

A new look at formulation of charge storage in capacitors and application to classical capacitor and fractional capacitor theory

Shantanu Das

Reactor Control System Design Section E & I Group B.A.R.C
Mumbai-400085

Honorary Senior Research Professor CMPRC Dept. of Physics Jadavpur University, Ex-UGC-Visiting
Fellow University of Calcutta, Kolkata & Adjunct Professor DIAT Pune.

Email: shantanu@barc.gov.in, shantanu.das@live.com

Dedicated to my Teachers and Professors of Kendiya Vidyalaya and BITS Pilani

ABSTRACT

In this study, we revisit the concept of classical capacitor theory-and derive possible new explanations to the definition charge stored in a capacitor. We introduce the capacity function with respect to time to describe the charge storage in a classical capacitor and a fractional capacitor. Here we will describe that charge stored at any time in a capacitor as ‘convolution integral’ of defined capacity function of a capacitor and voltage stress across it which comes from causality principle. This approach however is different to the conventional method, where we multiply the capacity and voltage functions to obtain charge stored. This new concept is in line with the observation of charge stored as a step function and the relaxation current in form of impulse function for ‘ideal geometrical capacitor’ of constant capacity; when an uncharged capacitor is impressed with a constant voltage stress. Also this new formulation is valid for a power-law decay current that is given by ‘universal dielectric relaxation law’ called as ‘Curie von-Schweidler law’, when an uncharged capacitor is impressed with a constant voltage stress. This universal dielectric relaxation law gives rise to fractional derivative relating voltage stress and relaxation current that is formulation of ‘fractional capacitor’. A ‘fractional capacitor’ we will discuss with this new concept of redefining the charge store definition i.e. via this ‘convolution integral’ approach, and obtain the loss tangent value. We will also show how for a ‘fractional capacitor’ by use of ‘fractional integration’ we can convert the fractional capacity a constant that is in terms of fractional units (Farads per sec to the power of fractional number) ; to normal units of Farads. From the defined capacity function, we will also derive integrated capacity of capacitor. We will also give possible physical explanation by taking example of porous and non-porous pitchers of constant volume holding water and thus, explaining the various interesting aspects of classical capacitor and fractional capacitor that we arrive with this new formulation; and also relate to a capacitor breakdown theory-due to electrostatic forces.

Keywords: causality, convolution integral, fractional derivative, fractional integration, Curie-von Schweidler law, fractional capacitor, geometric capacity, time varying capacity function, integrated capacity, loss tangent.

1. Introduction

The classical geometric capacitor or a constant capacitor (that we are used to since our school days) having constant value of Farad means that it has constant value at all the frequencies from DC value of zero Hertz to infinite Hertz. This is how we measure a capacitor value-by AC bridge circuit. That is we get the value of capacitor say $0.47\mu\text{F}$ in the AC bridge operation at 1KHz ; and take this value in our circuit for DC to high frequency application. Thus, we say this $0.47\mu\text{F}$ is valid for all frequencies-and is termed as ‘rated capacitor’. This is ideal capacitor as though the dielectric used is having loss less relative permittivity ϵ_r and is a constant (i.e. a purely real number with loss tangent value as zero) at all the frequencies. The capacity in this classical sense is given as $C_1 = \epsilon_0 \epsilon_r A / d$ i.e. by using geometric factor of ratio of area to the electrode separation. This we have learnt in textbooks. The ideal capacity that is constant at all the frequencies is called geometric capacity. This constant value C_1 in frequency domain is actually an impulse function in time domain i.e. $C_1 \delta(t)$. This time domain capacity i.e. $c(t) = C_1 \delta(t)$ for ideal capacitor we term as time dependent ‘capacity function’; which happens to be a delta function for ideal capacitor case. A general practical capacitor, which is not a constant in frequency domain, is having a function in time domain and we call it as capacity function in time, representing as $c(t)$. Where the frequency domain representation is via Laplace transformation i.e. $\mathcal{L}\{c(t)\} = C(s)$.

We will derive that charge stored in capacitor, as a function of time is not usual multiplication operation of capacity function and voltage stress i.e. $q(t) \neq c(t)v(t)$; instead, the charge is ‘convolution integral’ of the two i.e. $q(t) = c(t) * v(t)$. This comes from causality principle. However, the charge described in frequency domain as a function of frequency is multiplication operation of frequency domain functions of capacity-function and voltage-function, i.e. $Q(s) = C(s)V(s)$. We will revise this concept of capacitor in the paper, and derive various interesting concepts.

The Curie-von Schweidler law relates to the relaxation current in dielectric when a step DC voltage is applied and is given by $i(t) \sim t^{-n}$, where $t > 0$ and the power (exponent) i.e. n is called relaxation constant or decay constant, where $0 < n < 1$ [1]-[4], [12], [21], [22]. We note that n is non-integer. This relaxation law is taken as ‘universal law’, for dielectric relaxations. The Curie-von Schweidler behavior has been observed in many instances, since late 19th Century, such as those shown in dielectric studies and experiments [3] [4], [12], [13]-[17], [21], [22]. This power law relaxation of the ‘non-Debye’ type i.e. $i(t) \sim t^{-n}$ is interpreted as a many-body problem but can also be formulated as an infinite number of independent relaxing bodies meaning infinite number of relaxation rates varying from zero to infinity [4], [5], [6], [22]. The power law relaxation is observed in the experiments with super-capacitors [7]-[12], [21] [22]. These studies [7]-[11], [21], [22] with non-Debye relaxation function (i.e. power-law relaxation) also indicate the use of fractional calculus as constituent expression to describe super-capacitors. A very low value of exponent n is found in relaxation of Laponite studies averagely $n = 0.09$ [18]. In this Laponite study [18] though the exponent n was obtained on ‘self-discharge’ curves with various charging time history-showing memory effect, the expression obtained for self-discharge decay of voltage assumes fractional capacity-that in turn assumes Curie-von Schweidler law as current relaxation function. The Electrical Circuits that is composed of fractional order elements as circuit components are analyzed by Fractional Calculus [23]-[26].

The use of empirical power law i.e. Curie-von Schweidler Law of relaxation of current to a step input of voltage to get constituent relation with fractional derivative was proposed in [12] [21], by taking the concept of charge stored at any time as usual product of capacity function and voltage stressed i.e. $q(t) = c(t)v(t)$. We will revise the concept of capacitor in classical theory and apply the new concept of

charge stored at any time as convolution integral of capacity function and the voltage stress i.e. $q(t) = c(t) * v(t)$ and apply this concept in capacitors with observed Curie-von Schwedler relaxation current, and obtain same results as in [12] and [21]. We will also point out the differences with this new approach to the earlier approach in finding the capacity function and loss-tangent. Conventional classical theories describe the constituent equations i.e. the relationship between the field quantities by ordinary differential equations while the new theory requires fractional differential equations (rather integral equations) [4], [12], [21], [22].

The observation regarding capacitor breakdown is very interesting. We have practically seen especially for DC link capacitors (in power supply circuits), the capacitor breaks down even it were never exceeded its maximum voltage limit. Hence, the capacitor has other break down mechanism too, i.e. due to electrostatic forces that build up due to infinite charge stored when capacitor is afloat at a constant DC voltage that is very less than the maximum voltage limit. This explanation of accumulation of infinite charge is only possible by expressing capacitor dynamics by fractional calculus [12]-[14], [21]. It is experimentally verified fact that a capacitor given infinite time, may accept much more charge than to its rated capacitance measured at 1KHz [12], [21], [31]. We will deal with this breakdown concept with our new formulation of charge storage.

2. A brief about ideal capacitor

2.1: Ideal Loss less capacitor & Loss Tangent and its time varying ‘capacity function’

What we know about geometric capacitor or a constant capacitor of say value C_1 is a constant value of Farad at all the frequencies from DC value of zero Hertz to infinite Hertz. This is ideal capacitor as though the dielectric used ϵ_r is lossless and is constant at all frequencies; and the capacity is given as $C_1 = \epsilon_0 \epsilon_r A / d$ i.e. by using geometric factor of area to electrode separation ratio. This ideal capacity is constant at all the frequencies is called geometric capacity. Therefore, if we say s as complex frequency (Laplace variable) then this constant capacity is given as function in frequency domain $C(s) = C_1$. With $s = j\omega$, $j = \sqrt{-1}$, we have $C(\omega) = C_1$. The Laplace complex frequency is written in as $s = j\omega$ for writing sinusoidal or steady state frequency domain analysis [12], [19], [21]. From $C(\omega) = C_1$ we see that $C(\omega) = \text{Re}[C(\omega)] - j \text{Im}[C(\omega)] = C_1 - j(0)$ has only real part with imaginary part as zero at all frequencies. That gives loss tangent as $\tan \phi = \frac{\text{Im}[C(\omega)]}{\text{Re}[C(\omega)]} = 0$ Thus; ideal capacitor is a loss less capacitor. The dielectric loss is expressed as loss tangent for a complex dielectric quantity given as $\epsilon_r(\omega) = \text{Re}[\epsilon_r(\omega)] - j \text{Im}[\epsilon_r(\omega)]$ where loss tangent is given as $\tan \phi = \frac{\text{Im}[\epsilon_r(\omega)]}{\text{Re}[\epsilon_r(\omega)]}$.

Since the inverse Laplace transform of function i.e. $F(s) = 1$ gives time function i.e. $f(t) = \mathcal{L}^{-1}\{F(s)\} = \delta(t)$ i.e. a Dirac delta function at $t = 0$, we say the ‘time varying capacity function’ call it $c(t)$ of geometric capacitor (ideal-capacitor) is $c(t) = C_1 \delta(t)$ for $C(s) = C_1$. Therefore, we say that a constant ideal capacitor has a ‘capacity function’ $c(t)$ as Dirac delta function.

2.2. Causality principle applied for charge storage leading to convolution expression

Consider the capacitor device, which is supplied with an input function i.e. time varying voltage $v(t)$ and as a result the capacitor stores charge $q(t)$. Assume the input voltage $v(t')$ at a time t' is sustained for a short infinitesimal period call it dt' and we say output i.e. charge stored in capacitor at some later time

$t > t'$ is proportional to input, i.e. in the form $dq(t) = (c'(t, t'))(v(t')dt')$; with proportionality term as function $c'(t, t')$. Hence, the function $c'(t, t')$ describes the operation of charge storage in capacitor. Assuming this operation has no explicit dependence (i.e. no in-built clock that changes its behavior) then the relation between the input voltage and the charge storage will only depend on the time interval i.e. $t - t'$; and not on absolute time i.e. t . Therefore, we may replace the function $c'(t, t')$ with a function of single variable i.e. $c(t - t')$. This may be called response function of the capacitor, or fundamental impulse response. In this paper, we call this $c(t - t')$ as ‘capacity function’.

The assumption of ‘causality’, namely that cause precedes the effect, implies that output at any time t is obtained only due to input at or before t . Hence the expression $dq(t) = (c(t - t'))(v(t')dt')$ applies only for $t \geq t'$, or equivalently $c'(t, t') = 0$ for $t' > t$ or we say $c(t - t') = 0$ for $t' > t$. We get the total charge stored at time t by integration of $dq(t) = (c'(t, t'))(v(t')dt')$ from $t' = -\infty$ to $t' = t$ (since, $c'(t, t') = 0$ when $t' > t$: i.e. ‘cause cannot be preceding the effect’). We express as follows the causality principle

$$q(t) = \int_{-\infty}^t c(t - t')v(t')dt' \quad (1)$$

The above Eq. (1) is convolution operation of charge function and voltage function $q(t) = c(t) * v(t)$. Where in convolution operation is denoted as $(*)$ and the convolution of two functions $f_1(t)$ and $f_2(t)$ is described as $f_1(t) * f_2(t)$ is $\int_{-\infty}^t (f_1(t - t'))(f_2(t'))dt'$ or $\int_{-\infty}^t (f_1(t'))(f_2(t - t'))dt'$. We note that in frequency-transformed domain we have $\mathcal{L}\{f_1(t) * f_2(t)\} = \mathcal{L}\{f_1(t)\}\mathcal{L}\{f_2(t)\}$. If the application of voltage is at time $t' = 0$ then in Eq.(1) we have charge storage at time after $t > t'$ as $q(t) = \int_0^t c(t - t')v(t')dt'$. This is against conventional way of writing the charge i.e. $q(t) = c(t)v(t)$ or $c(t) = q(t)/v(t)$; which many text books and researchers use. This argument we will explain in the subsequent section.

3. Reviewing concept of charge storage in ideal capacitor in the classical theory

3.1: Impedance Function in Laplace domain for a ideal capacitor

We have standard expression of ‘impedance of a capacitor’ i.e. $Z(s)$ expressed in frequency domain as $Z(s) = V(s) / I(s) = (C_1 s)^{-1}$ with $V(s) = \mathcal{L}\{v(t)\}$, $I(s) = \mathcal{L}\{i(t)\}$. We also write $Z(\omega) = -j(\omega C_1)^{-1}$ by making $s = j\omega$, in $Z(s)$. Thus from impedance expression we have the capacity function expressed in Laplace frequency domain as a function as $C_1 = (s^{-1}I(s))/V(s)$ call it $C(s)$. We note that the constant C_1 is Laplace transformed quantity, i.e. $C_1 = \mathcal{L}\{c(t)\}$; and in this case of ‘constant capacity’ the capacity function in time is $c(t) = C_1 \delta(t)$. Therefore, we have in frequency domain representation of capacitor as function of Laplace variable s ; we call it as $C(s) = \mathcal{L}\{c(t)\}$. Therefore, for a general relation of capacity in frequency domain we have following expression

$$C(s) = \frac{s^{-1}I(s)}{V(s)}; \quad C(s) = \mathcal{L}\{c(t)\}, \quad V(s) = \mathcal{L}\{v(t)\}, \quad I(s) = \mathcal{L}\{i(t)\} \quad (2)$$

3.2: Getting charge function in time domain as convolution integral of capacity function and voltage function from the impedance function in Laplace domain

The numerator term in Eq. (2) i.e. $s^{-1}I(s)$ in time domain is $\int_0^t i(\tau) d\tau$ [19] that is charge the $q(t) = \int_0^t i(\tau) d\tau$, with its Laplace transform as $Q(s) = \mathcal{L}\{q(t)\}$. Therefore, from Eq. (2), we write charge in frequency domain as following expression $Q(s) = C(s)V(s)$. This is the expression in frequency domain. In the time domain by doing inverse Laplace transform we write $q(t) = c(t) * v(t) = \int_{-\infty}^t (c(t-t'))(v(t')) dt'$ [19]; that we had described in Section-2.2. Therefore the charge expression $q(t)$ is not multiplication of $c(t)$ and $v(t)$ but 'convolution' operation.

3.3: Charge function and current function in time domain when uncharged ideal capacitor is stressed with constant voltage by using convolution approach

Let an uncharged capacitor of constant capacity at $t=0$, of value C_1 be charged with a constant step voltage V_{BB} applied at $t=0$ i.e. $v(t) = V_{BB}(u(t))$; $t \geq 0$. By conventional approach using $q(t) = c(t)v(t)$ we say charge stored at any time for $t > 0$ is $q(t) = C_1V_{BB}$, whereas the charge is $q(t) = 0$ for $t < 0$. Thus the charge in time domain is a step function, we denote that as $q(t) = C_1V_{BB}(u(t))$; with $u(t)$ as unit-step function at $t=0$. Laplace transform of this unit-step is $\mathcal{L}\{u(t)\} = s^{-1}$. Using this we write $Q(s) = \mathcal{L}\{C_1V_{BB}u(t)\} = s^{-1}C_1V_{BB}$. The first derivative of charge i.e. $q^{(1)}(t)$ gives the charging current (or relaxation current) i.e. $i(t) = q^{(1)}(t) = (C_1V_{BB}u(t))^{(1)} = C_1V_{BB}(\delta(t))$. This is classical result that we all know is as per classical capacitor-theory that is charging current is impulse function at the time of application of voltage step, to an uncharged capacitor.

Now let us look at convolution integral, for $q(t) = c(t) * v(t) = \int_{-\infty}^t (c(t-t'))(v(t')) dt'$ for $t > 0$ i.e. where we have $v(t) = V_{BB}$, for $t \geq 0$. Only if we define $c(t)$ as function of time as the capacity function as $c(t) = C_1(\delta(t))$ we will be getting $q(t) = C_1V_{BB}$ for $t > 0$. That is $q(t) = c(t) * v(t)$ gives $q(t) = \int_0^t C_1(\delta(t-t'))(V_{BB}) dt'$ that is $q(t) = C_1V_{BB}$ for $t \geq 0$. We have used identity $\int (\delta(x_0 - x)) dx = 1$, i.e. property of delta function. Thus from expression $q(t) = c(t) * v(t)$ we get charge as step function at $t=0$, given as following expression $q(t) = C_1V_{BB}(u(t))$, for ideal capacitor.

3.4: For a loss-less ideal capacitor phase between charge function and voltage function is zero

Say we apply $v(t) = \cos at$ at $t=0$, for $t \geq 0$; then Laplace transform of $v(t)$ is $V(s) = s / (s^2 + a^2)$, to an uncharged constant capacitor $C(s) = C_1$. This gives $Q(s) = C_1(s / (s^2 + a^2))$ implying $q(t) = C_1 \cos at$; $t \geq 0$. Thus we observe for ideal capacitor, there is no phase difference between $v(t)$ and $q(t)$.

We do the same deduction following the convolution integration formulation $q(t) = c(t) * v(t)$, for ideal case $q(t) = \int_0^t C_1 (\delta(t-t')) (\cos at) dt'$ which is $q(t) = C_1 \cos at$ for $t \geq 0$; indicating same result.

3.5: Generalizing the charge function and current function for arbitrary voltage stress

Thus, we have general expression for any time varying voltage $v(t)$ applied at uncharged capacitor with geometrical capacity given by capacity function as $c(t) = C_1 (\delta(t))$, will have charge $q(t)$ for $t \geq 0$ as convolution integral $q(t) = \int_0^t C_1 (\delta(t-t')) (v(t')) dt' = C_1 (v(t))$ for $t \geq 0$.

Now we differentiate this expression of $q(t) = C_1 (v(t))$ to write following steps that is we derive for $i(t) = q^{(1)}(t) = (C_1 (v(t)))^{(1)}$ as

$$\begin{aligned} i(t) &= v(t) \frac{dC_1}{dt} + C_1 \frac{dv(t)}{dt} \\ &= (v(t)) (C_1 (\delta(t))) + C_1 \frac{dv(t)}{dt} = C_1 (v(0)\delta(t)) + C_1 \frac{dv(t)}{dt} = i(0) + i(t > 0) \end{aligned} \quad (3)$$

The first term at RHS of Eq. (3), indicate the value of current at $t=0$. The constant function starting at $t=0$ i.e. C_1 when differentiated gives $C_1 \delta(t)$. This unit delta functions at $t=0$, i.e. $\delta(t)$ when multiplied by $v(t)$ gives $v(0)\delta(t)$. This comes from property $\int (\delta(x_0 - x)) (f(x)) dx = f(x_0)$, and differentiation of this gives $(\delta(x_0 - x) f(x)) = \frac{d}{dx} f(x_0) = f(x_0) \delta(x)$. Thus at $t=0$ we have $i(0) = C_1 v(0)$ and $i(0) = 0$ for $t > 0$. Compositely we write $i(0) = C_1 v(0) (\delta(t))$, i.e. specifying its value at only $t=0$. The second term is $i(t > 0)$ for $t \neq 0$, that is $i(t) = C_1 (v^{(1)}(t))$. We write the following expression for $i(t)$

$$i(t) = C_1 v(0) (\delta(t)) + C_1 v^{(1)}(t) \quad (4)$$

This obtained expression Eq. (4) via the formulation $q(t) = c(t) * v(t)$ is consistent with expression obtained in [21], where $q(t) = c(t)v(t)$ is used.

As an example, we take $v(t) = V_{BB} u(t)$ a step input at time $t=0$, to an uncharged capacitor. We have $v^{(1)}(t) = 0$ for $t > 0$; and at $t=0$ we have, $v(0) = V_{BB}$. Using Eq. (4), we get $i(0) = C_1 V_{BB} (\delta(t))$; this makes $i(t) = C_1 V_{BB} (\delta(t))$, $t \geq 0$. This is for geometrical capacity charging current is impulse function.

3.6: Time varying capacity function as convolution integral of charge function and inverse voltage function

From the expression $C(s) = Q(s)/V(s)$ we write the time varying capacity $c(t)$ by use of convolution integral in the following steps $\mathcal{L}^{-1} \{C(s)\} = \mathcal{L}^{-1} \left\{ \left(Q(s) (V(s))^{-1} \right) \right\}$ that is $c(t) = (q(t)) * (v(t))^{-1}$. From this we say that capacity i.e. $c(t) \neq q(t)/v(t)$ i.e. not the usual ratio of charge to voltage in time domain, but it is given as convolution expression i.e. $c(t) = \int_{-\infty}^t \frac{q(t-t')}{v(t')} dt'$.

Let us verify, with $q(t) = (C_1 V_{BB})(u(t))$ i.e. at $t = 0$ and $q(t) = 0$ for $t < 0$, and $v(t) = V_{BB}u(t)$ i.e. a step voltage at $t = 0$, gives $c(t) = \int_{-\infty}^t \frac{C_1 V_{BB} u(t-t')}{V_{BB} u(t')} dt' = C_1 (u(t) * u(t)^{-1}) = C_1 (\delta(t))$. We have used inverse identity i.e. $f * f^{-1} = \delta$ here. Therefore, capacity at any time is the history of ratio of charge to voltage given by convolution integral $c(t) = \int_{-\infty}^t \frac{q(t-t')}{v(t')} dt'$.

We can verify with say $q(t) = (C_1 \cos at)u(t)$ for $t \geq 0$ with $v(t) = (C_1 \cos at)u(t)$ for $t \geq 0$ gives the following $c(t) = \int_{-\infty}^t \frac{C_1 \cos a(t-t')}{(\cos at')} dt' = C_1 ((\cos(at)) * (\cos(at))^{-1}) = C_1 (\delta(t))$. We have used inverse identity i.e. $f * f^{-1} = \delta$ here. We note here the formula used in [12], [21] is $c(t) = q(t) / v(t)$, whereas we used $c(t) = (q(t)) * (v(t))^{-1}$.

4. Fractional Derivative directly from Curie-von Schweidler Law-Fractional Capacitor

4.1: Impedance and Admittance in Laplace domain for a fractional capacitor

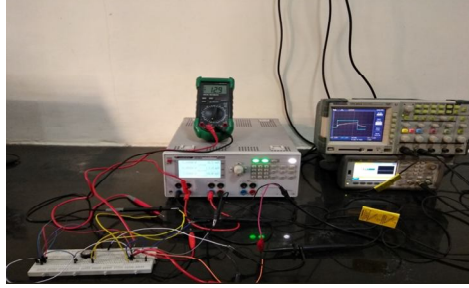
Practically on applying a step input voltage $v(t) = V_{BB}$ Volts at $t = 0$ to a capacitor which is initially uncharged; we get a power-law decay of current given by empirical Curie-von Schweidler as $i(t) \sim t^{-n}$; $0 < n < 1$ [1]-[3], [12], [21], [22]. That is $i(t) = K_n V_{BB} t^{-n}$ for $t > 0$. This is also indicated by experimental studies [12]-[17], [21], and [22]. The parameter K_n is proportionality constant, while in [12], [21] the proportionality constant is $1/h_1$. This is from observation and the evaluation of order of power-law function is $0.5 < n < 1$ [7]-[12], [21]. Let the uncharged capacitor be excited by a constant step input of V_{BB} Volts, i.e. written as $v(t) = V_{BB}(u(t))$, where $u(t)$ is unit step function at $t = 0$, thus The Laplace transform of step input as $V(s) = \mathcal{L}\{v(t)\} = \mathcal{L}\{V_{BB}(u(t))\} = s^{-1}V_{BB}$ and then taking Laplace transform of $i(t) = K_n V_{BB} t^{-n}$ i.e. of power-law decay current by using $\mathcal{L}\{t^m\} = m!s^{-(m+1)}$, [19] we write expression for $I(s) = \mathcal{L}\{i(t)\} = K_n V_{BB} (-n)! s^{n-1}$. Using the formula for generalization of factorial i.e. $(\alpha - 1)! = \Gamma(\alpha)$ [6], [20], [27] we get the expressions $I(s) = K_n \Gamma(1-n) s^n (V_{BB}/s)$. From here we get Transfer function [19] of capacitor admittance $Y(s) = I(s)/V(s) = C_n s^n$ with $C_n = K_n (\Gamma(1-n))$. This expression i.e. $Y(s) = I(s)/V(s)$ is 'admittance' expression in complex frequency (s) domain of a capacitor. Putting, $s = j\omega$ in $Y(s)$ we get $I(\omega) = (\cos \frac{n\pi}{2} + j \sin \frac{n\pi}{2}) \omega^n C_n V(\omega)$. This means current leads voltage in fractional capacitor by angle $\frac{n\pi}{2}$. For $n = 1$, i.e. for a classical geometrical ideal capacitor we have $I(\omega) = j\omega C_1 V(\omega)$, that is current leading voltage by angle of 90° .

4.2: Current voltage relation by fractional derivative for fractional capacitor

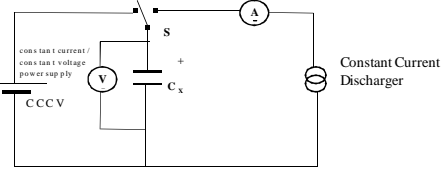
From $Y(s) = I(s)/V(s) = C_n s^n$ we write impedance expression $Z(s) = V(s)/I(s)$ for fractional-capacitor as $Z(s) = C_n^{-1} s^{-n}$; $0 < n < 1$. Thus we have $I(s) = C_n s^n (V(s))$ and by Laplace inversion

and by using the identity $\mathcal{L}^{-1}\{s^n F(s)\} = {}_0D_t^n[f(t)]$ i.e. fractional derivative operation [6], [20], [27] we get the constituent relation for capacity as following (the ${}_0\mathcal{I}_t^n$ is fractional integration and ${}_0D_t^n$ is fractional derivative operator)

$$i(t) = C_n \left({}_0D_t^n [v(t)] \right), \quad 0 < n < 1, \quad v(t) = C_n^{-1} \left({}_0\mathcal{I}_t^n [i(t)] \right) \quad (5)$$



Experimental set up to obtain the value of Fractional Capacity and Fractional order for Super Capacitor



Schematic of Experimental Set-up with CCCV Constant Current Constant Voltage Excitation and Constant Current Discharge

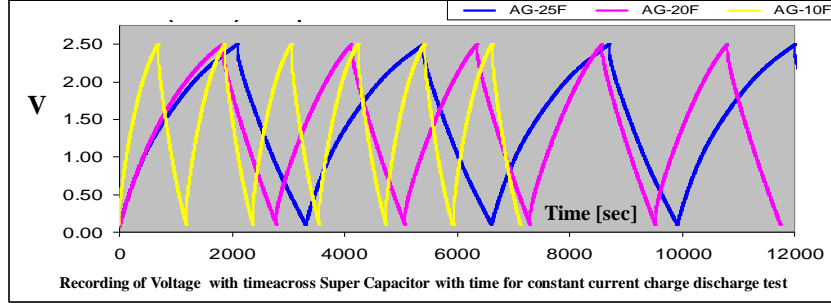


Figure-1: Experiments to determine the Fractional Order of Fractional Capacitor

The Figure-1 [7], [8], [32] gives the set-up to determine fractional order n and fractional capacity C_n . In experiments Eq. (5) is used, with a capacitor excited with constant current. If the capacitor were ideal, then as $n = 1$, the voltage rise and fall would have been linear, instead of curved.

4.3: Fractional units for fractional capacitor

The 'fractional capacity' C_n is in unit of Farad/sec $^{1-n}$; [12], [21] which is constant given by $C_n = K_n (\Gamma(1-n))$. This fractional derivative expression $i(t) = C_n ({}_0D_t^n [v(t)])$ gives a new capacitor theory [12], [21] and we utilize this formula to find characteristics of super-capacitors, variation of n with current excitation, and efficiency of energy discharged to energy stored [7]-[8], [32]. Classically the expression of capacitor is $i(t) = C_1 (D_t^{(1)} [v(t)])$ i.e. with $n = 1$ one-whole order (classical) derivative.

Curie-von Schweidler law gives a different approach for capacitor theory based on fractional calculus [12], [21], [22]. In experimental observations, we find that capacitor has fractional order impedance [7]-[17], [21]. This section gives us the understanding that this empirical law i.e. Curie-von Schweidler law gives a relation of voltage and current of capacitor by using fractional derivative. We will derive this $i(t) = C_n ({}_0D_t^n [v(t)])$ by the new approach of the definition of charge in the subsequent section.

5. Charge stored in a fractional capacitor using convolution integral of time varying capacity due to Curie-von Schweidler relaxation current

5.1: Expression of charge storage from Curie-von Schweidler relaxation current in a fractional capacitor

For Curie-von Schweidler law we have relaxation current as noted earlier empirically expressed as $i(t) = K_n V_{BB} t^{-n}$, $0 < n < 1$ for $t > 0$, i.e. when uncharged capacitor is applied with a step voltage $v(t) = V_{BB} (u(t))$ at $t = 0$. This empirical expression of current relaxation gives a relation of incremental charge Δq (or dq in infinitesimal small limit) when ‘pulse’ of a voltage of magnitude V_{BB} is applied for a duration Δt (or in infinitesimal small limit dt) given by following expressions

$$\Delta q = \frac{K_n V_{BB} \Delta t}{t^n} \quad dq = \frac{K_n V_{BB} dt}{t^n} \quad (6)$$

With this above Eq. (6) expression (and by $q(t) = \int_0^t dq$) we write the charge accumulated for this power law decay current as following

$$q(t) = \int_0^t dq = \int_0^t \frac{K_n V_{BB} dx}{x^n} = \frac{K_n V_{BB}}{(1-n)} t^{1-n}, \quad 0 < n < 1 \quad t > 0 \quad (7)$$

5.2: Expression for time varying capacity function from admittance relation of a fractional capacitor

From the expression in frequency domain i.e. $C(s) = (s^{-1} I(s)) / (V(s)) = (Q(s)) / (V(s))$ we have for $i(t) = K_n V_{BB} t^{-n}$ with $I(s) = K_n (\Gamma(1-n)) V_{BB} s^{n-1}$, and $V(s) = V_{BB} / s$, gives $C(s)$ as following

$$C(s) = \frac{(s^{-1} I(s))}{V(s)} = \frac{s^{-1} (K_n (\Gamma(1-n)) V_{BB} s^{n-1})}{V_{BB} s^{-1}} = \frac{K_n (\Gamma(1-n))}{s^{1-n}} = K_n \frac{(-n)!}{s^{1+(-n)}} \quad (8)$$

Now doing inverse Laplace transform by using $\mathcal{L}^{-1} \{ (m!) / s^{(1+m)} \} = t^m$ [19] of above Eq. (8) we get ‘time dependent’ capacity function $c(t)$ as following

$$c(t) = K_n t^{-n}; \quad 0 < n < 1, \quad t > 0 \quad (9)$$

5.3: Using convolution integral and time dependent capacity function evaluation of charge storage in time for a constant voltage applied to fractional capacitor

Using the convolution integral with this time dependent capacity function (9) and a step voltage applied at time zero, we get following expression for charge stored

$$\begin{aligned} c(t) &= K_n t^{-n}, \quad v(t) = V_{BB}; \quad t > 0 \\ q(t) &= (c(t)) * (v(t)) = \int_{-\infty}^t (c(t-t')) (v(t')) dt' = \int_0^t K_n ((t-t')^{-n}) (V_{BB}) dt' \\ &= -V_{BB} K_n \frac{(t-t')^{1-n}}{1-n} \Big|_{t'=0}^{t'=t} = \frac{V_{BB} K_n}{1-n} t^{1-n} \quad 0 < n < 1 \end{aligned} \quad (10)$$

The expression in Eq. (10) gives $q(t) = \frac{V_{BB}K_n}{1-n}t^{1-n}$ is obtained via our formula $q(t) = c(t) * v(t)$ is same as we got via $q(t) = \int_0^t dq$ above in Eq. (7).

6. Observations on breakdown mechanism of a fractional capacitor and loss tangent and comparison with earlier theory

6.1: When a fractional capacitor is float on a constant voltage the charge accumulated at large times is infinity-giving electrostatic break down

We note here from $q(t) = \frac{V_{BB}K_n}{1-n}t^{1-n}$ that for $0 < n < 1$, the charge store is $\lim_{t \rightarrow \infty} q(t) = \infty$ when the capacity function is $c(t) = K_n t^{-n}$, following Curie-von Schweidler decay current. Whereas for a classical capacity function i.e. given as $c(t) = C_1 \delta(t)$, the charge at large times is $\lim_{t \rightarrow \infty} q(t) = C_1 V_{BB}$. This observation i.e. $\lim_{t \rightarrow \infty} q(t) = \infty$ in our derivation is with convolution formula i.e. $q(t) = c(t) * v(t)$ is in line with the observations in [12], [21], where the used expression for charge is $q(t) = (c(t))(v(t))$. This is the new idea of breakdown of capacitors due to accumulation of enough charge (electrostatic breakdown) at a constant voltage even though voltage is less than the breakdown limit of dielectric proposed in [12], [21].

6.2: Evaluation of loss tangent by using earlier approach and the convolution approach of charge storage concept for fractional capacitor

In [12] and [21] the charge formula used is $c(t) = q(t) / v(t)$ and not via convolution approach that we discussed in this paper. In addition, with this formula $c(t) = q(t) / v(t)$ in [12] and [21] gets the time dependent capacity function with the constant h_1 is used in Curie von-Schweidler relaxation current i.e. $c(t) = \left(\frac{1}{h_1(1-n)}\right)t^{1-n}$ for $t > 0$ and $0 < n < 1$. The frequency domain representation for $c(t)$ obtained in [12] and [21] is $C(s) = \frac{(1-n)!}{h_1(1-n)}s^n$ and with $s = j\omega$ we get $C(\omega) = \left(\frac{\omega^n(1-n)!}{h_1(1-n)}\right)\left(\cos \frac{n\pi}{2} + j \sin \frac{n\pi}{2}\right)$. Here from this if we express loss tangent as $\tan \phi = \frac{\text{Im}[C(\omega)]}{\text{Re}[C(\omega)]} = -\tan\left(\frac{n\pi}{2}\right)$, which is not correct, as the loss tangent is $\tan \phi = \tan(1-n)\frac{\pi}{2}$; $0 < n < 1$ for a fractional capacitor. Therefore, in [12] and [21], the loss tangent is not calculated by the using capacity function $c(t) = \left(\frac{1}{h_1(1-n)}\right)t^{1-n}$ instead, phase difference ψ is calculated between current $I(\omega)$ and voltage $V(\omega)$ by using admittance expression $Y(s)|_{s=j\omega} = \frac{I(\omega)}{V(\omega)} = \left(\cos \frac{n\pi}{2} + j \sin \frac{n\pi}{2}\right)\omega^n C_n$ and then doing steady state (sinusoidal) analysis, and then writing loss tangent as $\tan \phi = \tan\left(\frac{\pi}{2} - \psi\right)$, which is $\tan \phi = \tan\left(\frac{(1-n)\pi}{2}\right)$. This expression $c(t) = \left(\frac{1}{h_1(1-n)}\right)t^{1-n}$ of [12] and [21] says that the time varying capacity function will be growing to infinity as time grows. Also in frequency domain, we will be getting infinite value at infinite frequency. This gives us notion of unrealistic property of capacity function, which is unstable.

Whereas we have from our new derivation for a fractional capacitor with capacity function $c(t) = K_n t^{-n}$ gives $C(\omega) = K_n \left(\Gamma(1-n)\right)\omega^{-(1-n)}\left(\cos \frac{(1-n)\pi}{2} - j \sin \frac{(1-n)\pi}{2}\right)$. Where the capacity function tends towards

zero for large time and large frequency. From this we get loss tangent as $\tan \phi = \text{Im}[C(\omega)] / \text{Re}[C(\omega)] = \tan\left(\frac{(1-n)\pi}{2}\right)$ which is also as reported in [12], [21]; obtained differently. However, [12] and [21] gives other expressions, same as that we will derive and report subsequently.

7. Further derivations regarding fractional capacitor in conjugation to classical capacitor

7.1: Getting charge function in time domain as convolution integral of capacity function and voltage function from the impedance function in Laplace domain for fractional capacitor

Now we do the steps as we did for classical capacitor, from the obtained impedance relation of fractional capacitor i.e. $Z(s) = s^{-n} C_n^{-1}$, $C_n = K_n \Gamma(1-n)$; $0 < n < 1$; with units $C_n \equiv \text{Farad} / \text{sec}^{1-n}$. With $C_n(s) = \mathcal{L}\{c_n(t)\} = C_n = K_n (\Gamma(1-n))$ as obtained in earlier section a constant in units of $\text{Farad} / \text{sec}^{1-n}$. We note that $C_n = K_n (\Gamma(1-n))$ is in units of $\text{Farad} / \text{sec}^{1-n}$; a “fractional form” of unit [12], [21], defining a “fractional capacity” as constant in the frequency domain. Thus, we expect that in time domain the fractional capacity call it $c_n(t)$ be given by delta function at $t=0$ i.e. $c_n(t) = (K_n (\Gamma(1-n))) (\delta(t)) = C_n \delta(t)$. The meaning of capacity function $c_n(t)$ in time domain is $c_n(t) = C_n (\delta(t))$ i.e. an impulse of height C_n (in units $\text{Farad} / \text{sec}^{1-n}$) at the time of application of voltage excitation (i.e. $t=0$). Whereas, in the frequency domain, the definition of fractional capacity is $C_n(s) = C_n$ i.e. $C_n(s) = \mathcal{L}\{C_n \delta(t)\} = C_n$ that is a constant (in unit of $\text{Farad} / \text{sec}^{1-n}$) value at all frequencies. We say here that classical geometrical capacitor presents a Farad value as impulse function at the time of application of voltage stress, while the fractional capacitor presents a $\text{Farad} / \text{sec}^{1-n}$ value at the time of application of voltage.

From this we write following steps, with $C_n(s) = \mathcal{L}\{c_n(t)\}$, $\mathcal{L}^{-1}\{s^{-n} F(s)\} = {}_0\mathcal{I}_t^n f(t)$ where ${}_0\mathcal{I}_t^n$ is defining fractional integration operation [6], [20], [27] of fractional order $0 < n < 1$

$$\begin{aligned}
 C_n(s) &= \frac{s^{-n} I(s)}{V(s)} = \frac{\mathcal{L}\{{}_0\mathcal{I}_t^n [i(t)]\}}{\mathcal{L}\{v(t)\}}; \quad 0 < n < 1 \quad {}_0\mathcal{I}_t^n [f(t)] = {}_0\mathcal{I}_t^{n-1} {}_0\mathcal{I}_t^1 [f(t)] \\
 &= \frac{\mathcal{L}\{{}_0\mathcal{I}_t^{n-1} {}_0\mathcal{I}_t^1 [i(t)]\}}{\mathcal{L}\{v(t)\}} = \frac{\mathcal{L}\{{}_0\mathcal{I}_t^{n-1} \int_0^t i(x) dx\}}{\mathcal{L}\{v(t)\}}; \quad q(t) = \int_0^t i(x) dx \\
 &= \frac{\mathcal{L}\{{}_0D_t^{1-n} [q(t)]\}}{\mathcal{L}\{v(t)\}}; \quad {}_0\mathcal{I}_t^{n-1} f(t) = {}_0D_t^{1-n} f(t) \tag{11} \\
 \mathcal{L}\{{}_0D_t^{1-n} [q(t)]\} &= (\mathcal{L}\{v(t)\}) (\mathcal{L}\{c_n(t)\}) \\
 \mathcal{L}\{c_n(t)\} &= \mathcal{L}\{{}_0D_t^{1-n} [q(t)]\} (\mathcal{L}\{v(t)\})^{-1} \\
 c_n(t) &= ({}_0D_t^{1-n} [q(t)]) * (v(t))^{-1}
 \end{aligned}$$

7.2: Defining capacity function as fractional integration of fractional capacity function thereby converting the units in fractional units to Farads-for a fractional capacitor

In Eq. (11) ${}_0D_t^{1-n}$ is fractional derivative operation with order $(1-n)$. Therefore, we write following formulas for fractional capacitor in with conjugation to classical capacitor theory

$$\begin{aligned}
c_n(t) &= \left({}_0D_t^{1-n} [q(t)] \right) * (v(t))^{-1}; \quad 0 < n < 1 \\
{}_0D_t^{1-n} [q(t)] &= (c_n(t)) * (v(t)) \\
q(t) &= {}_0D_t^{n-1} \left[(c_n(t)) * (v(t)) \right] = {}_0\mathcal{I}_t^{1-n} \left[(c_n(t)) * (v(t)) \right] \\
q(t) &= \left({}_0\mathcal{I}_t^{1-n} [c_n(t)] \right) * (v(t)) \quad \text{or} \quad q(t) = (c_n(t)) * \left({}_0\mathcal{I}_t^{1-n} [v(t)] \right) \\
c(t) &\stackrel{\text{def}}{=} {}_0\mathcal{I}_t^{1-n} \left[(c_n(t)) \right] \\
q(t) &= (c(t)) * (v(t))
\end{aligned} \tag{12}$$

In the steps of Eq. (12), we have $q(t) = {}_0\mathcal{I}_t^{1-n} \left[(c_n(t)) * (v(t)) \right]$ doing Laplace transform we get $Q(s) = s^{-(1-n)} \mathcal{L} \left\{ (c_n(t)) * (v(t)) \right\}$. Further, we get $Q(s) = s^{-(1-n)} (C_n(s)V(s)) = (s^{-(1-n)} C_n(s))(V(s))$. Writing $(s^{-(1-n)} C_n(s)) = C(s)$, i.e. ${}_0\mathcal{I}_t^{n-1} [c_n(t)] = c(t)$; we have $q(t) = \left({}_0\mathcal{I}_t^{1-n} [c_n(t)] \right) * (v(t))$.

In Eq. (12), thus we defined $c(t) = {}_0\mathcal{I}_t^{1-n} \left[(c_n(t)) \right]$. We note here that, by re-arrangement of term $s^{-(1-n)}$ in expression of $Q(s)$ we could have written $q(t) = (c_n(t)) * \left({}_0\mathcal{I}_t^{1-n} [v(t)] \right)$, this formula is also valid, that we have mentioned in Eq. (12). Using expression obtained earlier. $q(t) = \frac{K_n V_{BB}}{1-n} t^{1-n}$ in Eq. (12), we get ${}_0D_t^{1-n} [q(t)] = \frac{K_n V_{BB}}{1-n} \left(\frac{\Gamma(1-n+1)}{\Gamma(1-n+1-1+n)} t^{1-n-1+n} \right) = K_n V_{BB} (\Gamma(1-n))$ that is a constant function for $t > 0$. This we have got by formula of fractional derivative i.e. ${}_0D_x^\nu x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} x^{\beta-\nu}$ [6], [20], [27]. Thus, we write ${}_0D_t^{1-n} [q(t)] = K_n V_{BB} (\Gamma(1-n)) u(t)$ where $u(t)$ a unit-step function at $t = 0$. We write the following for $c_n(t)$ as described in Eq. (12).

$$\begin{aligned}
c_n(t) &= \left({}_0D_t^{1-n} [q(t)] \right) * (v(t))^{-1}; \quad {}_0D_t^{1-n} [q(t)] = K_n V_{BB} (\Gamma(1-n)) u(t) \\
&= \left(K_n V_{BB} (\Gamma(1-n)) (u(t)) \right) * \left(V_{BB} (u(t)) \right)^{-1} \\
&= K_n (\Gamma(1-n)) \delta(t)
\end{aligned} \tag{13}$$

We used identity i.e. $f * f^{-1} = \delta$, the inverse relation in Eq. (13). We consider the following relation Eq. (12) for time varying capacity function $c(t)$ from $c_n(t)$

$$c(t) = {}_0D_t^{(n-1)} [c_n(t)]; \quad t > 0, \quad 0 < n < 1; \quad {}_0D_t^{(n-1)} \equiv {}_0\mathcal{I}_t^{(1-n)} \tag{14}$$

i.e. time varying capacity function defined as fractional integral of the order $1-n$ for the fractional capacity function i.e. $c_n(t)$ i.e. in units of Farad / sec $^{1-n}$, which is constant in frequency domain as $C_n(\omega) = K_n (\Gamma(1-n))$ i.e. a fractional capacitor. Using $c_n(t) = K_n (\Gamma(1-n)) \delta(t)$ we write following

$$\begin{aligned}
c(t) &= {}_0D^{n-1} [c_n(t)] = {}_0\mathcal{I}_t^{1-n} \left[\left(K_n (\Gamma(1-n)) (\delta(t)) \right) \right] = K_n (\Gamma(1-n)) \left({}_0\mathcal{I}_t^{1-n} [\delta(t)] \right) \\
&= K_n (\Gamma(1-n)) \left(\frac{t^{1-n-1}}{\Gamma(1-n)} \right) = K_n t^{-n}
\end{aligned} \tag{15}$$

In Eq. (15), we have used formula for fractional integration of delta-function [6], [20], [27] i.e. ${}_0\mathcal{I}_x^\beta [\delta(x)] = \frac{1}{\Gamma(\beta)} x^{\beta-1}$. The expression $c(t) = K_n t^{-n}$ we had obtained earlier too.

We note that the fractional integration operation ${}_0\mathcal{I}_t^{1-n} [c_n(t)]$ in Eq. (14), Eq. (15) is converting units in Farad / sec¹⁻ⁿ for $c_n(t)$ into units of Farad for $c(t)$. This is because the fractional integration ${}_0\mathcal{I}_t^{1-n}$ is integration with respect to fractional differential element $(dt)^{1-n}$ i.e. ${}_0\mathcal{I}_t^{1-n} [c_n(t)] = \int_0^t (c_n(t)) dt^{(1-n)}$ [6], [20], [27]. Therefore, the capacity function $c(t) = K_n t^{-n}$ that we get for fractional capacitor is in units of Farad. This show for a fractional capacitor by the use of time varying capacity function we can convert the fractional capacity constant that is in units of fractional units of Farads per second to the power a fractional number, to units of Farads, by formula $c(t) = {}_0\mathcal{I}_t^{1-n} [c_n(t)]$.

7.3: General charge and current expression for fractional capacitor following universal dielectric relaxation law

We obtain a general expression of charge $q(t)$ for Curie-von Schweidler relaxing current in a capacitor, that is having capacity function as $c(t) = K_n t^{-n}$, $t > 0$ when stressed with a time varying voltage $v(t)$ applied at $t = 0$ is by convolution process as $q(t) = (c(t)) * (v(t)) = \int_{-\infty}^t (c(t-t')) (v(t')) dt'$, i.e. $q(t) = (K_n t^{-n}) * (v(t))$. The convolution integral from with $t' = 0$ is $q(t) = \int_0^t K_n \frac{v(t')}{(t-t')^n} dt'$. As we did for geometrical capacity in previous section, we differentiate this $q(t)$ to get $i(t)$ and write $i(t) = q^{(1)}(t) = K_n \frac{d}{dt} \int_0^t \frac{v(t')}{(t-t')^n} dt'$. We apply formula of integration by parts to evaluate $\int_0^t \frac{v(t')}{(t-t')^n} dt'$ that appears in as detailed in the following steps

$$\int_0^t \frac{v(t') dt'}{(t-t')^n} = \left[v(t') \int \frac{dt'}{(t-t')^n} \right]_{t'=0}^{t'=t} - \int_0^t \left(v^{(1)}(t') \int \frac{dt'}{(t-t')^n} \right) dt' = \frac{v(0)}{1-n} t^{1-n} - \int_0^t \frac{v^{(1)}(t') (t-t')^{1-n}}{1-n} dt' \tag{16}$$

Now we differentiate Eq. (16) and write the following steps

$$\frac{d}{dt} \int_0^t \frac{v(t') dx}{(t-t')^n} = \frac{d}{dt} \left(\frac{v(0)}{1-n} t^{1-n} - \int_0^t \frac{v^{(1)}(t')}{1-n} (t-t')^{1-n} dt' \right) = \frac{v(0)}{t^n} + \int_0^t \frac{v^{(1)}(t')}{(t-t')^n} dt' \tag{17}$$

This gives $i(t)$ as following relation

$$i(t) = K_n \frac{d}{dt} \int_0^t \frac{v(t')}{(t-t')^n} dx = K_n \frac{v(0)}{t^n} + K_n \int_0^t \frac{v^{(1)}(t') dt'}{(t-t')^n}; \quad K_n = \frac{C_n}{\Gamma(1-n)}; \quad 0 < n < 1 \tag{18}$$

The expression Eq. (18) obtained with the formula $q(t) = (c(t)) * (v(t))$, with $c(t) = K_n t^{-n}$ is consistent with obtained expression in [21].

For $v(t) = V_{BB}(u(t))$ i.e. a constant step voltage applied at time $t=0$ to a time varying capacity function given as $c(t) = K_n t^{-n}$ we have for $t > 0$, $v^{(1)}(t) = 0$ with $v(0) = V_{BB}$, the evaluation of $i(t)$ demonstrated below

$$i(t) = K_n \frac{v(0)}{t^n} + K_n \int_0^t \frac{v^{(1)}(t') dt'}{(t-t')^n} = K_n \frac{V_{BB}}{t^n} + K_n \int_0^t \frac{0 dt'}{(t-t')^n} = K_n \frac{V_{BB}}{t^n} \quad (19)$$

We get $i(t) = K_n V_{BB} t^{-n}$, for $t > 0$ i.e. we recover the Curie-von Schweidler law in Eq. (19). For a constant ideal capacitor case with capacity function as $c(t) = C_1 \delta(t)$, we have the relation that we derived earlier i.e. $i(t) = C_1 v(0)(\delta(t)) + C_1 (v^{(1)}(t))$.

8. Appearance of fractional derivative in Fractional Capacitor

We have formed a time varying capacity function with a dielectric whose relaxation to a step voltage at $t=0$ of constant magnitude follows a power law given by empirical expression of Curie-von Schweidler law. We have got current and charge expression for any arbitrary voltage function $v(t)$ applied at $t=0$ in previous section Eq. (18) as $i(t) = K_n v(0) t^{-n} + K_n \int_0^t \frac{v^{(1)}(t') dt'}{(t-t')^n}$ with $q(t) = (c(t)) * (v(t)) = \int_0^t K_n \frac{v(t')}{(t-t')^n} dt'$

The fractional derivative for $0 < n < 1$ is defined as following two ways [6], [20], [27]

$${}_0 D_t^n [f(t)] = \frac{1}{\Gamma(1-n)} \frac{d}{dt} \int_0^t \frac{f(x)}{(t-x)^n} dx = \frac{1}{\Gamma(1-n)} \left(\frac{f(0)}{t^n} + \int_0^t \frac{f^{(1)}(x)}{(t-x)^n} dx \right); \quad t > 0 \quad (20)$$

The first definition is of Riemann-Liouville type i.e. ${}_0 D_t^n [f(t)] = \frac{1}{\Gamma(1-n)} \frac{d}{dt} \int_0^t \frac{f(x)}{(t-x)^n} dx$, $0 < n < 1$ and in

the second expression of Eq. (20) the second term i.e. $\frac{1}{\Gamma(1-n)} \int_0^t \frac{f^{(1)}(x)}{(t-x)^n} dx$ is Caputo fractional derivative i.e.

${}_0^C D_t^n [f(t)] = \frac{1}{\Gamma(1-n)} \int_0^t \frac{f^{(1)}(x)}{(t-x)^n} dx$; $0 < n < 1$. Therefore, we have ${}_0 D_t^n [f(t)] = {}_0^C D_t^n [f(t)] + \frac{f(0)}{\Gamma(1-n)} t^{-n}$, i.e. relation between the two definitions of fractional derivative [6], [20], [27], for $0 < n < 1$.

Integrating the expression ${}_0 D_t^n [f(t)] = \frac{1}{\Gamma(1-n)} \frac{d}{dt} \int_0^t \frac{f(x)}{(t-x)^n} dx = \frac{1}{\Gamma(1-n)} \left(\int_0^t \frac{f(x)}{(t-x)^n} dx \right)^{(1)}$, once we write the following

$${}_0 \mathcal{I}_t^1 ({}_0 D_t^n [f(t)]) = \frac{1}{\Gamma(1-n)} \int_0^t \left(\int_0^t \frac{f(x)}{(t-x)^n} dx \right)^{(1)} dx = \frac{1}{\Gamma(1-n)} \int_0^t \frac{f(x)}{(t-x)^n} dx \quad (21)$$

We have used in Eq. (21) the identity $\mathcal{I}_t^1 (g^{(1)}(t)) \equiv g(t)$. Using the composition rule [6], [20], [27] i.e.

${}_0 \mathcal{I}_t^1 ({}_0 D_t^n [f(t)]) = {}_0 \mathcal{I}_t^{1-n} [f(t)] = {}_0 D_t^{n-1} [f(t)]$, we re-write Eq. (21) as following

$$\begin{aligned}
{}_0D_t^{n-1}[f(t)] &= {}_0\mathcal{I}_t^{1-n}[f(t)] = \frac{1}{\Gamma(1-n)} \int_0^t \frac{f(x)}{(t-x)^n} dx; \quad 0 < n < 1; \quad 1-n = \nu \\
{}_0D_t^{-\nu}[f(t)] &= {}_0\mathcal{I}_t^{\nu}[f(t)] = \frac{1}{\Gamma(\nu)} \int_0^t \frac{f(x)}{(t-x)^{1-\nu}} dx
\end{aligned} \tag{22}$$

Using the definitions of fractional derivative, we apply to current expression Eq. (18) also by manipulating with a constant i.e. $\Gamma(1-n)$ we get following expressions, with $K_n(\Gamma(1-n)) = C_n$

$$i(t) = K_n(\Gamma(1-n)) \left(\frac{1}{\Gamma(1-n)} \left(\frac{v(0)}{t^n} + \int_0^t \frac{v^{(1)}(t') dt'}{(t-t')^n} \right) \right) = C_n ({}_0D_t^n[v(t)]), \quad 0 < n < 1 \tag{23}$$

Applying the expression for fractional integration to the charge expression, we get following

$$q(t) = (c(t)) * (v(t)) = (K_n t^{-n}) * (v(t)) = \int_0^t K_n \frac{v(t') dt'}{(t-t')^n} = C_n ({}_0\mathcal{I}_t^{(1-n)}[v(t)]) = C_n ({}_0D_t^{n-1}[v(t)]) \tag{24}$$

We apply a constant step voltage $v(t) = V_{BB}$ at $t = 0$ to an uncharged fractional capacitor with capacity function $c(t) = \frac{C_n}{\Gamma(1-n)} t^{-n}$, applying the above formula Eq. (24) we get

$$q(t) = C_n ({}_0D_t^{n-1}[v(t)]) = C_n ({}_0D_t^{n-1}[V_{BB}]) = \frac{K_n V_{BB}}{(1-n)} t^{1-n}, \quad t > 0; \quad K_n(\Gamma(1-n)) = C_n \tag{25}$$

We have used the formula fractional derivative of constant i.e. ${}_0D_t^\alpha[C] = C \frac{\Gamma(1)}{\Gamma(1-\alpha)} t^{-\alpha}$, $\Gamma(1) = 1$, [6], [20], [27].

We wrote also in the steps of Eq. (12) the expression $q(t) = (c_n(t)) * ({}_0\mathcal{I}_t^{1-n}[v(t)])$. For fractional capacitor, we noted that $c_n(t) = C_n \delta(t)$ this we have $q(t) = (C_n \delta(t)) * ({}_0\mathcal{I}_t^{1-n}[v(t)])$. Expanding this convolution integral, we get $q(t) = \int_{-\infty}^t (C_n \delta(t-t')) ({}_0\mathcal{I}_t^{1-n}[v(t')]) dt'$. Now using property of delta function i.e. $\int \delta(x-a) f(x) dx = f(a)$, we get $q(t) = C_n ({}_0\mathcal{I}_t^{1-n}[v(t)])$. With $v(t) = V_{BB}$ applied at $t = 0$ we will get $q(t) = \frac{1}{(1-n)\Gamma(1-n)} C_n V_{BB} t^{1-n}$ same as in Eq. (25).

9. Integrated Capacity defined from Capacity function of a capacitor and explanation vis-à-vis a pitcher holding water

9.1: Defining integrated capacity from the time varying capacity function for ideal and fractional capacitor

We take example of a pitcher, which holds water, of volume V . Let the pitcher be made of metal walls so that there are no pores. It is fully filled with water from empty state, hence once full it has no capacity left. This is like ideal capacitor, where the volume of water V remains fixed as constant after filling, with no left over capacity. Thus, an ideal capacitor described by capacity function $c(t) = C_1 \delta(t)$, after it is

charged at $t=0$ with a constant voltage holds the constant charge $q(t) = C_1 V_{BB}$ at times $t > 0$. At time, $t > 0$ this capacitor has zero capacity function, i.e. $c(t) = 0$ that is like no more capacity left to fill, like pitcher. Thus, we have maximum charge holding capacity in this case as $q_{\max} = \lim_{t \uparrow \infty} q(t) = C_1 V_{BB}$. Therefore we can say the capacity function $c(t)$ at $t > 0$ indicates the left over capacity to fill from maximum charge say $q_{\max} = \lim_{t \uparrow \infty} q(t)$.

Now let the walls of the pitcher be made of clay with an infinitely porous material. As the pitcher gets the water the volume V , the pitcher walls too starts seepage of water into its pores. Thus, extra water keeps entering pores of the porous pitcher walls. This water filling process in the porous walls we call fractional capacity. Now due to infinite nature of these pores, we have a situation, that infinite amount of water keeps seeping into the walls. This is analogous to charging porous walls with water as charging a fractional capacitor where we derived $q_{\max} = \lim_{t \uparrow \infty} q(t) = \infty$. Yet as we go on with charging process, the remaining capacity of holding the charge from maximum value (in this case infinity) keeps on decreasing but will never be going to zero, and thus we got the capacity function for a fractional capacitor as, $c(t) = K_n t^{-n}$ where $\lim_{t \uparrow \infty} c(t) = 0$. The charge of a fractional capacitor as in the case of filling the porous walls gets the form that we derived as $q(t) = \frac{K_n V_{BB}}{(1-n)} t^{1-n}$, $0 < n < 1$ for $t > 0$ increasing with time. This phenomena leads to electrostatic break down of capacitors [12], [21], even if the constant voltage V_{BB} is lower than dielectric breakdown limit. Thus a fractional capacitor with $c(t) = K_n t^{-n}$ will break down when the electrostatic forces are high enough due to large accumulation of charge at large times, even if V_{BB} is lower than dielectric breakdown limit. While the ideal geometric capacitor with $c(t) = C_1 \delta(t)$ will have $\lim_{t \uparrow \infty} q(t) = C_1 V_{BB}$ and will never breakdown when V_{BB} is less than dielectric breakdown limit.

We define integral capacity from the capacity function $c(t)$ as $c_{\text{int}}(t) = \int_0^t c(t') dt'$ for $t > 0$. The integration of the capacity function w.r.t. time from time of application of voltage excitation (in our case is $t = 0$). Thus for a classical capacitor with capacity function defined as $c(t) = C_1 \delta(t)$ we get integrated capacity as $c_{\text{int}}(t) = \int_0^t (C_1 \delta(t')) dt' = C_1$ for $t > 0$. We observe $\lim_{t \uparrow \infty} c_{\text{int}}(t) = C_1$ a constant value. This integrated capacity is what is discussed in classical theory that we derived from capacity function. Now for the case of fractional capacitor where the capacity-function as $c(t) = K_n t^{-n}$, the integrated capacity is for $t > 0$ $c_{\text{int}}(t) = \int_0^t K_n t'^{(-n)} dt' = \frac{K_n}{(1-n)} t^{1-n}$. This is same as used in [12], [35]. We note that $\lim_{t \uparrow \infty} c_{\text{int}}(t) = \infty$, for a fractional capacitor.

9.2: Difference in usage of integrated capacity and the time varying capacity function for obtaining loss-tangent value

Thus, the term ‘integrated capacity’ $c_{\text{int}}(t)$ of capacitor is analogous to ‘total’ water holding capacity of pitcher. The total water holding capacity of pitcher with metal walls is constant is equivalent to classical capacitor case, while the total water holding capacity of walls of porous pitcher is infinity is equivalent to the fractional capacitor case. We mention here the expressions for $C_{\text{int}}(\omega) = \mathcal{L}\{c_{\text{int}}(t)\}_{s=j\omega}$ cannot be

used to determine the loss tangent, while from capacity function with $C(\omega) = \mathcal{L}\{c(t)\}\big|_{s=j\omega}$ is used to determine loss tangent value.

10. Conclusion

In this paper we discussed that charge stored in a capacitor, as a function of time is not the usual multiplication operation of capacity and voltage; instead, the charge is convolution integral of these two functions, derived from causality principles. With this formulation, we showed for a fractional capacitor (where the current and voltage are related by fractional derivative), the charge goes to infinity for large times, when the fractional capacitor is placed on a constant voltage; and is line with earlier fractional order models and observations. With this formulation with convolution integral, we also showed relaxation current is in the form of impulse function for ideal geometrical capacitor of constant capacity, when stressed by a constant voltage and for fractional capacitor with power-law decay current that is given by universal dielectric relaxation law called as Curie von-Schweidler law. Practically, this new 'generalized- formulation' has use while getting the charge stored in a capacitor as a function of time with varying voltage stress across it, and to convert the fractional capacity units to usual capacity units in Farads.

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