A new look at formulation of charge storage in capacitors and application to classical capacitor and fractional capacitor theory

ABSTRACT

In this study, we revisit the concept of classical capacitor theory-and derive possible new explanations to the definition of capacitance, charge stored in capacitor. We introduce the capacity function with respect to time to describe the charge storage in a classical capacitor and fractional capacitor. Here we will describe that charge stored at any time in a capacitor say $q(t)$ as 'convolution integral' of defined capacity function $c(t)$ of a capacitor and voltage stress $v(t)$ across it i.e. $q(t) = c(t) * v(t)$. This approach however is different to the conventional method, where we multiply the capacity and voltage functions to obtain charge stored i.e. $q(t) = c(t)v(t)$. This new concept is in line with the observation of charge stored as a step function and the relaxation current in form of impulse function for 'ideal geometrical capacitor' of constant capacity when an uncharged capacitor is impressed with a constant voltage stress. Also this new formulation is valid for a power-law decay current that is given by 'universal dielectric relaxation law' called as 'Curie von-Schweidler law', when an uncharged capacitor is impressed with a constant voltage stress i.e. $i(t) \propto t^{-n}; \quad 0 < n < 1$ .This universal dielectric relaxation law gives rise to fractional derivative relating voltage stress and relaxation current that is formulation of 'fractional capacitor'. A 'fractional capacitor' we will discuss with this new concept of redefining the charge store definition i.e. via this 'convolution integral' approach, and obtain the loss tangent value. We will also show how for a 'fractional capacitor' by use of 'fractional integration' we can convert the fractional capacity a constant that is in terms of fractional units of $\mathrm{Farad}/\sec^{1-n}$ to units of Farad. From the defined capacity function, we will also derive integrated capacity of capacitor. We will also give possible physical explanation by taking example of porous and non-porous pitchers of constant volume holding water and thus, explaining the various interesting aspects of classical capacitor and fractional capacitor that we arrive with this new formulation. We note that circuit theory with classical calculus and fractional calculus remains unaltered with this new approach of defining charge storage via 'convolution integral'.

Keywords: convolution integral, fractional derivative, fractional integration, Curie-von Schweidler law, fractional capacity, geometrical capacity, time varying capacity function, integrated capacity, loss tangent.

1. Introduction

The classical geometric capacitor or a constant capacitor (that we are used to since our school days) having constant value of Farad means that it has constant value at all the frequencies from DC value of zero Hertz to infinite Hertz. This is ideal capacitor as though the dielectric used is having loss less relative permittivity ε_r and is constant (a purely real number with loss tangent value as zero) at all the frequencies. The capacity in this classical sense is given as $C_1 = \varepsilon_0 \varepsilon_r A/d$ i.e. by using geometric factor of ratio of area to the electrode separation. This we have learnt in textbooks. The ideal capacity that is constant at all the frequencies is called geometric capacity. This constant value C_1 in frequency domain is actually an impulse function in time domain i.e. $C_1\delta(t)$. A general practical capacitor, which is not a constant in frequency domain, is having a function in time domain and we call it as capacity function in time, representing as $c(t)$ .Where the frequency domain representation is via Laplace transformation i.e. $\mathcal{L}\{c(t)\} = C(s)$. We will derive that charge stored in capacitor, as a function of time is not usual multiplication operation of capacity function and voltage stress i.e. $q(t) \neq c(t)v(t)$; instead, the charge is 'convolution integral' of the two i.e. $q(t) = c(t) * v(t)$. However, the charge described in frequency domain as a function of frequency is multiplication operation of frequency domain functions of capacity-function and voltage-function, i.e. $Q(s) = C(s)V(s)$. We will revise this concept of capacitor in the paper, and derive various interesting concepts.

The Curie-von Schweidler law relates to the relaxation current in dielectric when a step DC voltage is applied and is given by $i(t) \sim t^{-n}$, where $t > 0$ and the power (exponent) i.e. n is called relaxation constant or decay constant, where $0 < n < 1$ [1]-[4], [12], [21], [22]. We note that n is non-integer. This relaxation law is taken as 'universal law', for dielectric relaxations. The Curie-von Schweidler behavior has been observed in many instances, since late 19th Century, such as those shown in dielectric studies and experiments [3] [4], [12], [13]-[17], [21], [22]. This power law relaxation of the 'non-Debye' type i.e. $i(t) \sim t^{-n}$ is interpreted as a many-body problem but can also be formulated as an infinite number of independent relaxing bodies meaning infinite number of relaxation rates varying from near zero to infinity [4], [5], [6], [22]. The power law relaxation is observed in the experiments with super-capacitors [7]-[11]. These studies [7]-[11] with non-Debye relaxation function (i.e. power-law relaxation) also indicate the use of fractional calculus as constituent expression to describe super-capacitors

The use of empirical power law i.e. Curie-von Schweidler Law of relaxation of current to a step input of voltage to get constituent relation with fractional derivative was proposed in [12] [21], by taking the concept of charge stored at any time as usual product of capacity function and voltage stressed i.e. $q(t) = c(t)v(t)$. We will revise the concept of capacitor in classical theory and apply the new concept of charge stored at any time as convolution integral of capacity function and the voltage stress i.e. $q(t) = c(t) * v(t)$ and also apply this concept in capacitors with observed Curie-von Schewdler relaxation current, and obtain same results as in [12] and [21]. We will also point out the differences with this new approach to the earlier approach in finding the capacity function and loss-tangent.

2. A brief about ideal capacitor

2.1: Ideal Loss less capacitor & Loss Tangent

What we know about geometric capacitor or a constant capacitor of say value C_1 is a constant value of Farad at all the frequencies from DC value of zero Hertz to infinite Hertz. This is ideal capacitor as though the dielectric used ε_r is lossless and is constant at all frequencies; and the capacity is given as

$C_1 = \varepsilon_0 \varepsilon_r A / d$ i.e. by using geometric factor of area to electrode separation ratio. This ideal capacity is constant at all the frequencies is called geometric capacity. Therefore, if we say s as complex frequency (Laplace variable) then this constant capacity is given as following function

$$C(s) = C_1 \qquad s = \mathrm{i}\omega \qquad \mathrm{i} = \sqrt{-1} \qquad C(\omega) = C_1 \tag{1}$$

The Laplace complex frequency is written in (1) as $s = \mathrm{i}\omega$ for writing sinusoidal or steady state frequency domain analysis [12], [19], [21]. From (1) we see that $C(\omega) = \mathrm{Re}\left[C(\omega)\right] - \mathrm{i}\,\mathrm{Im}\left[C(\omega)\right] = C_1 - \mathrm{i}(0)$ has only real part with imaginary part as zero at all frequencies; that gives loss tangent as $\tan\phi = \frac{\mathrm{Im}[C(\omega)]}{\mathrm{Re}[C(\omega)]} = 0$ Thus, ideal capacitor (1) is a loss less capacitor. The dielectric loss is expressed as loss tangent for a complex dielectric quantity given as $\varepsilon_r(\omega) = \mathrm{Re}\left[\varepsilon_r(\omega)\right] - \mathrm{i}\,\mathrm{Im}\left[\varepsilon_r(\omega)\right]$ where loss tangent is given as $\tan\phi = \frac{\mathrm{Im}[\varepsilon_r(\omega)]}{\mathrm{Re}[\varepsilon_r(\omega)]}$.

2.2: Representation of time varying capacity function as delta function for a loss-less ideal capacitor

Since the inverse Laplace transform of function $F(s) = 1$ gives time function i.e. $f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \delta(t)$ i.e. a Dirac delta function at $t = 0$, we say the 'time varying capacity function' call it $c(t)$ of geometric capacitor (ideal-capacitor) is following (2) by taking inverse Laplace transform of (1), i.e. $c(t) = \mathcal{L}^{-1}\left\{C(s)\right\}$ we write the following expression for a time varying capacity function for ideal loss-less capacitor

$$c(t) = C_1 \delta(t) \tag{2}$$

Therefore, we say that a constant ideal capacitor has a 'capacity function' $c(t)$ as Dirac delta function. For example if the capacity of a capacitor is a function of frequency say as $C(s) = C_m s^{-m}$; then the time varying capacity function $c(t)$ for this capacitor $c(t) = \mathcal{L}^{-1}\left\{C(s)\right\}$ is following

$$c(t) = \frac{C_m}{(m-1)!} t^{m-1}; \qquad t > 0 \tag{3}$$

If the capacity function is constant $c(t) = C_0$ for $t \geq 0$ only if the frequency function is $C(s) = C_0 s^{-1}$. Therefore, we say $c(t) = C_0$, $t \geq 0$ is not a constant capacitor or a lossless capacitor. This capacitor with capacity function $c(t) = C_0$, $t \geq 0$ in frequency domain in complex notation is $C(\omega) = 0 - \mathrm{i}\omega^{-1} C_0$, $\mathrm{i} = \sqrt{-1}$ with loss tangent as infinity.

2.3: The charge function in time is convolution integral of capacity function and voltage function

When we apply a voltage function $v(t)$ to an uncharged capacity we write the charge stored at any time as convolution integral as follows

$$\begin{aligned} q(t) &= c(t) * v(t) \\ &= \int_{-\infty}^{t} \big(c(t-x)\big)\big(v(x)\big)\mathrm{d}x = \int_{-\infty}^{t} \big(c(y)\big)\big(v(t-y)\big)\mathrm{d}y \end{aligned} \tag{4}$$

This is against conventional way of writing the charge i.e.

$$q(t) = c(t)v(t); \qquad c(t) = \frac{q(t)}{v(t)} \tag{5}$$

This argument in (4) we will explain in the subsequent section.

In reality the capacity of a capacitor, say of $1\mu\mathrm{F}$ means this value is at particular frequency of measurement standard is at $1\mathrm{kHz}$ (also depends on application) [12]. Practically due to losses in ε_r the value of capacity of capacitor is varying in frequency; therefore in reality we have time varying capacity function $c(t)$ describing a capacitor.

3. Reviewing concept of charge storage in constant capacitor in the classical theory

3.1: Impedance Function in Laplace domain for a ideal capacitor

We have standard expression of 'impedance of a capacitor' i.e. $Z(s)$ expressed in frequency domain as following in following expression (6) with $V(s) = \mathcal{L}\{v(t)\}$, $I(s) = \mathcal{L}\{i(t)\}$

$$Z(s) = \frac{V(s)}{I(s)} = \frac{1}{C_1 s} \qquad\qquad Z(\omega) = \frac{V(\omega)}{I(\omega)} = -\mathrm{i}\frac{1}{\omega C_1} \tag{6}$$

Thus from (6), we have the capacity function expressed in Laplace frequency domain as a function as

$$C_1 = \frac{s^{-1}I(s)}{V(s)} \tag{7}$$

We note that the constant C_1 is Laplace transformed quantity, i.e. $C_1 = \mathcal{L}\{c(t)\}$; and in this case of 'constant capacity' the capacity function in time is $c(t) = C_1\delta(t)$ (2). Therefore, we have in frequency domain representation of capacitor as function of Laplace variable s, so we call it as $C(s) = \mathcal{L}\{c(t)\}$. Therefore, for a general relation of capacity in frequency domain we have following expression

$$C(s) = \frac{s^{-1}I(s)}{V(s)}; \quad C(s) = \mathcal{L}\{c(t)\}, \quad V(s) = \mathcal{L}\{v(t)\}, \quad I(s) = \mathcal{L}\{i(t)\} \tag{8}$$

3.2: Getting charge function in time domain as convolution integral of capacity function and voltage function from the impedance function in Laplace domain

The numerator term in (8) i.e. $s^{-1}I(s)$ in time domain is $\int_0^t i(x)\mathrm{d}x$ [19] that is charge the $q(t)$, i.e. $q(t) = \int_0^t i(x)\mathrm{d}x$ with its Laplace transform as $Q(s) = \mathcal{L}\{q(t)\}$. Therefore, from (8), we write charge in frequency domain as following expression

$$Q(s) = C(s)V(s) \tag{9}$$

This (9) is the expression in frequency domain. In the time domain, we write the charge equation as convolution integral [19], i.e. using $\mathcal{L}^{-1}\{(F_1(s))(F_2(s))\} = (\mathcal{L}^{-1}\{F_1(s)\}) * (\mathcal{L}^{-1}\{F_2(s)\}) = f_1(t) * f_2(t)$ where $F_j(s) = \mathcal{L}\{f_j(t)\}$, $j = 1,2$ i.e. $q(t) = \mathcal{L}^{-1}\{C(s)V(s)\}$ gives the following expression

$$\begin{aligned} q(t) &= c(t) * v(t) \\ &= \int_{-\infty}^{t} \big(c(t-x)\big)\big(v(x)\big)\mathrm{d}x \end{aligned} \tag{10}$$

Where in (10) convolution operation is denoted as $(*)$ and the convolution of two functions $f_1(t)$ and $f_2(t)$ is defined as following convolution integral

$$f_1(t) * f_2(t) = \int_{-\infty}^{t} \big(f_1(t-x)\big)\big(f_2(x)\big)\mathrm{d}x = \int_{-\infty}^{t} \big(f_1(x)\big)\big(f_2(t-x)\big)\mathrm{d}x \tag{11}$$

3.3: Charge function and current function in time domain when uncharged ideal capacitor is stressed with constant voltage by using convolution approach

Let an uncharged capacitor of constant capacity at $t=0$, of value C_1 is charged with a constant step voltage V_{BB} applied at $t=0$ i.e. $v(t) = V_{BB}\big(u(t)\big);\ \ t \ge 0$. By conventional approach using $q(t) = c(t)v(t)$ we say charge stored at any time for $t>0$ is $q(t) = C_1 V_{BB}$, whereas the charge is $q(t)=0$ for $t<0$. Thus the charge in time domain is a step function, we denote that as $q(t) = C_1 V_{BB}\big(u(t)\big)$; with $u(t)$ as unit-step function at $t=0$. Laplace transform of this step charge is following

$$\mathcal{L}\{u(t)\} = \frac{1}{s} \qquad\qquad Q(s) = \mathcal{L}\{C_1 V_{BB} u(t)\} = \frac{C_1 V_{BB}}{s} \tag{12}$$

The first derivative of charge i.e. $q^{(1)}(t)$ gives the charging current (or relaxation current) i.e.

$$\begin{aligned} i(t) = q^{(1)}(t) &= \frac{\mathrm{d}q(t)}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} C_1 V_{BB} u(t) = C_1 V_{BB}\big(\delta(t)\big) \end{aligned} \tag{13}$$

This (13) is classical result that we all know is as per classical capacitor-theory that is charging current is impulse function at the time of application of voltage step, to an uncharged capacitor. This impulse current also comes from circuit equation i.e. $\frac{1}{C_1}\int i(t)\mathrm{d}t = V_{BB}\big(u(t)\big)$; and the classical theory deals with geometrical capacitor given by $C_1 = \varepsilon_0 \varepsilon_r A/d$.

Now let us look at convolution integral, for $q(t) = c(t)*v(t) = \int_{-\infty}^{t}\big(c(t-x)\big)\big(v(x)\big)\mathrm{d}x$ for $t>0$ i.e. where we have $v(t) = V_{BB}$, for $t \ge 0$. Only if we define $c(t)$ as function of time as the capacity function i.e. $c(t) = C_1\big(\delta(t)\big)$ we will be getting $q(t) = C_1 V_{BB}$ for $t>0$ demonstrated in following steps

$$\begin{aligned} q(t) = c(t)*v(t) &= \int_{-\infty}^{t}\big(c(t-x)\big)\big(v(x)\big)\mathrm{d}x; \quad c(x) = C_1\big(\delta(x)\big), \quad v(x) = V_{BB}, \quad x \ge 0 \\ &= \int_{0}^{t} C_1\big(\delta(t-x)\big)\big(V_{BB}\big)\mathrm{d}x = C_1 V_{BB}\int_0^t \big(\delta(t-x)\big)\mathrm{d}x; \qquad t \ge 0 \\ &= C_1 V_{BB}; \qquad t \ge 0 \end{aligned} \tag{14}$$

We have used identity $\int \big(\delta(x_0 - x)\big)\mathrm{d}x = 1$, i.e. property of delta function. Thus from above (14) we get charge as step function at $t=0$, given as following expression

$$q(t) = C_1 V_{BB}\left(u(t)\right) \tag{15}$$

The meaning of capacity function $c(t)$ in time domain is $c(t) = C_1\left(\delta(t)\right)$ i.e. an impulse of height C_1 (in units Farad) at the time of application of voltage excitation (i.e. $t = 0$), refer Figure-1. Whereas, in the frequency domain, the definition of capacity i.e. for geometrical capacity is, $C(s) = C_1$ i.e. $C(s) = \mathcal{L}\left\{C_1\delta(t)\right\} = C_1$ that is a constant (in unit of Farad) value at all frequencies that we have discussed earlier (1).

3.4: For a loss less ideal capacitor phase between charge function and voltage function is zero

Therefore, with $V(s) = V_{BB}/s$ we get $Q(s) = C(s)V(s) = C_1 V_{BB}/s$. Thus when we say a capacitor is having a constant value, it implies that its capacity function is an 'impulse' function at the time of application of voltage stress; in time domain, say at $t = 0$. The constant capacity C_1 is written as capacity function of time as $c(t) = C_1\left(\delta(t)\right)$. For any other time say at time, say $t = t_0$ of application of voltage-stress the classical geometrical constant capacitor is expressed as capacity function $c(\tau) = C_1\left(\delta(\tau)\right)$, with $\tau = t - t_0$. From now on we will state $t = 0$ as time of application of voltage stress to uncharged capacitor with capacity function as $c(t)$.

Say we apply $v(t) = \cos at$ at $t = 0$, for $t \geq 0$; then Laplace transform of $v(t)$ is $V(s) = s/(s^2 + a^2)$, to an uncharged constant capacitor $C(s) = C_1$. This gives $Q(s) = C_1\left(s/(s^2 + a^2)\right)$ implying $q(t) = C_1 \cos at;\quad t \geq 0$. Thus we observe for a constant capacitor, there is no phase difference between $v(t)$ and $q(t)$. We do the same deduction following the convolution integration formulation (16). Also, refer Figure-1 for curves of $v(t)$ and $q(t)$ that have no delay implying no phase difference for a constant capacity case.

$$\begin{aligned} q(t) = c(t) * v(t) &= \int_{-\infty}^{t}\left(c(t-x)\right)\left(v(x)\right)\mathrm{d}x; \quad c(x) = C_1\left(\delta(x)\right), \quad v(x) = \cos ax; \quad x \geq 0 \\ &= \int_{0}^{t} C_1\left(\delta(t-x)\right)\left(\cos ax\right)\mathrm{d}x; \qquad t \geq 0 \\ &= C_1 \cos at; \qquad t \geq 0 \end{aligned} \tag{16}$$

We have used identity $\int\left(\delta(x_0 - x)\right)\left(f(x)\right)\mathrm{d}x = f(x_0)$, i.e. property of delta function.

3.5: Generalizing the charge function and current function for arbitrary voltage stress

Thus, we have general expression for any time varying voltage $v(t)$ applied at uncharged capacitor with geometrical capacity given by capacity function as $c(t) = C_1\left(\delta(t)\right)$, will have charge $q(t)$ for $t \geq 0$ as following (17) convolution integral

$$\begin{aligned} q(t) = c(t) * v(t) &= \int_{-\infty}^{t}\left(c(t-x)\right)\left(v(x)\right)\mathrm{d}x; \quad c(x) = C_1\left(\delta(x)\right), \quad v(x); \quad x \geq 0 \\ &= \int_{0}^{t} C_1\left(\delta(t-x)\right)\left(v(x)\right)\mathrm{d}x; \qquad t \geq 0 \\ &= C_1\left(v(t)\right); \qquad t \geq 0 \end{aligned} \tag{17}$$

Now we differentiate the expression above (17) of $q(t)$ to write following expression

$$
\begin{aligned}
i(t) = q^{(1)}(t) &= \frac{\mathrm{d}q(t)}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(C_1\left(v(t)\right)\right), \qquad t \geq 0 \\
&= v(t)\frac{\mathrm{d}C_1}{\mathrm{d}t} + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} \\
&= \left(v(t)\right)\left(C_1\left(\delta(t)\right)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} = C_1\left(v(0)\delta(t)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} \\
&= i(0) + i(t), \qquad t \geq 0
\end{aligned}
\tag{18}
$$

The first term at RHS of (18), indicate the value of current at $t = 0$. The constant function starting at $t = 0$ i.e. C_1 when differentiated gives $C_1\delta(t)$. This unit delta functions at $t = 0$, i.e. $\delta(t)$ when multiplied by $v(t)$ gives $v(0)\delta(t)$. This comes from property $\int\left(\delta(x_0 - x)\right)\left(f(x)\right)\mathrm{d}x = f(x_0)$, differentiation of this gives $\left(\delta(x_0 - x)\,f(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\,f(x_0) = f(x_0)\delta(x)$. Thus at $t = 0$ we have $i(0) = C_1 v(0)$ and $i(0) = 0$ for $t > 0$. Compositely we write $i(0) = C_1 v(0)\left(\delta(t)\right)$, i.e. specifying its value at only $t = 0$. The second term is $i(t)$ for $t \neq 0$, that is $i(t) = C_1\left(v^{(1)}(t)\right)$ (refer Figure-1). We write the following expression for $i(t)$ as

$$
i(t) = C_1 v(0)\left(\delta(t)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{19}
$$

The obtained expression (19) via the formulation $q(t) = c(t) * v(t)$ is consistent with expression obtained in [21], where $q(t) = c(t)v(t)$ is used.

As an example, we take $v(t) = V_{BB}u(t)$ a step input at time $t = 0$, to an uncharged capacitor. We have $v^{(1)}(t) = 0$ for $t > 0$; and at $t = 0$ we have, $v(0) = V_{BB}$. Using (19), we get $i(0) = C_1 V_{BB}\left(\delta(t)\right)$; this makes $i(t) = C_1 V_{BB}\left(\delta(t)\right), \quad t \geq 0$ .This is for geometrical capacity charging current is impulse function.

Generally the capacitance is not a constant parameter of the capacitor, it varies in frequency and therefore in time too. The constant capacitor concept is approximation when we assume the relative permittivity ε_r to be constant (note that geometrical capacity we define as $C_1 = \varepsilon_0\varepsilon_r A/d$) [12], [21]. We note that only a loss free capacitor has a constant capacitance in frequency domain. Losses manifest themselves in frequency domain as a phase angle, ϕ by which $q(t)$ lags $v(t)$, or given as loss tangent i.e. $\tan\phi$ in the charge expression of capacitor i.e.

$$
q(t) = c(t) * v(t) \qquad \text{or} \qquad Q(s) = \left(C(s)\right)\left(V(s)\right) \tag{20}
$$

For $C(\omega) = C_1$, the constant geometrical capacitor with capacity function as $c(t) = C_1\delta(t)$ (1) (2), we have $\tan\phi = \frac{\operatorname{Im}C(\omega)}{\operatorname{Re}C(\omega)} = 0$; that is ideal lossless capacitor.

Therefore, we say that charge stored in capacitor, as a function of time is not multiplication operation of capacity and voltage i.e. $q(t) \neq c(t)v(t)$; instead, the charge is convolution integral, i.e. $q(t) = c(t) * v(t)$ However, the charge as a function of frequency is multiplication operation of frequency domain functions of capacity and voltage, i.e. $Q(s)=C(s)V(s)$. A time varying capacity say $c(t)\propto t^{-n}$, $0<n<1$ has a delay between $v(t)$ and $q(t)$ that is shown in Figure-1 (for time varying capacity case), where $q(t)$ lags $v(t)$. We will see that $c(t)\propto t^{-n}$, $0<n<1$ will have non-zero loss-tangent i.e. it represents a lossy capacitor-and thus phase delay between $q(t)$ and $v(t)$.

3.6: Time varying capacity function as convolution integral of charge function and inverse voltage function

From the expression (20), $C(s)=Q(s)/V(s)$ we write the time varying capacity $c(t)$ by use of convolution integral in the following steps

$$\begin{aligned}
C(s) &= \frac{Q(s)}{V(s)} \\
\mathcal{L}^{-1}\left\{C(s)\right\} &= \mathcal{L}^{-1}\left\{\left(Q(s)\left(V(s)\right)^{-1}\right)\right\} \\
c(t) &= \mathcal{L}^{-1}\left\{Q(s)\right\} * \mathcal{L}^{-1}\left\{(V(s)^{-1}\right\} \\
&= \left(q(t)\right) * \left(v(t)\right)^{-1}
\end{aligned} \tag{21}$$

From above derivation (21), we say that capacity i.e. $c(t)\neq q(t)/v(t)$ i.e. not the usual ratio of charge to voltage in time domain, but it is given as convolution expression i.e.

$$\begin{aligned}
c(t) &= \left(q(t)\right) * \left(v(t)\right)^{-1} \\
&= \int_{-\infty}^{t} \frac{q(t-x)}{v(x)} \mathrm{d}x = \int_{-\infty}^{t} \frac{q(x)}{v(t-x)} \mathrm{d}x
\end{aligned} \tag{22}$$

Let us verify, with $q(t)=\left(C_1 V_{BB}\right)\left(u(t)\right)$ i.e. at $t=0$ and $q(t)=0$ for $t<0$, and $v(t)=V_{BB}u(t)$ i.e. a step voltage at $t=0$, gives following steps

$$\begin{aligned}
c(t) &= \left(q(t)\right) * \left(v(t)\right)^{-1} \\
&= \int_{-\infty}^{t} \frac{C_1 V_{BB} u(t-x)}{V_{BB} u(x)} \mathrm{d}x = \int_{-\infty}^{t} \frac{C_1 V_{BB} u(x)}{V_{BB} u(t-x)} \mathrm{d}x \\
&= C_1\left(u(t) * u(t)^{-1}\right) \\
&= C_1\left(\delta(t)\right)
\end{aligned} \tag{23}$$

We have used inverse identity i.e. $f * f^{-1} = \delta$ in (23).

Therefore, capacity at any time is the history of ratio of charge to voltage given by convolution integral (22). We can verify with say $q(t)=\left(C_1 \cos at\right)u(t)$ for $t\geq 0$ with $v(t)=\left(C_1 \cos at\right)u(t)$ for $t\geq 0$ gives the following

$$
\begin{aligned}
c(t) &= \left(q(t)\right) * \left(v(t)\right)^{-1} \\
&= \int_{-\infty}^{t} \frac{C_1 \cos a(t-x)}{\left(\cos ax\right)} \mathrm{d}x = \int_{-\infty}^{t} \frac{C_1 \cos ax}{\cos a(t-x)} \mathrm{d}x \\
&= C_1 \left(\left(\cos(at)\right) * \left(\cos(at)\right)^{-1} \right) \\
&= C_1 \left(\delta(t)\right)
\end{aligned} \tag{24}
$$

We have used inverse identity i.e. $f * f^{-1} = \delta$ in (24). We note here the formula used in [12], [21] is $c(t) = q(t) / v(t)$, whereas we used $c(t) = \left(q(t)\right) * \left(v(t)\right)^{-1}$.

4. Fractional Derivative directly from Curie-von Schweidler Law-Fractional Capacitor

4.1: Impedance and Admittance in Laplace domain for a fractional capacitor

Practically on applying a step input voltage $v(t) = V_{BB}$ Volts at $t = 0$ to a capacitor which is initially uncharged; we get a power-law decay of current given by empirical Curie-von Schweidler as $i(t) \sim t^{-n}$; $0 < n < 1$ [12], [21]. That we write in following way as indicated by experimental studies [12]-[17], [21], and [22]

$$
i(t) = K_n \frac{V_{BB}}{t^n} \qquad t > 0 \tag{25}
$$

The parameter K_n is proportionality constant, while in [12], [21] the proportionality constant is $1/h_1$. This is from observation and the evaluation of order of power-law function is $0.5 < n < 1$ [7]-[12], [21]. Let the uncharged capacitor be excited by a constant step input of V_{BB} Volts, i.e. written as $v(t) = V_{BB}\left(u(t)\right)$, where $u(t)$ is unit step function at $t = 0$. The Laplace transform of step input is following

$$
V(s) = \mathcal{L}\left\{v(t)\right\} = \mathcal{L}\left\{V_{BB}\left(u(t)\right)\right\} = \frac{V_{BB}}{s} \tag{26}
$$

and then taking Laplace transform of (25) i.e. of power-law decay current by using $\mathcal{L}\left\{t^m\right\} = m!\, s^{-(m+1)}$, [19] we write following expression for $I(s) = \mathcal{L}\left\{i(t)\right\}$

$$
\begin{aligned}
I(s) &= \mathcal{L}\left\{i(t)\right\} = \mathcal{L}\left\{K_n V_{BB} t^{-n}\right\} \\
&= K_n V_{BB} \left(\frac{(-n)!}{s^{-n+1}} \right)
\end{aligned} \tag{27}
$$

Using the formula for generalization of factorial i.e. $(\alpha - 1)! = \Gamma(\alpha)$ [6], [20], we get the following expressions

$$
\begin{aligned}
I(s) &= K_n \frac{\Gamma(1-n) V_{BB}}{s^{1-n}} \\
&= K_n \frac{\Gamma(1-n)}{s^{-n}} \left(\frac{V_{BB}}{s} \right)
\end{aligned} \tag{28}
$$

We get Transfer function [19] of capacitor as following expression for admittance $Y(s)$

$$Y(s)=\frac{I(s)}{V(s)}=\frac{K_n\dfrac{\Gamma(1-n)}{s^{-n}}\left(\dfrac{V_{BB}}{s}\right)}{\left(\dfrac{V_{BB}}{s}\right)} \tag{29}$$
$$=K_n\left(\Gamma(1-n)\right)s^n=C_ns^n \qquad C_n=K_n\left(\Gamma(1-n)\right)$$

This expression i.e. $Y(s)=I(s)/V(s)$ in (29) is 'admittance' expression in complex frequency (s) domain of a capacitor. Putting, $s=\mathrm{i}\omega$ in (29) we get $I(\omega)=\left(\cos\frac{n\pi}{2}+\mathrm{i}\sin\frac{n\pi}{2}\right)\omega^nC_nV(\omega)$. This means current leads voltage in fractional capacitor by angle $\frac{n\pi}{2}$. For $n=1$, i.e. for a classical geometrical ideal capacitor we have $I(\omega)=\mathrm{i}\omega C_1V(\omega)$, that is current leading voltage by angle of 90^0.

4.2: Current voltage relation by fractional derivative for fractional capacitor

From here (29), we write impedance expression $Z(s)=V(s)/I(s)$ for fractional-capacitor as following

$$Z(s)=\frac{1}{C_ns^n}, \quad 0<n<1 \tag{30}$$

From the obtained expression (29) i.e. $I(s)=C_ns^n\left(V(s)\right)$ and by Laplace inversion by using the identity $\mathcal{L}^{-1}\left\{s^nF(s)\right\}={}_0D_t^n\left[f(t)\right]$ i.e. fractional derivative operation [6], [20], we get the constituent relation for capacity as following

$$i(t)=C_n\left({}_0D_t^n\left[v(t)\right]\right), \qquad 0<n<1 \tag{31}$$

4.3: Fractional units for fractional capacitor

The 'fractional capacity' C_n is in unit of $\mathrm{Farad}/\mathrm{sec}^{1-n}$; [12], [21] which is constant given by $C_n=K_n\left(\Gamma(1-n)\right)$. This fractional derivative expression (31) gives a new capacitor theory [12], [21] and we utilize this above formula (31) to find characteristics of super-capacitors, variation of n with current excitation, and energy discharged to energy stored [13]-[17]. Classically the expression of capacitor is $i(t)=C_1\left(D_t^{(1)}\left[v(t)\right]\right)$ i.e. with one-whole order (classical) derivative.

Curie-von Schweidler law gives a different approach for capacitor theory based on fractional calculus [12], [21], [22]. In experimental observations, we find that capacitor has fractional order impedance [7]-[17], [21]. This section gives us the understanding that this empirical law i.e. Curie-von Schweidler law gives a relation of voltage and current of capacitor by using fractional derivative. We will derive this (31) by the new approach of the definition of charge in the subsequent section.

5. Charge stored in a fractional capacitor using convolution integral of time varying capacity due to Curie-von Schweidler relaxation current

5.1: Expression of charge storage from Curie-von Schweidler relaxation current in a fractional capacitor

For Curie-von Schweidler law we have relaxation current as noted earlier (25) empirically expressed as $i(t)=K_nV_{BB}t^{-n}$, $0<n<1$ for $t>0$, i.e. when uncharged capacitor is applied with a step voltage $v(t)=V_{BB}\left(u(t)\right)$ at $t=0$. This empirical expression of current relaxation gives a relation of incremental charge Δq (or $\mathrm{d}q$ in infinitesimal small limit) when 'pulse' of a voltage of magnitude V_{BB} is applied for a duration Δt (or in infinitesimal small limit $\mathrm{d}t$) given by following expressions

$$\Delta q=\frac{K_nV_{BB}\Delta t}{t^n} \qquad \mathrm{d}q=\frac{K_nV_{BB}\mathrm{d}t}{t^n} \tag{32}$$

With this above (32) expression (and by $q(t)=\int_0^t \mathrm{d}q$) we write the charge accumulated for this power law decay current as following

$$\begin{aligned} q(t)=\int_0^t \mathrm{d}q=\int_0^t \frac{K_nV_{BB}\mathrm{d}x}{x^n} \\ =\frac{K_nV_{BB}}{(1-n)}t^{1-n}, \quad 0<n<1 \quad t>0 \end{aligned} \tag{33}$$

5.2: Expression for time varying capacity function from admittance relation of a fractional capacitor

From the expression in frequency domain (8) i.e. $C(s)=\left(s^{-1}I(s)\right)/\left(V(s)\right)=\left(Q(s)\right)/\left(V(s)\right)$ we have for $i(t)=K_nV_{BB}t^{-n}$ with $I(s)=K_n\left(\Gamma(1-n)\right)V_{BB}s^{n-1}$, and $V(s)=V_{BB}/s$, gives $C(s)$ as following

$$\begin{aligned} C(s)=\frac{\left(s^{-1}I(s)\right)}{V(s)}=\frac{s^{-1}\left(K_n\left(\Gamma(1-n)\right)V_{BB}s^{n-1}\right)}{V_{BB}s^{-1}} \\ =\frac{K_n\left(\Gamma(1-n)\right)}{s^{1-n}}; \qquad m!=\Gamma(1+m) \\ =K_n\frac{(-n)!}{s^{1+(-n)}} \end{aligned} \tag{34}$$

Now doing inverse Laplace transform by using $\mathcal{L}^{-1}\left\{(m!)/s^{(1+m)}\right\}=t^m$ of above (34) we get 'time dependent' capacity function $c(t)$ as following

$$c(t)=K_nt^{-n}; \quad 0<n<1, \quad t>0 \tag{35}$$

5.3: Using convolution integral and time dependent capacity function evaluation of charge storage in time for a constant voltage applied to fractional capacitor

Using the convolution integral with this time dependent capacity function (35) step voltage applied at time zero, i.e. we get following expression for charge stored

$$
\begin{aligned}
q(t) &= \left(c(t)\right) * \left(v(t)\right) = \int_{-\infty}^{t} \left(c(t-x)\right)\left(v(x)\right) \mathrm{d}x, \qquad c(x) = K_n x^{-n}, \quad v(x) = V_{BB}; \quad t > 0 \\
&= \int_{0}^{t} K_n \left((t-x)^{-n}\right)\left(V_{BB}\right) \mathrm{d}x, \qquad 0 < n < 1 \\
&= -V_{BB} K_n \left. \frac{(t-x)^{1-n}}{1-n} \right|_{x=0}^{x=t} = \frac{V_{BB} K_n}{1-n} t^{1-n}
\end{aligned}
\tag{36}
$$

The expression above (36) obtained expression $q(t) = \frac{V_{BB} K_n}{1-n} t^{1-n}$ obtained via our formula $q(t) = c(t) * v(t)$ is same as we got via $q(t) = \int_0^t \mathrm{d}q$ above in (33).

6. Observations on breakdown mechanism of a fractional capacitor and loss tangent and comparison with earlier theory

6.1: When a fractional capacitor is float on a constant voltage the charge accumulated at large times is infinity-giving electrostatic break down

We note here from (36) that for $0 < n < 1$, the charge store is $\lim_{t \uparrow \infty} q(t) = \infty$ when the capacity function is $c(t) = K_n t^{-n}$, following Curie-von Schweidler decay current. Whereas for a classical capacity function i.e. given as $c(t) = C_1 \delta(t)$, the charge at large times is $\lim_{t \uparrow \infty} q(t) = C_1 V_{BB}$ (15). This observation i.e. $\lim_{t \uparrow \infty} q(t) = \infty$ in our derivation is with convolution formula i.e. $q(t) = c(t) * v(t)$ is in line with the observations in [12], [21], where the used expression for charge is $q(t) = \left(c(t)\right)\left(v(t)\right)$. This is the new idea of breakdown of capacitors due to accumulation of enough charge (electrostatic breakdown) at a constant voltage even though voltage is less than the breakdown limit of dielectric proposed in [12], [21].

6.2: Evaluation of loss tangent by using earlier approach and the convolution approach of charge storage concept for fractional capacitor

In [12] and [21] the charge formula used is $c(t) = q(t) / v(t)$ and not via convolution approach that we discussed in this paper. In addition, with this formula $c(t) = q(t) / v(t)$ in [12] and [21] gets the time dependent capacity function as following where the constant h_1 is used in Curie von-Schweidler relaxation current i.e. $h_1 \equiv \left(K_n\right)^{-1}$

$$
c(t) = \frac{t^{1-n}}{h_1 (1-n)}, \qquad t > 0; \qquad 0 < n < 1 \tag{37}
$$

The frequency domain representation for $c(t)$ obtained in [12] and [21] is following

$$
\begin{aligned}
C(s) &= \frac{(1-n)!}{h_1 (1-n)} s^n, \qquad 0 < n < 1, \quad s = \mathrm{i}\omega \\
C(\omega) &= \left(\frac{\omega^n (1-n)!}{h_1 (1-n)} \right) \left(\cos \tfrac{n\pi}{2} + \mathrm{i} \sin \tfrac{n\pi}{2} \right)
\end{aligned}
\tag{38}
$$

Here from (38) if we expresses loss tangent as $\tan\phi = \frac{\mathrm{Im}[C(\omega)]}{\mathrm{Re}[C(\omega)]} = -\tan\left(\frac{n\pi}{2}\right)$, which is not correct, as the loss tangent is $\tan\phi = \tan(1-n)\frac{\pi}{2}; \quad 0<n<1$ for a fractional capacitor.

Therefore, in [12] and [21], the loss tangent is not calculated by the using capacity function $c(t)$ (37), instead, phase difference ψ is calculated between current $I(\omega)$ and voltage $V(\omega)$ by using admittance expression $Y(s)\big|_{s=\mathrm{i}\omega}$ (29) and then doing steady state (sinusoidal) analysis, and then writing loss tangent as $\tan\phi = \tan\left(\frac{\pi}{2}-\psi\right)$, which is $\tan\phi = \tan\left(\frac{(1-n)\pi}{2}\right)$.

This above expression (37) and (38) of [12] and [21] says that the time varying capacity function will be growing to infinity as time grows. Also in frequency domain, we will be getting infinite value at infinite frequency. This gives us notion of unrealistic property of capacity function, which is unstable.

Whereas we have from our new derivation (35) the following for a fractional capacitor

$$\begin{aligned} &c(t) = K_n t^{-n}; \qquad t>0, \quad 0<n<1 \\ &C(s) = K_n\left(\Gamma(1-n)\right)s^{-(1-n)}; \qquad s=\mathrm{i}\omega \\ &C(\omega) = K_n\left(\Gamma(1-n)\right)\omega^{-(1-n)}\left(\cos\tfrac{(1-n)\pi}{2} - \mathrm{i}\sin\tfrac{(1-n)\pi}{2}\right) \end{aligned} \tag{39}$$

where the capacity function tends towards zero for large time and large frequency. From above (39) we get loss tangent as

$$\tan\phi = \frac{\mathrm{Im}\left[C(\omega)\right]}{\mathrm{Re}\left[C(\omega)\right]} = \tan\left(\tfrac{(1-n)\pi}{2}\right) \tag{40}$$

which is also as reported in [12], [21]; obtained differently than demonstrated in (39). However, [12] and [21] gives other expressions, same as that we will derive and report subsequently.

7. Further derivations regarding fractional capacitor in conjugation to classical capacitor

7.1: Getting charge function in time domain as convolution integral of capacity function and voltage function from the impedance function in Laplace domain for fractional capacitor

Now we do the steps as we did for classical capacitor, from the obtained impedance relation of fractional capacitor i.e.

$$Z(s) = s^{-n}\frac{1}{C_n}; \qquad 0<n<1 \qquad C_n = K_n\Gamma(1-n) \qquad C_n \equiv \mathrm{Farad}/\sec^{1-n} \tag{41}$$

with $C_n(s) = \mathcal{L}\{c_n(t)\} = C_n = K_n\left(\Gamma(1-n)\right)$ as obtained in earlier section (30), a constant in units of $\mathrm{Farad}/\sec^{1-n}$. We note that $C_n = K_n\left(\Gamma(1-n)\right)$ is in units of $\mathrm{Farad}/\sec^{1-n}$; a “fractional form” of unit [12], [21], defining a “fractional capacity” as constant in the frequency domain. Thus, we expect that in time domain the fractional capacity call it $c_n(t)$ be given by delta function at $t=0$ i.e. following

$$c_n(t) = \left(K_n\left(\Gamma(1-n)\right)\right)\left(\delta(t)\right) = C_n\delta(t) \tag{42}$$

The meaning of capacity function $c_n(t)$ in time domain is $c_n(t) = C_n\left(\delta(t)\right)$ i.e. an impulse of height C_n (in units $\mathrm{Farad}/\mathrm{sec}^{1-n}$) at the time of application of voltage excitation (i.e. $t=0$). Whereas, in the frequency domain, the definition of fractional capacity is $C_n(s) = C_n$ i.e. $C_n(s) = \mathcal{L}\left\{C_n\delta(t)\right\} = C_n$ that is a constant (in unit of $\mathrm{Farad}/\mathrm{sec}^{1-n}$) value at all frequencies.

We say here that classical geometrical capacitor presents a Farad value as impulse function at the time of application of voltage stress, while the fractional capacitor presents a $\mathrm{Farad}/\mathrm{sec}^{1-n}$ value at the time of application of voltage.

From this (42) we write following steps, with $C_n(s) = \mathcal{L}\left\{c_n(t)\right\}$, $\mathcal{L}^{-1}\left\{s^{-n}F(s)\right\} = {}_0\mathcal{I}_t^n f(t)$ where ${}_0\mathcal{I}_t^n$ is defining fractional integration operation [6], [20] of fractional order $0<n<1$

$$
\begin{aligned}
C_n(s) &= \frac{s^{-n}I(s)}{V(s)} = \frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^n\left[i(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}}; \qquad 0<n<1 \qquad {}_0\mathcal{I}_t^n\left[f(t)\right] = {}_0\mathcal{I}_t^{n-1}\,{}_0\mathcal{I}_t^1\left[f(t)\right] \\
&= \frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^{n-1}\,{}_0\mathcal{I}_t^1\left[i(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}} = \frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^{n-1}\int_0^t i(x)\mathrm{d}x\right\}}{\mathcal{L}\left\{v(t)\right\}}; \qquad q(t) = \int_0^t i(x)\mathrm{d}x \\
&= \frac{\mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}}; \qquad {}_0\mathcal{I}_t^{n-1}f(t) = {}_0D_t^{1-n}f(t) \\
&\mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\} = \left(\mathcal{L}\left\{v(t)\right\}\right)\left(\mathcal{L}\left\{c_n(t)\right\}\right) \\
&\mathcal{L}\left\{c_n(t)\right\} = \mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\}\left(\mathcal{L}\left\{v(t)\right\}\right)^{-1} \\
&c_n(t) = \left({}_0D_t^{1-n}\left[q(t)\right]\right)*\left(v(t)\right)^{-1}
\end{aligned}
\tag{43}
$$

7.2: Defining capacity function as fractional integration of fractional capacity function thereby converting the units in fractional units to Farads-for a fractional capacitor

In (43) ${}_0D_t^{1-n}$ is fractional derivative operation with order $(1-n)$.Therefore, we write following formulas for fractional capacitor in with conjugation to classical capacitor theory

$$
\begin{aligned}
&c_n(t) = \left({}_0D_t^{1-n}\left[q(t)\right]\right)*\left(v(t)\right)^{-1}; \qquad 0<n<1 \\
&{}_0D_t^{1-n}\left[q(t)\right] = \left(c_n(t)\right)*\left(v(t)\right) \\
&q(t) = {}_0D_t^{n-1}\left[\left(c_n(t)\right)*\left(v(t)\right)\right] = {}_0\mathcal{I}_t^{1-n}\left[\left(c_n(t)\right)*\left(v(t)\right)\right] \\
&q(t) = \left({}_0\mathcal{I}_t^{1-n}\left[c_n(t)\right]\right)*\left(v(t)\right) \qquad \text{or} \qquad q(t) = \left(c_n(t)\right)*\left({}_0\mathcal{I}_t^{1-n}\left[v(t)\right]\right) \\
&c(t) \overset{\text{def}}{=} {}_0\mathcal{I}_t^{1-n}\left[\left(c_n(t)\right)\right] \\
&q(t) = \left(c(t)\right)*\left(v(t)\right)
\end{aligned}
\tag{44}
$$

In the steps of (44), we have $q(t) = {}_0\mathcal{I}_t^{1-n}\left[\left(c_n(t)\right)*\left(v(t)\right)\right]$ doing Laplace transform we will get $Q(s) = s^{-(1-n)}\mathcal{L}\left\{\left(c_n(t)\right)*\left(v(t)\right)\right\}$. Further we get $Q(s) = s^{-(1-n)}\left(C_n(s)V(s)\right) = \left(s^{-(1-n)}C_n(s)\right)\left(V(s)\right)$. Writing $\left(s^{-(1-n)}C_n(s)\right) = C(s)$, i.e. ${}_0\mathcal{I}_t^{n-1}\left[c_n(t)\right] = c(t)$; we have $q(t) = \left({}_0\mathcal{I}_t^{1-n}\left[c_n(t)\right]\right)*\left(v(t)\right)$. In (44) thus we defined $c(t) \overset{\text{def}}{=} {}_0\mathcal{I}_t^{1-n}\left[\left(c_n(t)\right)\right]$. We note here that, by re-arrangement of term $s^{-(1-n)}$ in expression of $Q(s)$ we could have written $q(t) = \left(c_n(t)\right)*\left({}_0\mathcal{I}_t^{1-n}\left[v(t)\right]\right)$, this formula is also valid, that we have mentioned in (44). Using (36) i.e. $q(t) = \frac{K_nV_{BB}}{1-n}t^{1-n}$ in (44), we get ${}_0D_t^{1-n}\left[q(t)\right] = \frac{K_nV_{BB}}{1-n}\left(\frac{\Gamma(1-n+1)}{\Gamma(1-n+1-1+n)}t^{1-n-1+n}\right) = K_nV_{BB}\left(\Gamma(1-n)\right)$ that is a constant function for $t>0$.This we have got by formula of fractional derivative i.e. ${}_0D_x^{\upsilon}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\upsilon)}x^{\beta-\upsilon}$ [6], [20]. Thus, we write ${}_0D_t^{1-n}\left[q(t)\right] = K_nV_{BB}\left(\Gamma(1-n)\right)u(t)$ where $u(t)$ a unit-step function at $t=0$. We write the following for $c_n(t)$ as described in (44).

$$\begin{aligned} c_n(t) &= \left({}_0D_t^{1-n}\left[q(t)\right]\right)*\left(v(t)\right)^{-1}; \quad {}_0D_t^{1-n}\left[q(t)\right] = K_nV_{BB}\left(\Gamma(1-n)\right)u(t) \\ &= \left(K_nV_{BB}\left(\Gamma(1-n)\right)\left(u(t)\right)\right)*\left(V_{BB}\left(u(t)\right)\right)^{-1} \\ &= K_n\left(\Gamma(1-n)\right)\delta(t) \end{aligned} \tag{45}$$

We used identity i.e. $f*f^{-1}=\delta$, the inverse relation in (45).

We consider the following relation (44) for time varying capacity function $c(t)$ from $c_n(t)$

$$c(t) = {}_0D_t^{(n-1)}\left[c_n(t)\right]; \quad t>0, \quad 0<n<1; \qquad {}_0D_t^{(n-1)} \equiv {}_0\mathcal{I}_t^{(1-n)} \tag{46}$$

i.e. time varying capacity function defined as fractional integral of the order $1-n$ for the fractional capacity function i.e. $c_n(t)$ i.e. in units of $\text{Farad}/\sec^{1-n}$, which is constant in frequency domain as $C_n(\omega) = K_n\left(\Gamma(1-n)\right)$ i.e. a fractional capacitor. Using $c_n(t) = K_n\left(\Gamma(1-n)\right)\delta(t)$ as obtained above (42), we write following

$$\begin{aligned} c(t) &= {}_0D^{n-1}\left[c_n(t)\right] \\ &= {}_0\mathcal{I}_t^{1-n}\left[\left(K_n\left(\Gamma(1-n)\right)\left(\delta(t)\right)\right)\right] \\ &= K_n\left(\Gamma(1-n)\right)\left({}_0\mathcal{I}_t^{1-n}\left[\delta(t)\right]\right); \qquad {}_0\mathcal{I}_x^{\beta}\left[\delta(x)\right] = \frac{1}{\Gamma(\beta)}x^{\beta-1} \\ &= K_n\left(\Gamma(1-n)\right)\left(\frac{t^{1-n-1}}{\Gamma(1-n)}\right) = K_nt^{-n} \end{aligned} \tag{47}$$

In (47) we have used formula for fractional integration of delta-function [6], [20], as mentioned i.e. ${}_0\mathcal{I}_x^{\beta}\left[\delta(x)\right] = \frac{1}{\Gamma(\beta)}x^{\beta-1}$. The expression $c(t) = K_nt^{-n}$ we had obtained earlier too (35).

We note that the fractional integration operation ${}_0\mathcal{I}_t^{1-n}\left[c_n(t)\right]$ in (46), (47) is converting units in $\text{Farad}/\sec^{1-n}$ for $c_n(t)$ into units of Farad for $c(t)$. This is because the fractional integration ${}_0\mathcal{I}_t^{1-n}$ is integration with respect to fractional differential element $(\mathrm{d}t)^{1-n}$ i.e. ${}_0\mathcal{I}_t^{1-n}\left[c_n(t)\right]=\int_0^t \left(c_n(t)\right)\mathrm{d}t^{(1-n)}$. Therefore, the capacity function $c(t)=K_n t^{-n}$ that we get for fractional capacitor is in units of Farad . This show for a fractional capacitor by the use of time varying capacity function we can convert the fractional capacity constant that is in units of fractional units of Farads per second to the power a fractional number, to units of Farads, by formula $c(t)={}_0\mathcal{I}_t^{1-n}\left[c_n(t)\right]$.

7.3: General charge and current expression for fractional capacitor following universal dielectric relaxation law

We obtain a general expression of charge $q(t)$ for Curie-von Schweidler relaxing current in a capacitor, that is having capacity function as $c(t)=K_n t^{-n}$ (47) when stressed with a time varying voltage $v(t)$ applied at $t=0$ is by convolution process as $q(t)=\left(K_n t^{-n}\right)*\left(v(t)\right)$ elaborated below

$$
\begin{aligned}
&q(t)=\left(c(t)\right)*\left(v(t)\right)=\int_{-\infty}^{t}\left(c(t-x)\right)\left(v(x)\right)\mathrm{d}x\\
&c(x)=K_n x^{-n} \qquad x>0
\end{aligned}
\tag{48}
$$

The convolution integral from (48), with $x=0$ is following

$$
q(t)=\int_0^t K_n\frac{v(x)}{(t-x)^n}\mathrm{d}x \tag{49}
$$

As we did for geometrical capacity in previous section, we differentiate (49) this $q(t)$ to get $i(t)$ and write following

$$
\begin{aligned}
i(t)=q^{(1)}(t)&=\frac{\mathrm{d}q(t)}{\mathrm{d}t}\\
&=K_n\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{v(x)}{(t-x)^n}\mathrm{d}x
\end{aligned}
\tag{50}
$$

We apply formula of integration by parts i.e.

$$
\int_0^t\left(f_1(x)\right)\left(f_2(x)\right)\mathrm{d}x=\left[f_1(x)\int f_2(x)\mathrm{d}x\right]_{x=0}^{x=t}-\int_0^t\left(\left(f_1^{(1)}(x)\right)\int_0^t\left(f_2(x)\right)\mathrm{d}x\right)\mathrm{d}x \tag{51}
$$

to evaluate $\int_0^t\frac{v(x)}{(t-x)^n}\mathrm{d}x$ that appears in (50) as detailed in the following steps

$$
\begin{aligned}
\int_0^t\frac{v(x)\mathrm{d}x}{(t-x)^n}&=\left[v(x)\int\frac{\mathrm{d}x}{(t-x)^n}\right]_{x=0}^{x=t}-\int_0^t\left(v^{(1)}(x)\int\frac{\mathrm{d}x}{(t-x)^n}\right)\mathrm{d}x\\
&=v(x)\left(-\frac{(t-x)^{1-n}}{1-n}\right)\Bigg|_{x=0}^{x=t}-\int_0^t v^{(1)}(x)\left(\frac{(-1)(t-x)^{1-n}}{1-n}\right)\mathrm{d}x\\
&=\frac{v(0)}{1-n}t^{1-n}-\int_0^t\frac{v^{(1)}(x)}{1-n}(t-x)^{1-n}\mathrm{d}x
\end{aligned}
\tag{52}
$$

Now we differentiate (52) and write the following steps

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(x)\mathrm{d}x}{(t-x)^n} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v(0)}{1-n}t^{1-n} - \int_0^t \frac{v^{(1)}(x)}{1-n}(t-x)^{1-n}\,\mathrm{d}x\right)\\ &= v(0)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^{1-n}}{1-n}\right) - \int_0^t \frac{v^{(1)}(x)}{1-n}\frac{\mathrm{d}\left((-1)(t-x)^{1-n}\right)}{\mathrm{d}t}\mathrm{d}x\\ &= \frac{v(0)}{t^n} - \int_0^t \frac{v^{(1)}(x)}{1-n}\left((-1)(1-n)(t-x)^{1-n-1}\right)\mathrm{d}x\\ &= \frac{v(0)}{t^n} + \int_0^t \frac{v^{(1)}(x)}{(t-x)^n}\mathrm{d}x\end{aligned} \tag{53}$$

This gives $i(t)$ as following relation

$$\begin{aligned} i(t) &= K_n \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(x)}{(t-x)^n}\mathrm{d}x\\ &= K_n\frac{v(0)}{t^n} + K_n\int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}; \quad K_n = \frac{C_n}{\Gamma(1-n)}; \qquad 0<n<1\end{aligned} \tag{54}$$

The expression (54) obtained with the formula $q(t) = \left(c(t)\right)*\left(v(t)\right)$, with $c(t) = K_n t^{-n}$ is consistent with obtained expression in [21].

For $v(t) = V_{BB}\left(u(t)\right)$ i.e. a constant step voltage applied at time $t=0$ to a time varying capacity function given as $c(t) = K_n t^{-n}$ we have for $t>0$, $v^{(1)}(t) = 0$ with $v(0) = V_{BB}$, the evaluation of $i(t)$ demonstrated below

$$\begin{aligned} i(t) &= K_n\frac{v(0)}{t^n} + K_n\int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n} \qquad v(0) = V_{BB}; \quad v^{(1)}(x) = 0, \quad x>0\\ &= K_n\frac{V_{BB}}{t^n} + K_n\int_0^t \frac{(0)\mathrm{d}x}{(t-x)^n} = K_n\frac{V_{BB}}{t^n}\end{aligned} \tag{55}$$

We get $i(t) = K_n V_{BB} t^{-n}$, for $t>0$ i.e. we recover the Curie-von Schweidler law in (55). For a constant capacitor case with capacity function as $c(t) = C_1\delta(t)$, we have the relation that we derived earlier (18) (19); i.e. $i(t) = C_1 v(0)\left(\delta(t)\right) + C_1\left(v^{(1)}(t)\right)$. The Figure-1 gives summary of our discussion about a constant capacity and a time varying capacity function.

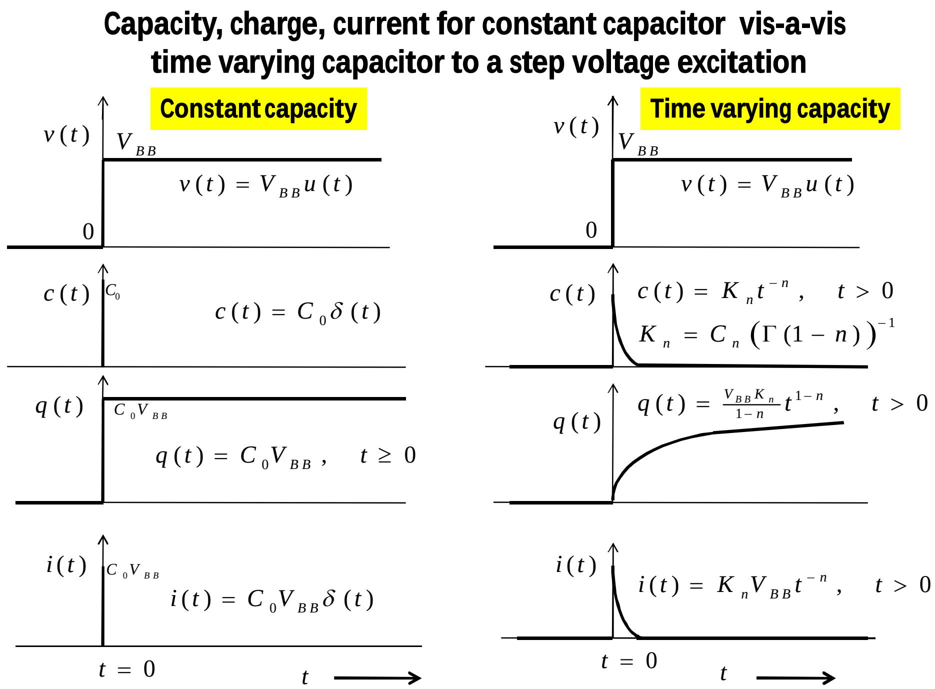

Figure-1: Summary of discussion about constant capacity vis-à-vis time varying capacity

8. Appearance of fractional derivative in Fractional Capacitor

We have formed a time varying capacity function with a dielectric whose relaxation to a step voltage at $t=0$ of constant magnitude follows a power law given by empirical expression of Curie-von Schweidler law. We have got current and charge expression for any arbitrary voltage function $v(t)$ applied at $t=0$ in above section (54) as following

$$
\begin{aligned}
i(t) &= K_n \frac{v(0)}{t^n} + K_n \int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n} \\
q(t) &= \big(c(t)\big) * \big(v(t)\big) = \big(K_n t^{-n}\big) * \big(v(t)\big) \\
&= \int_0^t K_n \frac{v(x)}{(t-x)^n}\mathrm{d}x
\end{aligned}
\tag{56}
$$

The fractional derivative for $0 < n < 1$ is defined as following two ways [6], [20]

$$
\begin{aligned}
{}_0D_t^n\big[f(t)\big] &= \frac{1}{\Gamma(1-n)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(x)}{(t-x)^n}\mathrm{d}x \\
&= \frac{1}{\Gamma(1-n)} \left(\frac{f(0)}{t^n} + \int_0^t \frac{f^{(1)}(x)}{(t-x)^n}\mathrm{d}x \right); \quad t > 0
\end{aligned}
\tag{57}
$$

The first definition is of Riemann-Liouville type i.e. ${}_0D_t^n\big[f(t)\big] = \frac{1}{\Gamma(1-n)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{f(x)}{(t-x)^n}\mathrm{d}x, \quad 0<n<1$ and in the second expression of (57) the second term i.e. $\frac{1}{\Gamma(1-n)}\int_0^t \frac{f^{(1)}(x)}{(t-x)^n}\mathrm{d}x$ is Caputo fractional derivative i.e.

${}_{0}^{C}D_{t}^{n}\left[f(t)\right]=\frac{1}{\Gamma(1-n)}\int_{0}^{t}\frac{f^{(1)}(x)}{(t-x)^{n}}\mathrm{d}x;\quad 0<n<1$. Therefore, we have ${}_{0}D_{t}^{n}\left[f(t)\right]={}_{0}^{C}D_{t}^{n}\left[f(t)\right]+\frac{f(0)}{\Gamma(1-n)}t^{-n}$, i.e. relation between the two definitions of fractional derivative [6], [20].

Integrating the expression ${}_{0}D_{t}^{n}\left[f(t)\right]=\frac{1}{\Gamma(1-n)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{t}\frac{f(x)}{(t-x)^{n}}\mathrm{d}x$, once we write the following

$$\begin{aligned}{}_{0}\mathcal{I}_{t}^{1}\left({}_{0}D_{t}^{n}\left[f(t)\right]\right)&=\int_{0}^{t}\left(\frac{1}{\Gamma(1-n)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{t}\frac{f(x)}{(t-x)^{n}}\mathrm{d}x\right)\right)\mathrm{d}x\\&=\frac{1}{\Gamma(1-n)}\int_{0}^{t}\left(\int_{0}^{t}\frac{f(x)}{(t-x)^{n}}\mathrm{d}x\right)^{(1)}\mathrm{d}x\\&=\frac{1}{\Gamma(1-n)}\int_{0}^{t}\frac{f(x)}{(t-x)^{n}}\mathrm{d}x\end{aligned}\tag{58}$$

We have used in (58) the identity $\mathcal{I}_{t}^{1}\left(g^{(1)}(t)\right)\equiv g(t)$. Using the composition rule [6], [20] i.e. ${}_{0}\mathcal{I}_{t}^{1}\left({}_{0}D_{t}^{n}\left[f(t)\right]\right)={}_{0}\mathcal{I}_{t}^{1-n}\left[f(t)\right]={}_{0}D_{t}^{n-1}\left[f(t)\right]$, we re-write (58) as following

$$\begin{aligned}{}_{0}D_{t}^{n-1}\left[f(t)\right]&={}_{0}\mathcal{I}_{t}^{1-n}\left[f(t)\right]=\frac{1}{\Gamma(1-n)}\int_{0}^{t}\frac{f(x)}{(t-x)^{n}}\mathrm{d}x;\quad 0<n<1;\qquad 1-n=\upsilon\\{}_{0}D_{t}^{-\upsilon}\left[f(t)\right]&={}_{0}\mathcal{I}_{t}^{\upsilon}\left[f(t)\right]=\frac{1}{\Gamma(\upsilon)}\int_{0}^{t}\frac{f(x)}{(t-x)^{1-\upsilon}}\mathrm{d}x\end{aligned}\tag{59}$$

Using the definitions of fractional derivative (57), we apply to current expression (54) also by manipulating with a constant i.e. $\Gamma(1-n)$ we get following (60) expressions

$$\begin{aligned}i(t)&=K_{n}\frac{v(0)}{t^{n}}+K_{n}\int_{0}^{t}\frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^{n}}\qquad 0<n<1\\&=K_{n}\left(\Gamma(1-n)\right)\left(\frac{1}{\Gamma(1-n)}\left(\frac{v(0)}{t^{n}}+\int_{0}^{t}\frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^{n}}\right)\right),\qquad K_{n}\left(\Gamma(1-n)\right)=C_{n}\\&=C_{n}\left({}_{0}D_{t}^{n}\left[v(t)\right]\right),\qquad 0<n<1\end{aligned}\tag{60}$$

Applying the expression for fractional integration (59) to the charge expression, we get following

$$\begin{aligned}q(t)&=\left(c(t)\right)*\left(v(t)\right)=\left(K_{n}t^{-n}\right)*\left(v(t)\right);\quad t>0\\&=\int_{0}^{t}K_{n}\frac{v(x)}{(t-x)^{n}}\mathrm{d}x\\&=K_{n}\left(\Gamma(1-n)\right)\left(\frac{1}{\Gamma(1-n)}\int_{0}^{t}\frac{v(x)}{(t-x)^{n}}\mathrm{d}x\right);\qquad K_{n}\left(\Gamma(1-n)\right)=C_{n}\\&=C_{n}\left({}_{0}\mathcal{I}_{t}^{(1-n)}\left[v(t)\right]\right)\\&=C_{n}\left({}_{0}D_{t}^{n-1}\left[v(t)\right]\right)\qquad t>0\qquad 0<n<1\end{aligned}\tag{61}$$

We apply a constant step voltage $v(t)=V_{BB}$ at $t=0$ to an uncharged fractional capacitor with capacity function $c(t)=\frac{C_n}{\Gamma(1-n)}t^{-n}$, applying the above formula (61) we get

$$
\begin{aligned}
q(t) &= C_n\left({}_0D_t^{n-1}\left[v(t)\right]\right) \qquad t>0 \qquad 0<n<1 \\
&= C_n\left({}_0D_t^{n-1}\left[V_{BB}\right]\right) \qquad {}_0D_t^{\alpha}\left[\mathrm{C}\right]=\mathrm{C}\frac{\Gamma(1)}{\Gamma(1-\alpha)}t^{-\alpha} \\
&= C_nV_{BB}\frac{\Gamma(1)}{\Gamma\left(1+(1-n)\right)}t^{1-n} = \frac{C_n}{(1-n)\left(\Gamma(1-n)\right)}V_{BB}t^{1-n}, \qquad K_n\left(\Gamma(1-n)\right)=C_n \\
&= \frac{K_nV_{BB}}{(1-n)}t^{1-n}
\end{aligned}
\tag{62}
$$

The same expression we showed earlier (36) and in Figure-1.

We wrote also in the steps (44) the expression $q(t)=\left(c_n(t)\right)*\left({}_0\mathcal{I}_t^{1-n}\left[v(t)\right]\right)$. For fractional capacitor, we noted that $c_n(t)=C_n\delta(t)$ this we have $q(t)=\left(C_n\delta(t)\right)*\left({}_0\mathcal{I}_t^{1-n}\left[v(t)\right]\right)$ Expanding this convolution integral we get $q(t)=\int_{-\infty}^{t}\left(C_n\delta(x-t)\right)\left({}_0\mathcal{I}_x^{1-n}\left[v(x)\right]\right)\mathrm{d}x$. Now using property of delta function $\int\delta(x-a)f(x)\,\mathrm{d}x=f(a)$, we get $q(t)=C_n\left({}_0\mathcal{I}_t^{1-n}\left[v(t)\right]\right)$. With $v(t)=V_{BB}$ applied at $t=0$ we will get $q(t)=\frac{1}{(1-n)\left(\Gamma(1-n)\right)}C_nV_{BB}t^{1-n}$ same as in (62).

9. Integrated Capacity defined from Capacity function of a capacitor and explanation vis-à-vis a pitcher holding water

9.1: Defining integrated capacity from the time varying capacity function for ideal and fractional capacitor

We take example of a pitcher, which holds water, of volume V. Let the pitcher be made of metal walls so that there are no pores. It is fully filled with water from empty state, hence once full it has no capacity left. This is like ideal capacitor, where the volume of water V remains fixed as constant after filling, with no left over capacity. Thus, an ideal capacitor described by capacity function $c(t)=C_1\delta(t)$, after it is charged at $t=0$ with a constant voltage holds the constant charge $q(t)=C_1V_{BB}$ at times $t>0$ and at time, $t>0$ this capacitor has zero capacity function, i.e. $c(t)=0$ that is like no more capacity left to fill, like pitcher. Thus, we have maximum charge holding capacity in this case as $q_{\max}=\lim_{t\uparrow\infty}q(t)=C_1V_{BB}$. Therefore we can say the capacity function $c(t)$ at $t>0$ indicates the left over capacity to fill from maximum charge say $q_{\max}=\lim_{t\uparrow\infty}q(t)$.

Now let the walls of the pitcher be made of clay with an infinitely porous material. As the pitcher gets the water volume V the pitcher walls too starts seepage of water into its pores. Thus, extra water keeps entering pores of the porous pitcher walls. This water filling process in the porous walls we call fractional capacity. Now due to infinite nature of these pores, we have a situation, that infinite amount of water keeps seeping into the walls. This is analogous to charging porous walls with water as charging a fractional capacitor where we derived $q_{\max} = \lim_{t\uparrow\infty} q(t) = \infty$. Yet as we go on with charging process, the remaining capacity of holding the charge from maximum value (in this case infinity) keeps on decreasing but will never be going to zero, and thus we got the capacity function for a fractional capacitor as, $c(t) = K_n t^{-n}$ where $\lim_{t\uparrow\infty} c(t) = 0$. The charge of a fractional capacitor as in the case of filling the porous walls gets the form that we derived as in (36) (62), $q(t) = \frac{K_n V_{BB}}{(1-n)} t^{1-n}$ for $t > 0$ increasing with time. This phenomena leads to electrostatic break down of capacitors [12], [35], even if the constant voltage V_{BB} is lower than dielectric breakdown limit. Thus a fractional capacitor with $c(t) = K_n t^{-n}$ will break down when the electrostatic forces are high enough due to large accumulation of charge at large times, even if V_{BB} is lower than dielectric breakdown limit. While the ideal geometric capacitor with $c(t) = C_1\delta(t)$ will have $\lim_{t\uparrow\infty} q(t) = C_1 V_{BB}$ and will never breakdown when V_{BB} is less than dielectric breakdown limit.

We define integral capacity as following from the capacity function $c(t)$

$$c_{\text{int}}(t) = \int_0^t c(x)\mathrm{d}x; \quad t > 0 \tag{63}$$

The above (63) in integration of the capacity function w.r.t time from time of application of voltage excitation (in our case is $t = 0$). Thus for a classical capacitor with capacity function defined as $c(t) = C_1\delta(t)$ we get integrated capacity as

$$c_{\text{int}}(t) = \int_0^t \left(C_1\delta(x)\right)\mathrm{d}x = C_1, \qquad t > 0 \tag{64}$$

We observe $\lim_{t\uparrow\infty} c_{\text{int}}(t) = C_1$ a constant value. This integrated capacity is what is discussed in classical theory that we derived from capacity function. Now for the case of fractional capacitor where the capacity function is $c(t) = K_n t^{-n}$, the integrated capacity is

$$c_{\text{int}}(t) = \int_0^t K_n x^{-n}\mathrm{d}x = \frac{K_n}{(1-n)} t^{1-n}; \qquad t > 0 \tag{65}$$

This is same as (37) used in [12], [35]. We note in (65) $\lim_{t\uparrow\infty} c_{\text{int}}(t) = \infty$.

9.2: Difference in usage of integrated capacity and the time varying capacity function for obtaining loss-tangent value

Thus, the term ‘integrated capacity’ $c_{\text{int}}(t)$ of capacitor is analogous to ‘total’ water holding capacity of pitcher. The total water holding capacity of pitcher with metal walls is constant is equivalent to classical capacitor case (64), while the total water holding capacity of walls of porous pitcher is infinity is equivalent to (65) the fractional capacitor case. We mention here the expressions for

$C_{\mathrm{int}}(\omega) = \mathcal{L}\left\{c_{\mathrm{int}}(t)\right\}\Big|_{s=\mathrm{i}\omega}$ cannot be used to determine the loss tangent, while from capacity function with $C(\omega) = \mathcal{L}\left\{c(t)\right\}\Big|_{s=\mathrm{i}\omega}$ is used to determine loss tangent value.

10. Experimental results showing fractional capacitor

The Curie-von Schweidler empirical law of power law relaxation, i.e. $i(t) \propto t^{-n}$ states that $0 < n < 1$ .This is validated via experiments on dielectric relaxations. A 100V step input applied to a completely discharged capacitor of $0.47\mu\mathrm{F}$ having metalized paper dielectric, and the current decay is recorded with time. The graphs of log-log plot i.e. $\log\left(i(t)\right)$ vs. $\log(t)$ show a straight line of average slope -0.86 [12]-[17]. This experiment indicates a Curie-von Schweidler law, with $i(t) \propto t^{-n}$, having $n = 0.86$. The exponent n is in the range of $0.85 < n < 1$ in several dielectric relaxation experiments [12]-[17]. The experiments with super-capacitors [7], [8], show range as $0.5 < n < 1$. A very low value of exponent n is found in relaxation of Laponite studies averagely $n = 0.09$ [18]. In this Laponite study [18] though the exponent n was obtained on ‘self-discharge’ curves with various charging time history-showing memory effect, the expression obtained for self-discharge decay of voltage assumes fractional capacity-that in turn assumes Curie-von Schweidler law as current relaxation function.

11. Summary

In the tabular form (Table-1), we present the various concepts (formulas) that we discussed with this new approach of charge store in classical capacitor and fractional capacitor.

S. No.	Parameter	Classical Geometrical (Constant) Capacity ($n=1$)	Fractional Capacity $0<n<1$
1	Relaxing current to constant step voltage V_{BB} applied to an un-charge capacitor at $t=0$	$i(t)=C_1V_{BB}\delta(t)$ $C_1\equiv \text{Farad}$	$i(t)=K_nV_{BB}t^{-n}=\frac{C_nV_{BB}}{\Gamma(1-n)}t^{-n}, \quad t>0$ $K_n\Gamma(1-n)=C_n, \quad C_n\equiv \text{Farad}/\sec^{1-n}$
2	Relaxing Current in frequency domain	$I(s)=C_1V_{BB}$ $I(\omega)=C_1V_{BB}$	$I(s)=K_nV_{BB}\left(\Gamma(1-n)\right)s^{n-1}=C_nV_{BB}s^{n-1}$ $I(\omega)=\frac{C_nV_{BB}}{\omega^{1-n}}\left(\cos\left(\frac{(1-n)\pi}{2}\right)-\mathrm{i}\sin\left(\frac{(1-n)\pi}{2}\right)\right)$ $K_n\Gamma(1-n)=C_n, \quad C_n\equiv \text{Farad}/\sec^{1-n}$
3	Capacity function in time domain and frequency domain with loss tangent	$c(t)=C_1\delta(t) \quad \text{Farad}$ $C(s)=C_1 \quad \text{Farad}$ $C(\omega)=C_1-\mathrm{i}(0)$ $\text{Loss-tangent} \quad \tan\phi=0$	$c_n(t)=K_n\left(\Gamma(1-n)\right)\delta(t)=C_n\delta(t) \quad \text{Farad}/\sec^{1-n}$ $C_n(s)=C_n=K_n\left(\Gamma(1-n)\right) \quad \text{Farad}/\sec^{1-n}$ $c(t)={}_0D_t^{n-1}\left[c_n(t)\right]={}_0\mathcal{I}_t^{1-n}\left[c_n(t)\right]; \quad \text{Farad}$ $c(t)=K_nt^{-n}=\frac{C_n}{\Gamma(1-n)}t^{-n} \quad \text{Farad}$ $C(s)=K_n\left(\Gamma(1-n)\right)s^{n-1}=C_ns^{n-1} \quad \text{Farad}$ $C(\omega)=\frac{C_n}{\omega^{1-n}}\left(\cos\left(\frac{(1-n)\pi}{2}\right)-\mathrm{i}\sin\left(\frac{(1-n)\pi}{2}\right)\right)$ $\text{Loss-tangent} \quad \tan\phi=\tan\left(\frac{(1-n)\pi}{2}\right)$ $K_n\Gamma(1-n)=C_n, \quad C_n\equiv \text{Farad}/\sec^{1-n}$
4	Charge function to a constant step voltage V_{BB} applied at $t=0$	$q(t)=c(t)*v(t)$ $=C_1V_{BB}; \quad t\geq 0$	$q(t)=c(t)*v(t)$ $=\frac{K_nV_{BB}}{1-n}t^{1-n}=\frac{C_nV_{BB}}{(1-n)\Gamma(1-n)}t^{1-n}; \quad t\geq 0$ $K_n\Gamma(1-n)=C_n, \quad \text{Farad}/\sec^{1-n}$

5	Current to an arbitrary voltage function $v(t)$ applied to uncharged capacitor at $t=0$	$i(t)=C_1v(0)\delta(t)+C_1\dfrac{\mathrm{d}v(t)}{\mathrm{d}t}$	$i(t)=K_nv(0)t^{-n}+K_n\int_0^t\frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}$ $=\frac{C_nv(0)}{\Gamma(1-n)}t^{-n}+\frac{C_n}{\Gamma(1-n)}\int_0^t\frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}$ $K_n\Gamma(1-n)=C_n \quad \text{Farad}/\sec^{1-n}$
6	Current voltage relation	$i(t)=C_1\left({}_0D_t^1v(t)\right)$ $C_1\equiv\text{Farad}$	$i(t)=C_n\left({}_0D_t^nv(t)\right)$ $K_n\Gamma(1-n)=C_n \quad \text{Farad}/\sec^{1-n}$
7	Charge voltage relation for arbitrary voltage $v(t)$ function applied at $t=0$	$q(t)=\left(c(t)\right)*\left(v(t)\right)$ $=\left(C_1\delta(t)\right)*v(t)$ $=C_1v(t);\quad t\geq 0$ $C_1\equiv\text{Farad}$	$q(t)=\left(c(t)\right)*\left(v(t)\right)$ $=\left(K_nt^{-n}\right)*\left(v(t)\right)$ $=C_n\left({}_0D_t^{n-1}v(t)\right)=C_n\left({}_0\mathcal{I}_t^{1-n}v(t)\right),\quad t\geq 0$ $K_n\Gamma(1-n)=C_n \quad \text{Farad}/\sec^{1-n}$

Table-1: Summary of the discussions regarding formulas for classical capacitor and fractional capacitor

12. Conclusion

In this paper we discussed that charge stored in a capacitor, as a function of time is not the usual multiplication operation of capacity and voltage; instead, the charge is convolution integral of capacity function and voltage stressed across the capacitor. However, the charge as a function of frequency is multiplication operation of frequency domain functions of capacity and voltage. We say that capacity is not the usual ratio of charge to voltage in time domain, but it is given as convolution expression. We discussed in this paper that for a fractional capacitor, the charge goes to infinity for large times, when the fractional capacitor is placed on a constant voltage; whereas, for a classical capacitor the charge at large time is a constant value. This observation in our derivation is with convolution formula defining the charge stored in capacitor and is consistence with other fractional capacitor models. This new concept is in line with the observation of charge stored as step function, and relaxation current in form of impulse function for ideal geometrical capacitor of constant capacity, when stressed by a constant voltage and for fractional capacitor with power-law decay current that is given by universal dielectric relaxation law called as Curie von-Schweidler law. This universal dielectric relaxation law gives rise to fractional derivative relating voltage stress and relaxation current that is formulation of ‘fractional capacitor’. A ‘fractional capacitor’ we discussed is with this new concept of redefining the charge store definition i.e. via this convolution integral approach, and we have obtained the loss tangent value, from the described capacity function. We also showed for a fractional capacitor by the use of time varying capacity function we can convert the fractional capacity constant that is in fractional units of Farads per second to the power a fractional number, to units of Farads.

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