\mu+d)-(r+\vartheta+\theta)]x}{\lambda^2} + \frac{p_1 x H_{p_1}}{\phi H} + \frac{\rho\sigma x H_r}{\lambda\phi H}. \quad (23)$$

To eliminate dependency on p_1, we further conjecture that

$$H(t,r,p_1) = \frac{{p_1}^{1-\phi}}{1-\phi} I(t,r), \quad (24)$$

where

$$I(T,r) = \frac{1-\phi}{{p_1}^{1-\phi}}. \quad (25)$$

We obtain the following from (24)

$$H_r = \frac{{p_1}^{1-\phi}}{1-\phi} I_r, H_{p_1} = {p_1}^{-\phi} I. \quad (26)$$

Using (24) and (26) in (23) gives

$$\pi^*_{d,\vartheta,\theta} = \left[\frac{[(\mu+d)-(r+\vartheta+\theta)]x}{\lambda^2} + \frac{(1-\phi)x}{\phi} + \frac{\rho\sigma x I_r}{\lambda\phi I}\right]. \quad (27)$$

We conjecture further that

$$I(t,r) = \frac{r^{1-\phi}}{1-\phi}J(t), \quad (28)$$

to eliminate dependency on r such that at the terminal time T,

$$J(T) = \frac{(1-\phi)^2}{(rp_1)^{1-\phi}}. \quad (29)$$

From (28) obtains,

$$I_r = r^{-\phi}J. \quad (30)$$

Therefore equation (27) becomes

$$\pi^*_{d,\vartheta,\theta} = x\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + \frac{1-\phi}{\phi} + \frac{(1-\phi)\rho\sigma}{\lambda\phi r}\right], \quad (31)$$

the optimal investment in the risky asset.

3.2. When there is consumption

The derivation of Hamilton-Jacobi-Bellman (HJB) partial differential starts with the Bellman;

$$G(t,r,p_1,x) = sup_\pi\left\{\frac{K^{1-\phi}}{1-\phi} + \frac{1}{1+\zeta}E[G(t+\Delta t, r', x')]\right\}. \quad (32)$$

The actual utility over time interval of length Δt is $\frac{C^{1-\phi}}{1-\phi}\Delta t$ and the discounting over such period is expressed as $\frac{1}{1+\zeta\Delta t}$, $\zeta > 0$.

Therefore, the Bellman equation becomes;

$$G(t,r,p_1,x) = sup_\pi\left\{\frac{K^{1-\phi}}{1-\phi}\Delta t + \frac{1}{1+\vartheta\Delta t}E[G(t+\Delta t, r', p_1', x')]\right\}. \quad (33)$$

The multiplication of (13) by $(1+\zeta\Delta t)$ and rearranging terms obtains;

$$\vartheta G(t,r,p_1,x)\Delta t = sup_\pi\left\{\frac{K^{1-\phi}}{1-\phi}\Delta t\,(1+\zeta\Delta t) + E(\Delta G)\right\}. \quad (34)$$

Dividing (14) by Δt and taking limit to zero, obtains the Bellman equation;

$$\zeta G = sup_\pi\left\{\frac{K^{1-\phi}}{1-\phi} + \frac{1}{dt}E(dG)\right\}. \quad (35)$$

Applying the maximum principle obtains the corresponding Hamilton-Jacob-Bellman equation (HJB) as

$$\frac{k^{1-b}}{1-b} + G_t + \mu p_1 G_{p_1} + \alpha(\beta - r)G_r + \{[(\mu+d) - (r+\vartheta+\theta)]\pi + rx - k\}G_x + \rho\sigma x p_1 G_{rp_1} + \rho\sigma\lambda\pi G_{rx} + \lambda^2\pi p_1 x G_{p_1} + \frac{1}{2}\left[\sigma^2 G_{rr} + \lambda^2 {p_1}^2 G_{p_1p_1} + \lambda^2\pi^2 G_{xx}\right] - \zeta G = 0. \quad (36)$$

$G_t, G_{p_1} and\ G_x$ are first partial derivatives $G_{rp_1}, G_{rx}, G_{p_1x}, G_{rr}, G_{p_1p_1} and\ G_{xx}$ are second order partial derivatives.

Differentiating (36) with respect to π gives the optimal investment in the risky asset as;

$$\pi^* = \frac{-[(\mu+d)-(r+\vartheta+\theta)]G_x}{\lambda^2 G_{xx}} - \frac{\rho\sigma}{\lambda}\frac{G_{rx}}{G_{xx}} - \frac{p_1 G_{p_1}}{G_{xx}}. \quad (37)$$

To cope with this, it is conjectured that a solution of the form

$$G(t,r,p_1,x) = \frac{x^{1-\phi}}{1-\phi}J(t,r,p_1), \quad (38)$$

such that

$$J(T,r,p_1) = 1. \quad (39)$$

Eliminates the dependency on x.

From (38) obtain

$$G_x = x^{-b}J, G_{xx} = -\phi x^{-\phi-1}J, G_s = \frac{x^{1-\phi}}{1-\phi}J_{p_1}, G_{rx} = x^{-\phi}J_r. \quad (40)$$

Applying the equivalents of G_x, G_{rx} , G_{p_1x}, and G_{xx} from (40) to (37) yields

$$\pi^* = \frac{[(\mu+d)-(r+\vartheta+\theta)]x}{\phi\lambda^2} + \frac{\rho\sigma x J_r}{\phi\lambda J} + \frac{p_1 x J_{p_1}}{\phi\ J} . \quad (41)$$

To continue we conjecture that

$$J(t,r,p_1) = H(t,r)\frac{{p_1}^{1-\phi}}{1-\phi} , \quad (42)$$

such that

$$H(T,r) = \frac{1-b}{{p_1}^{1-b}} , \quad (43)$$

at the terminal time T, and dependency on p_1 eliminated.
Obtained from (42) are

$$J_r = \frac{{p_1}^{1-\phi}}{1-\phi}H_r; J_{p_1} = {p_1}^{-\phi}H. \quad (44)$$

The application of the equivalents of J_r and J_{p_1} from (44) and (42) to (41) gives the following

$$\pi^* = \frac{[(\mu+d)-(r+\vartheta+\theta)]x}{\phi\lambda^2} + \frac{1-\phi}{\phi}x + \frac{\rho\sigma x}{\phi\lambda}\frac{H_r}{H}, \quad (45)$$

for the optimal investment is the risky asset.
To eliminate the dependency on r, the conjecture that

$$H(t,r) = I(t)\frac{r^{1-\phi}}{1-b}, \quad (46)$$

is used such that

$$I(T) = \frac{(1-\phi)^2}{(rp_1)^{1-\phi}}, \quad (47)$$

at the terminal time T.
From (46) obtains

$$H_r = r^{-\phi}I. \quad (48)$$

Applying the equivalent of H_r from (48) and (46) to (45) yields

$$\pi^* = \frac{[(\mu+d)-(r+\vartheta+\theta)]x}{\phi\lambda^2} + \frac{(1-\phi)x}{\phi} + \frac{(1-\phi)\rho\sigma x}{r\phi\lambda}$$
$$= \frac{x}{\phi}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + (1-\phi)\left(1+\frac{\rho\sigma}{r\lambda}\right)\right] . \quad (49)$$

3.3. The effect of the consumption

We shall assume that $\phi \neq 1$ and $\phi > 0$.
Let π^{*NC} and π^{*C} denote the optimal investment in the risky asset when there is no consumption and when there is consumption respectively. Therefore we have the following:

1. When there is no consumption; equation (31) gives;

$$\pi^{*NC}_{d,\vartheta,\theta} = x\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + \frac{(1-\phi)}{\phi} + \frac{(1-\phi)\rho\sigma}{\lambda r\phi}\right]. \quad (50)$$

2. When there is consumption, equation (49) becomes,

$$\pi^{*C}_{d,\vartheta,\theta} = \frac{x}{\phi}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + (1-\phi)\left(1+\frac{\rho\sigma}{r\lambda}\right)\right]. \quad (51)$$

Taking ratio gives:

$$\frac{\pi^{*NC}_{d,\vartheta,\theta}}{\pi^{*C}_{d,\vartheta,\theta}} = \frac{\left[x\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2}+\frac{1-\phi}{\phi}+\frac{(1-\phi)\rho\sigma}{\lambda r\phi}\right]\right.}{\left[\frac{x}{\phi}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2}+(1-\phi)\left(1+\frac{\rho\sigma}{r\lambda}\right)\right]\right.}. \quad (52)$$

Notice:

1. $$\lim_{\phi\to 1}\left[\frac{\pi^{*NC}_{d,\vartheta,\theta}}{\pi^{*C}_{d,\vartheta,\theta}}\right] = 1. \quad (53)$$

2. $$\lim_{\phi\to\infty}\left[\frac{\pi^{*NC}_{d,\vartheta,\theta}}{\pi^{*C}_{d,\vartheta,\theta}}\right] = 1 - \left[\frac{r[(\mu+d)-(r+\vartheta+\theta)]}{\lambda(\lambda r+\rho\sigma)}\right] . \quad (54)$$

Since the investor holds the risky asset as long as $[(\mu+d)-(r+\vartheta+\theta)] > 0$ and λ, r, ρ, σ are all positive constants, the

$$\left[\frac{r[(\mu+d)-(r+\vartheta+\theta)]}{\lambda(\lambda r+\rho\sigma)}\right] = k , \quad (55)$$

is positive So,

$$\lim_{\phi\to\infty}\left[\frac{\pi^{*NC}_{d,\vartheta,\theta}}{\pi^{*C}_{d,\vartheta,\theta}}\right] = 1 - k. \quad (56)$$

This implies that the limit of the investment in risky asset when there is no consumption is less than that of when there is consumption. Put in another way, when there is consumption, more fund is required for investment in the risky asset to keep the investor solvent.

3.4. Findings

1. When there is no consumption:
   Equation (31)

$$\pi^{*NC}_{d,\vartheta,\theta} = x\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + \frac{1-\phi}{\phi} + \frac{(1-\phi)\rho\sigma}{\lambda\phi r}\right]$$

shows that the investment in the risky a fraction of the total amount available for investment which becomes dependent on x, ρ, σ, λ,r and ϕ whenever $[(\mu+d)-(r+\vartheta+\theta)] = 0$.

2. When there is consumption:
   It can be seen from equation (49)

$$\pi^{*C}_{d,\vartheta,\theta} = \frac{x}{\phi}\left[\frac{[(\mu+d)-(r+\vartheta+\theta)]}{\lambda^2} + (1-\phi)\left(1+\frac{\rho\sigma}{r\lambda}\right)\right].$$

that the optimal investment is a ratio of the total amount available for investment and the relative risk aversion coefficient.

3. From the effect of consumption, more fund is required for investment on the risky asset when There is consumption to keep the investor solvent.

4. Conclusions

This work investigated the effect of consumption on the investment strategy of an investor. It assumed that the price process of the risk less asset has a rate of return that is driven Ornstein-Uhlenbeck model. Using the maximum principle and conjectures on elimination of variables obtained the optimal investment strategy of investor who has power utility preference where taxes, transaction costs and dividend payments are charged and paid.

It was found that the investment in the risky a fraction of the total amount available for investment which becomes dependent on x, ρ, σ, λ,r and ϕ whenever $[(\mu+d)-(r+\vartheta+\theta)] = 0$, when there was no consumption, while when there was consumption, the optimal investment in the risky asset was a ratio of the total amount available for investment and the relative risk aversion coefficient. Also, consumption resulted that more fund is required for investment on the risky asset if the investor is to remain in business.

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