

# A New Generalized Transmuted Inverse Exponential Distribution: Properties and Application

Uchenna U. Uwadi<sup>1\*</sup> and Elebe E. Nwaezza<sup>1</sup>

<sup>1</sup>Department of Mathematics, Computer Science, Statistics and Informatics, Federal University Ndufu Alike  
Ikwo, Nigeria.

Original Research Article

## Abstract

In this study, we proposed a new generalised transmuted inverse exponential distribution with three parameters and have transmuted inverse exponential and inverse exponential distributions as sub models. The hazard function of the distribution is nonmonotonic, unimodal and inverted bathtub shaped making it suitable for modelling lifetime data. We derived the moment, moment generating function, quantile function, maximum likelihood estimates of the parameters, Renyi entropy and order statistics of the distribution. A real life data set is used to illustrate the usefulness of the proposed model.

**Keywords:** Generalized transmuted-G; inverse exponential distribution; entropy; maximum likelihood.

## 1 Introduction

Probability distributions have so many applications in the description of real world phenomenon. Classical distributions have been developed over decades to model data in engineering, actuarial, environmental and medical sciences, finance, demography and economics. Several probability distributions have been proposed for modelling of lifetime data in the literature, one of such distributions is inverse exponential distribution. The inverse exponential distribution was first introduced by [1], for modelling lifetime data. The cumulative density function (cdf) of inverse exponential distribution is

$$G(x) = \exp\left(-\frac{\beta}{x}\right) \quad x > 0 \quad (1)$$

Where  $\beta > 0$  is the scale parameter. The corresponding density function (pdf) to (1) is

$$g(x) = \frac{\beta}{x^2} \exp\left(-\frac{\beta}{x}\right) \quad x > 0 \quad (2)$$

---

\*Corresponding author: E-mail: uchenna.uwadi@funai.edu.ng;

Many univariate distributions have been extended for accuracy and flexibility purposes in modelling. Based on this understanding, a number of authors have proposed and generalised the inverse exponential distribution. The kumaraswamy-Inverse Exponential distribution [2], Generalized inverse exponential distribution [3], Exponentiated generalised inverse exponential distribution [4], transmuted inverse exponential distribution [5], Transmuted generalised inverted exponential distribution [6], Beta inverted exponential distribution [7] and Beta generalised inverted exponential distribution [8].

In this paper, we introduced a New Generalized Transmuted inverse-Exponential distribution (NGT-IE) which is a generalised form of inverse exponential distribution to make inverse exponential distribution flexible. This distribution can be used to model data with an inverted bathtub failure rate. It provides greater flexibility than generalised inverse exponential and inverse exponential distribution.

The rest of this paper is organised as: section 2 states the distribution function, density function; hazard function of New Generalized Transmuted Inverse Exponential (NGT-IE) distribution. We derived its mathematical properties such as the moments and quantile function in section 3, while section 4 deals with maximum likelihood estimates of the parameters of the new distribution. The expressions for entropy and order statistics of New Generalized Transmuted Inverse Exponential (NGT-IE) were obtained in section 5 and 6 respectively. Lastly, in section 7 we provided the real life application of the proposed distribution.

## 2 New Generalized Transmuted Inverse Exponential (NGT-IE) Distribution

The cdf of Another Generalized Transmuted class (AGT-G) of distributions according to [9] is

$$F(x) = (1 + \lambda) \left[ 1 - (\bar{G}(x))^\alpha \right] - \lambda \left[ 1 - (G(x))^\alpha \right]^2 \quad (3)$$

With the corresponding pdf

$$f(x) = \alpha g(x) (\bar{G}(x))^{\alpha-1} \left\{ 1 + \lambda - 2\lambda \left[ 1 - (\bar{G}(x))^\alpha \right] \right\} \quad (4)$$

Where  $\alpha > 0$ ,  $|\lambda| \leq 1$  and  $G(x)$  is the cdf of the baseline pdf.  $\bar{G}(x)$  is the survival function of the baseline pdf and its is given by

$$\bar{G}(x) = 1 - G(x) \quad (5)$$

Substituting (1), (2) and (5) into (3) and (4) we obtained the cdf

$$F(x) = (1 + \lambda) \left[ 1 - \left( 1 - \exp\left(-\frac{\beta}{x}\right) \right)^\alpha \right] - \lambda \left[ 1 - \left( 1 - \exp\left(-\frac{\beta}{x}\right) \right)^\alpha \right]^2 \quad (6)$$

and pdf

$$f(x) = \frac{\alpha\beta}{x^2} \exp\left(-\frac{\beta}{x}\right) \left[ 1 - \exp\left(-\frac{\beta}{x}\right) \right]^{\alpha-1} \left\{ 1 + \lambda - 2\lambda \left[ 1 - \left( 1 - \exp\left(-\frac{\beta}{x}\right) \right)^\alpha \right] \right\} \quad (7)$$

---

of the New Generalized Transmuted Inverse Exponential  $NGT - IE(x, \alpha, \beta, \lambda)$  distribution.

Where  $\alpha > 0$ ,  $\beta > 0$ ,  $|\lambda| \leq 1$

## 2.1 Sub-Models of NGT-IE Distribution

The sub-models that can be derived from NGT-IE distribution are as follows

- I. If  $\alpha = 1$ , equation (7) reduces to transmuted inverse exponential (T-IE) distribution with parameters  $\beta$  and  $\lambda$  [5]
- II. If  $\lambda = 0$  and  $\alpha = 1$ , (6) reduces to one parameter inverse exponential distribution with parameter  $\beta$  [1]

## 2.2 Hazard Function

The hazard function  $h(x)$  otherwise known as hazard rate of New Generalized Transmuted Inverse Exponential  $NGT - IE(x, \alpha, \beta, \lambda)$  is given by

$$h(x) = \frac{f(x)}{1 - F(x)}$$

$$h(x) = \frac{\frac{\alpha\beta}{x^2} \exp\left(-\frac{\beta}{x}\right) \left[1 - \exp\left(-\frac{\beta}{x}\right)\right]^{\alpha-1} \left\{1 + \lambda - 2\lambda \left[1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha\right]\right\}}{1 - (1 + \lambda) \left[1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha\right] + \lambda \left[1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha\right]^2} \quad (8)$$

## 2.3 Shapes of Density and Hazard Functions

The possible shapes of the density and hazard functions of New Generalized Transmuted Inverse Exponential ( $NGT - IE$ ) distribution for selected values of  $\alpha, \beta$  and  $\lambda$  are illustrated in figures below. The hazard function in Fig. 2 shows that  $NGT - IE$  distribution allows for increasing, unimodal decreasing failure rate.

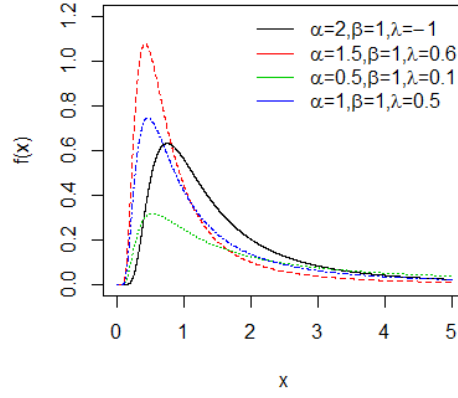


**Fig. 1. Plots of density function of  $NGT - IE$**

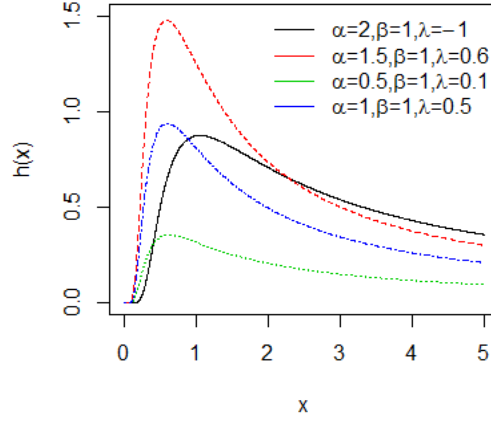


**Fig. 2. The plots of the hazard rate function of  $NGT - IE$**

The shapes of the density and hazard rate function can be described analytically. The critical points of the density function of  $NGT - IE$  distribution are the roots of the (9).

$$\frac{d \log f(x)}{dx} = \frac{2}{x} + \frac{\beta}{x^2} - \frac{\beta(\alpha-1) \exp\left(-\frac{\beta}{x}\right)}{x^2 \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)} - \frac{2\alpha\lambda\beta \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^{\alpha-1} \exp\left(-\frac{\beta}{x}\right)}{x^2 \left\{ (1+\lambda) - 2\lambda \left[ 1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha \right] \right\}} \quad (9)$$

There may be more than one root to (9). If  $x = x_0$  is the root of (9), then it corresponds to a local maximum, local minimum or point of inflexion depending whether  $\psi(x_0) < 0$ ,  $\psi(x_0) > 0$ , or  $\psi(x) = 0$  where

$$\psi(x) = \frac{d^2 \log f(x)}{dx^2}.$$

The critical point of the hazard function is given by (10)

$$\begin{aligned} \frac{d \log h(x)}{dx} &= \alpha \beta \exp\left(-\frac{\beta}{x}\right) \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^{\alpha-1} \left\{ \frac{2\lambda \left(1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha - (1+\lambda)\right)}{x^2 \left\{1 - (1+\lambda) \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha + 2\lambda \left[1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha\right]^2\right\}} \right\} \\ &\quad - \frac{2}{x} + \frac{\beta}{x^2} - \frac{\beta(\alpha-1) \exp\left(-\frac{\beta}{x}\right)}{x^2 \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)} - \frac{2\alpha\lambda\beta \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^{\alpha-1} \exp\left(-\frac{\beta}{x}\right)}{x^2 \left\{(1+\lambda) - 2\lambda \left[1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha\right]\right\}} \end{aligned} \quad (10)$$

There may be more than one root to (10). If  $x = x_0$  is the root of (10), then it corresponds to a local maximum, local minimum or point of inflexion depending whether  $\phi(x_0) < 0$ ,  $\phi(x_0) > 0$ , or  $\phi(x) = 0$

where  $\phi(x) = \frac{d^2 \log h(x)}{dx^2}$ .

### 3 Moments and Quantiles

In this section, we present the  $r$ th moment, moment generating function (mgf) and the quantile function of the  $NGT - IE(x, \alpha, \beta, \lambda)$  distribution.

#### 3.1 Moments

Theorem 1: If  $X$  has the  $NGT - IE(x, \alpha, \beta, \lambda)$  with  $|\lambda| \leq 1$ , then the  $r$ th non central moment of  $X$  is

$$E(X^r) = \alpha \beta^r \Gamma(1-r) \left\{ (1-\lambda) \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} (j+1)^{r-1} + 2\lambda \sum_{j=0}^{\infty} (-1)^j \binom{2\alpha-1}{j} (j+1)^{r-1} \right\} \quad (11)$$

Proof:

By definition

$$\begin{aligned} E(X^r) &= \int_{-\infty}^{\infty} x^r f(x) dx \\ E(X^r) &= \int_0^{\infty} x^r \frac{\alpha \beta}{x^2} \exp\left(-\frac{\beta}{x}\right) \left[1 - \exp\left(-\frac{\beta}{x}\right)\right]^{\alpha-1} \left\{ (1+\lambda) - 2\lambda \left[1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha\right] \right\} dx \\ E(X^r) &= \alpha \beta (1+\lambda) \int_0^{\infty} x^{r-2} \exp\left(-\frac{\beta}{x}\right) \left[1 - \exp\left(-\frac{\beta}{x}\right)\right]^{\alpha-1} dx - 2\alpha \beta \lambda \int_0^{\infty} x^{r-2} \exp\left(-\frac{\beta}{x}\right) \\ &\quad \times \left[1 - \exp\left(-\frac{\beta}{x}\right)\right]^{\alpha-1} \left[1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha\right] dx \end{aligned}$$

Considering the power series expansion

---


$$(1-z)^b = \sum_{j=0}^{\infty} (-1)^j \binom{b}{j} z^j \quad \text{where } |z| \leq 1$$

$$E(X^r) = \alpha\beta(1-\lambda)\beta^{r-1}\Gamma(1-r)\sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} (j+1)^{r-1} + 2\alpha\beta\lambda\beta^{r-1}\Gamma(1-r)\sum_{j=0}^{\infty} (-1)^j \binom{2\alpha-1}{j} (j+1)^{r-1}$$

$$E(X^r) = \alpha\beta^r\Gamma(1-r)\left\{(1-\lambda)\sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} (j+1)^{r-1} + 2\lambda\sum_{j=0}^{\infty} (-1)^j \binom{2\alpha-1}{j} (j+1)^{r-1}\right\}$$

### 3.2 Moment Generating Function (MGF)

Theorem 2: If  $X$  has the  $NGT-IE(x, \alpha, \beta, \lambda)$  with  $|\lambda| \leq 1$ , then the Moment generating function (mgf) of  $X$  is

$$M_x(t) = \alpha\beta Z_{jk} \left[ (1-\lambda) \binom{\alpha-1}{j} + 2\lambda \binom{2\alpha-1}{j} \right] \quad (12)$$

Where  $Z_{jk} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^j \frac{t^k}{k!} [\beta(j+1)]^{k-1} \Gamma(1-k)$

Proof:

By definition

$$\begin{aligned} M_x(t) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ M_x(t) &= \int_0^{\infty} e^{tx} \left\{ \frac{\alpha\beta}{x^2} \exp\left(-\frac{\beta}{x}\right) \left[1 - \exp\left(-\frac{\beta}{x}\right)\right]^{\alpha-1} \left\{ (1+\lambda) - 2\lambda \left[1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^{\alpha}\right] \right\} \right\} dx \\ M_x(t) &= \alpha\beta \left\{ (1+\lambda) \int_0^{\infty} \frac{e^{tx}}{x^2} \exp\left(-\frac{\beta}{x}\right) \left[1 - \exp\left(-\frac{\beta}{x}\right)\right]^{\alpha-1} - 2\lambda \int_0^{\infty} \frac{e^{tx}}{x^2} \exp\left(-\frac{\beta}{x}\right) \left[1 - \exp\left(-\frac{\beta}{x}\right)\right]^{\alpha} \right\} dx \\ &\quad \times \left[1 - \exp\left(-\frac{\beta}{x}\right)\right]^{\alpha-1} \left[1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^{\alpha}\right] \end{aligned}$$

Using binomial expansion

$$\left[1 - \exp\left(-\frac{\beta}{x}\right)\right]^{\alpha-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha-1}{j} \exp\left(-\frac{\beta j}{x}\right)$$

And the expansion of  $\exp(tx)$

---


$$\exp(tx) = 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!}$$

We now have that

$$\begin{aligned} M_x(t) &= \alpha\beta(1-\lambda) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left\{ (-1)^j \binom{\alpha-1}{j} \frac{t^k}{k!} \beta(j+1)^{k-1} \Gamma(1-k) \right\} \\ &+ 2\alpha\beta\lambda \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left\{ (-1)^j \binom{2\alpha-1}{j} \frac{t^k}{k!} \beta(j+1)^{k-1} \Gamma(1-k) \right\} \end{aligned}$$

Therefore

$$\begin{aligned} M_x(t) &= \alpha\beta Z_{jk} \left[ (1-\lambda) \binom{\alpha-1}{j} + 2\lambda \binom{2\alpha-1}{j} \right] \\ \text{Where } Z_{jk} &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^j \frac{t^k}{k!} [\beta(j+1)]^{k-1} \Gamma(1-k) \end{aligned}$$

### 3.3 Quantile Function

The quantile function of corresponding to  $NGT - IE(x, \alpha, \beta, \lambda)$  distribution is the real solution of (13) obtained by inverting (6).

$$x = Q(q) = \frac{\beta}{\ln \left\{ 1 - \left[ \frac{(\lambda-1) + \sqrt{(1-\lambda)^2 + 4\lambda(1-q)}}{2\lambda} \right]^{\frac{1}{\alpha}} \right\}^{-1}}, 0 < q < 1 \quad (13)$$

Where  $q$  is uniformly distributed within the interval  $U(0,1)$  by letting  $q = \frac{1}{2}$  in (13) we obtain the median of  $NGT - IE(x, \alpha, \beta, \lambda)$

$$x_{0.5} = \frac{\beta}{\ln \left\{ 1 - \left[ \frac{(\lambda-1) + \sqrt{\lambda^2 + 1}}{2\lambda} \right]^{\frac{1}{\alpha}} \right\}^{-1}}, \lambda \neq 0 \quad (14)$$

The effect of the extra two parameters on the skewness and kurtosis is examined using measures based on quantiles. The skewness and kurtosis are obtained by Bowley's skewness (S) and Moor's kurtosis (K) respectively given by

$$S = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \quad (15)$$

$$K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \quad (16)$$

The measures of skewness and kurtosis given by (15) and (16) are less sensitive to outliers and they also exist for distributions without moments. Figure 3 shows the effect of  $\lambda$  and  $\alpha$  on the  $NGT - IE(x, \alpha, \beta, \lambda)$ . Figure 3 shows that skewness and kurtosis of  $NGT - IE(x, \alpha, \beta, \lambda)$  reduces when  $\lambda$  increases for fixed value of  $\alpha$ ; and skewness reduces when  $\alpha$  increases for fixed value of  $\lambda$ .

#### 4 Estimation

The method of maximum likelihood is used for the estimation of the parameters of the  $NGT - IE(x, \alpha, \beta, \lambda)$  distribution. The likelihood function is given by (17).

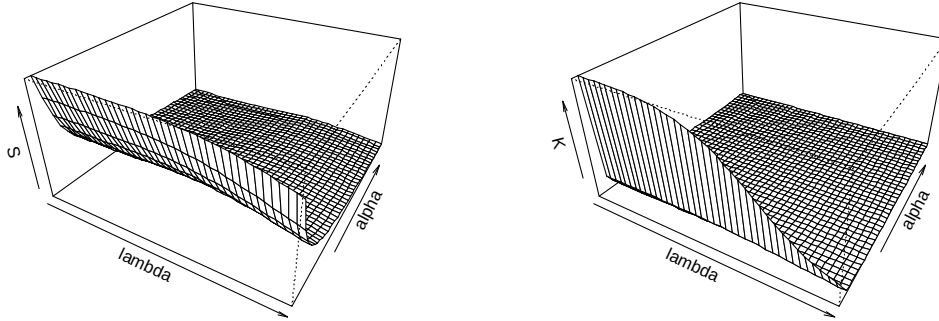


**Fig. 3. Plot of skewness and kurtosis of  $NGT - IE$  the distribution**

$$L(\alpha, \beta, \lambda / x) = \prod_{i=1}^n \left\{ \frac{\alpha\beta}{x^2} \exp\left(-\frac{\beta}{x}\right) \left[1 - \exp\left(-\frac{\beta}{x}\right)\right]^{\alpha-1} \left\{1 + \lambda - 2\lambda \left[1 - \left(1 - \exp\left(-\frac{\beta}{x}\right)\right)^\alpha\right]\right\} \right\} \quad (17)$$

Taking the natural log of both sides and letting  $\ln L(\alpha, \beta, \lambda / x) = l$  we then have

$$\begin{aligned} l = & n \ln \alpha + n \ln \beta - 2 \sum_{i=1}^n \ln x_i - \beta \sum_{i=1}^n \frac{1}{x_i} + (\alpha - 1) \sum_{i=1}^n \ln \left[1 - \exp\left(-\frac{\beta}{x_i}\right)\right] \\ & + \sum_{i=1}^n \ln \left\{1 + \lambda - 2\lambda \left[1 - \left(1 - \exp\left(-\frac{\beta}{x_i}\right)\right)^\alpha\right]\right\} \end{aligned} \quad (18)$$

The maximum likelihood estimates  $\hat{\omega} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})^T$  of  $NGT - IE(x, \alpha, \beta, \lambda)$  is obtained by differentiating (18) partially with respect to  $\alpha, \beta, \lambda$  and equating to zero.

$$\begin{aligned}\frac{\partial l}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n \ln \left[ 1 - \exp \left( -\frac{\beta}{x_i} \right) \right] + \sum_{i=1}^n \left\{ \frac{2\lambda \left( 1 - \exp \left( -\frac{\beta}{x_i} \right) \right)^\alpha \ln \left( 1 - \exp \left( -\frac{\beta}{x_i} \right) \right)}{1 + \lambda - 2\lambda \left[ 1 - \left( 1 - \exp \left( -\frac{\beta}{x_i} \right) \right)^\alpha \right]} \right\} \\ \frac{\partial l}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^n \frac{1}{x_i} + (\alpha - 1) \sum_{i=1}^n \left[ \frac{\exp \left( -\frac{\beta}{x_i} \right)}{x_i \left( 1 - \exp \left( -\frac{\beta}{x_i} \right) \right)} \right] + \sum_{i=1}^n \left[ \frac{2\lambda \alpha \left( 1 - \exp \left( -\frac{\beta}{x_i} \right) \right)^{\alpha-1} \exp \left( -\frac{\beta}{x_i} \right)}{x_i \left[ 1 + \lambda - 2\lambda \left[ 1 - \left( 1 - \exp \left( -\frac{\beta}{x_i} \right) \right)^\alpha \right] \right]} \right] \\ \frac{\partial l}{\partial \lambda} &= \sum_{i=1}^n \left\{ \frac{1 - 2 \left[ 1 - \left( 1 - \exp \left( -\frac{\beta}{x_i} \right) \right)^\alpha \right]}{1 + \lambda - 2\lambda \left[ 1 - \left( 1 - \exp \left( -\frac{\beta}{x_i} \right) \right)^\alpha \right]} \right\}\end{aligned}$$

The solutions to the nonlinear system of equations above will yield the ML estimators  $\hat{\alpha}, \hat{\beta}$  and  $\hat{\lambda}$ . Using large sample property of MLE estimators, estimators of  $\omega$  can be treated as being approximately trivariate normal. The asymptotic distribution of the elements of the  $3 \times 3$  information matrix for the  $NGT - IE$  distribution is  $\sqrt{n}(\hat{\omega} - \omega) \square N_3(0, V^{-1})$ , where  $V$  is the expected information matrix. The expected information matrix is

$$V^{-1} = -E \begin{bmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 l}{\partial \beta \partial \alpha} & \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \lambda} \\ \frac{\partial^2 l}{\partial \lambda \partial \alpha} & \frac{\partial^2 l}{\partial \lambda \partial \beta} & \frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix} \quad (19)$$

Solving the inverse matrix of the observed information matrix (19) gives the asymptotic variance and covariance of the MLE,  $\hat{\alpha}, \hat{\beta}$  and  $\hat{\lambda}$ . Hence approximate  $100(1 - \alpha)\%$ , asymptotic confidence intervals for  $\alpha, \beta$  and  $\lambda$  are

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{11}}, \quad \hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{22}}, \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{33}}$$

where  $Z_{\frac{\alpha}{2}}$  is the upper  $\alpha$ th percentile of the standard normal distribution.

## 5 Renyi Entropy

The entropy of a random variable  $X$  with density function  $f(x)$  is a measure of variation of uncertainty. For any real parameter  $\gamma > 0$  and  $\gamma \neq 1$ . The Renyi entropy is given by

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \int_0^\infty f^\gamma(x) dx \right\} \quad (20)$$

The Renyi entropy of  $NGT - IE(x, \alpha, \beta, \lambda)$  is obtained by substituting (7) in (20)

$$f^\gamma(x) = \left( \frac{\alpha\beta}{x^2} \right)^\gamma \exp\left(-\frac{\gamma\beta}{x}\right) \left( 1 - \exp\left(-\frac{\beta}{x}\right) \right)^{\gamma(\alpha-1)} (1+\lambda)^\gamma \left\{ 1 - \frac{2\lambda}{(1+\lambda)} \left[ 1 - \left( 1 - \exp\left(-\frac{\beta}{x}\right) \right)^\alpha \right] \right\}^\gamma \quad (21)$$

Applying the general binomial expansion on  $\left\{ 1 - \frac{2\lambda}{(1+\lambda)} \left[ 1 - \left( 1 - \exp\left(-\frac{\beta}{x}\right) \right)^\alpha \right] \right\}^\gamma$  (21), (20) becomes

$$\begin{aligned} I_R(\gamma) &= \frac{1}{1-\gamma} \log \left\{ (\alpha\beta)^\gamma (1+\lambda)^\gamma \sum_{i,k=0}^\infty \sum_{j=0}^i (-1)^{i+j+k} \binom{\gamma}{i} \binom{i}{j} \binom{\gamma\alpha + j\alpha - \gamma}{k} \left( \frac{2\lambda}{1+\lambda} \right)^{i\alpha} \int_0^\infty x^{-2\gamma} \exp\left(-\frac{\beta(k+\gamma)}{x}\right) dx \right\} \\ I_R(\gamma) &= \frac{\gamma \log(\alpha\beta)}{1-\gamma} + \frac{\gamma \log(1+\lambda)}{1-\gamma} - \frac{(2\gamma+1) \log \beta}{(1-\gamma)} + \frac{1}{1-\gamma} \log \left\{ \sum_{i,k} \sum_j w_{i,j,k}(\alpha, \lambda, \gamma) (k+\gamma)^{-2\gamma+1} \Gamma(2\gamma-1) \right\} \quad (18) \\ w_{i,j,k}(\alpha, \lambda, \gamma) &= (-1)^{i+j+k} \binom{\gamma}{i} \binom{i}{j} \binom{\gamma\alpha + j\alpha - \gamma}{k} \left( \frac{2\lambda}{1+\lambda} \right)^i \end{aligned}$$

Please check highlighted equation numbers

## 6 Order Statistics

Consider a random sample  $x_1, x_2, x_3, \dots, x_n$  drawn from  $NGT - IE(x, \alpha, \beta, \lambda)$  distribution with pdf and cdf as defined in (7) and (6) respectively. The  $i^{th}$  order statistics denoted by  $x_{i:n}$  is given by

$$f(x_{i:n}) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} (1-F(x))^{n-i} = K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x) F(x)^{i+j-1} \quad (22)$$

where  $K = \frac{n!}{(i-1)!(n-i)!}$ ,  $F(x)$  and  $f(x)$  are as defined in (6) and (7) respectively.

Substituting for  $F(x)$  and  $f(x)$  in (22) and applying the general binomial expansion, we have

$$f(x_{i:n}) = K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[ \sum_{p=0}^{\infty} w_{k,l} x^{-2} \exp\left(-\frac{\beta(p+1)}{x}\right) - \sum_{q=0}^{\infty} h_{k,m} x^{-2} \exp\left(-\frac{\beta(q+1)}{x}\right) \right] \quad (23)$$

Where

$$w_{k,l} = w_{k,l}(\alpha, \beta, \lambda) = \alpha \beta \sum_{k=0}^{i+j+1} \sum_{l=0}^{i+j+k-1} (-1)^{k+l+p} \lambda^k (1+\lambda)^{i+j-k} \binom{i+j-1}{k} \binom{i+j+k-1}{l} \binom{\alpha(l+1)-1}{p}$$

And

$$h_{k,m} = h_{k,m}(\alpha, \beta, \lambda) = 2\lambda \alpha \beta \sum_{k=0}^{i+j-1} \sum_{m=0}^{i+j+k} (-1)^{k+m+q} \lambda^k (1+\lambda)^{i+j-k-1} \binom{i+j-1}{k} \binom{i+j+k}{m} \binom{\alpha(m+1)-1}{q}$$

## 7 Application

In this section application of NGT-IE to real life data set is provided. The data set is from [10] with respect to breaking strength of carbon fibres. The maximum likelihood method of estimation of parameters is used for estimation. The fit is compared with other models based on the maximised log-likelihood, Komogorov-Smirnov test (K-S) and Akaike Information Criterion (AIC). The histogram of the data set, pdf and cdf of the fitted models are also presented for graphical illustration of the goodness of fit.

Please mention ref [11,12] inside the text

The estimate of the parameters of NGT-IE for the data set is as shown in the table 1. From table 1 the NGT-IE provides a better fit compared Inverse exponential (IE) distribution, Inverse Rayleigh (IR) distribution and generalized inverse exponential (GIE) distribution. The NGT-IE distribution has the smallest K-S and AIC statistics and the largest log-likelihood value. This proves that the three parameter NGT-IE seems to be a better model than other models considered in the table.

## 8 Conclusion

In this paper, we proposed a new generalisation of the in inverse exponential distribution, called the New generalised transmuted inverse exponential distribution (NGT-IE). We derived the explicit expressions for moments, moment generating functions, quantile function, maximum likelihood estimate of the parameters, Renyi entropy and order statistics. An application to show that the proposed model provides a better fit than other competing models is done. We hope that this distribution will attract more applications in statistics.

**Table 1. Estimated parameters of the NGT-IE, IE, IR and GIE distribution for the data set**

| Models   | Parameter estimates<br>(Standard errors) |                  |           | Statistics     |         |        |
|----------|------------------------------------------|------------------|-----------|----------------|---------|--------|
|          | $\alpha$                                 | $\beta$          | $\lambda$ | K-S Statistics | Log L   | AIC    |
| NGT – IE | 8.175<br>(1.930)                         | 5.215<br>(0.750) | -0.575    | 0.117          | -149.12 | 304.23 |
| IE       | –                                        | 2.139<br>(0.214) | –         | 0.345          | -199.39 | 400.79 |
| IR       | –                                        | 3.276<br>(0.329) | –         | 0.182          | -175.24 | 352.48 |
| GIE      | 9.059                                    | 6.197            | –         | 0.130          | -151.24 | 306.48 |

---

|         |         |
|---------|---------|
| (2.088) | (0.604) |
|---------|---------|

---

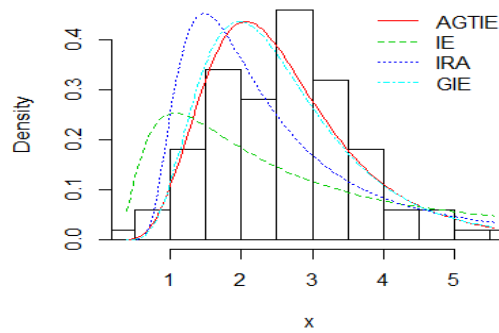


**Fig. 4. The fitted pdfs for the data set**

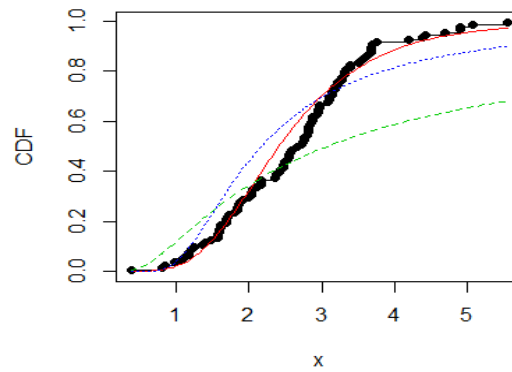


**Fig. 5. The fitted cdfs for the data set**

## Competing Interests

Authors have declared that no competing interests exist.

## References

- [1] Keller A Z, Kamath A R. Reliability analysis of CNC machine tools. Reliability Engineering. 1982; 3: 449-473
- [2] Oguntunde P E, Babatunde O S, Ogunmola A O. Theoretical analysis of the kumaraswamy inverse exponential distribution. International Journal of Statistics and Applications. 2014; 4(2): 113-116
- [3] Abouanmoh A A, Alshingiti, A M. Reliability estimation of generalised inverted exponential distribution. Journal of statistical computation and simulation. 2009; 79(11): 1301-1315
- [4] Oguntunde P E, Adejumo A.O, Balogun O S. Statistical properties of the exponentiated generalized inverted exponential distribution. Applied Mathematics. 2014; 4(2): 47-55
- [5] Oguntunde P E, Adejumo AO. The transmuted inverse exponential distribution. International Journal of Advanced Statistics and probability. 2015; 3(1):1-7

- 
- [6] Elbata I. Transmuted generalised inverted exponential distribution. *Economic Quality Control*. 2013; 28(3): 125-133
  - [7] Singh B, Geol R. The beta inverted exponential distribution: properties and application. *International Journal of Applied Science and Mathematics*. 2015; 2(5):2394-2894
  - [8] Bakoban K A, Abu-Zinadah H H. The beta generalized inverted exponential distribution with real data applications. *Revstat Statistical Journal*. 2017; 15(1): 65-68
  - [9] Merovci F, Alizadeh M, Hamedani G G. Another transmuted family of distributions: Properties and Applications. *Austrian Journal of Statistics*. 2016; 45: 71-93
  - [10] Nichols M D, Padgett W J. A bootstrap control chart for Weibull percentiles. *Quality and Reliability Engineering International*. 2006; 22: 141-151
  - [11] Haq, M. A., Elgarhy, M., Hashmi, S., Ozel, G., & ul Ain, Q. (2018). Transmuted Weibull Power Function Distribution: its Properties and Applications. *Journal of Data Science*, 16(2), 397-418.
  - [12] Haq, M. A., Butt, N. S., Usman, R. M., & Fattah, A. A. (2016). Transmuted power function distribution. *Gazi University Journal of Science*, 29(1), 177-185.