Original Research Article

BAYESIAN ANALYSIS OF A SHAPE PARAMETER OF THE WEIBULL-FRECHET DISTRIBUTION

Abstract

In this paper, we estimate a shape parameter of the Weibull-Frechet distribution by considering the Bayesian approach under two non-informative priors using three different loss functions. We derive the corresponding posterior distributions for the shape parameter of the Weibull-Frechet distribution assuming that the other three parameters are known. The Bayes estimators and associated posterior risks have also been derived using the three different loss functions. The performance of the Bayes estimators are evaluated and compared using a comprehensive simulation study to find out the combination of a loss function and a prior having the minimum Bayes risk and hence producing the best results. In conclusion, this study reveals that in order to estimate the parameter in question, we should use quadratic loss function under either of the two non-informative priors used in this study.

Keywords: Weibull-Frechet, Bayesian, MLE, prior, Uniform, Jeffrey, Loss functions.

1. Introduction

The Fréchet distribution is mostly used in extreme value theory and it has applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, wind speeds and sea waves. To get details on the Fréchet distribution and its applications, readers can study [25]. Moreover, applications of this distribution in various fields are given in [22], where it has been proven that the frechet distribution is used for modeling the statistical behavior of materials properties for a variety of engineering applications. [32] discussed the sociological models based on Fréchet random variables. [37] applied the Fréchet model for analyzing the wind speed data. [30] studied the Fréchet progressive type-II censored data with binomial removals.

A random variable X is said to follow a Fréchet distribution with parameters θ *and* λ if its probability density function (*pdf*) is given by

$$f(x) = \lambda\theta^{\lambda} x^{-\lambda-1} e^{-\left(\frac{\theta}{x}\right)^{\lambda}} \tag{1.1}$$

and the corresponding cumulative distribution function (*cdf*) is given as

$$F(x) = e^{-\left(\frac{\theta}{x}\right)^{\lambda}} \tag{1.2}$$

For $x > 0, \theta > 0, \lambda > 0$ where θ *and* λ are the scale and shape parameters of the Fréchet respectively.

Many authors have developed generalizations of the Fréchet distribution. For instance, [32] pioneered the exponentiated Fréchet, [31] and [17] studied the beta Fréchet, [27] proposed

the transmuted Fréchet, [26] introduced the Marshall-Olkin Fréchet, [35] defined the gamma extended Fréchet, [18] studied the transmuted exponentiated Fréchet, [29] introduced the Kumaraswamy-Fréchet, [1] investigated the transmuted Marshall-Olkin Fréchet distributions, [2] studied the transmuted complementary Weibull geometric distribution and [3] studied the Weibull- Fréchet distribution. Of interest to us in this paper is the Weibull-Fréchet distribution (*WFrD*) proposed by [3]. This is because the parameters, properties and applications of this four parameter distribution have been studied and compared with some other distributions and the result showed that it is more fitted compared to kumaraswamy Frechet (*KFr*), exponentiated Frechet (*EFr*), beta Frechet (*BFr*), gamma extended Frechet (*GEFr*), transmitted marshallOlkin Frechet (*TMOFr*) and Frechet (*Fr*) distributions ([3]).
The probability density function (*pdf*) and cumulative distribution function (*cdf*) of the Weibull-Fréchet distribution are given by (for x > 0)

$$f(x) = \alpha\beta\lambda\theta^{\lambda}x^{-\lambda-1}e^{-\left(\frac{\theta}{x}\right)^{\lambda}}\left(1-e^{-\left(\frac{\theta}{x}\right)^{\lambda}}\right)^{-\beta-1}e^{-\alpha\left(e^{\left(\frac{\theta}{x}\right)^{\lambda}}-1\right)^{-\beta}} \quad (1.3)$$

and

$$F(x) = 1-e^{-\alpha\left(e^{\left(\frac{\theta}{x}\right)^{\lambda}}-1\right)^{-\beta}} \quad (1.4)$$

respectively, where $\theta > 0$ is a scale parameter and $\alpha, \beta, \lambda > 0$ are the shape parameters of the Fréchet distribution respectively according to [3].
There are two main philosophical approaches to statistics. The first is called the classical approach which was founded by Professor R.A. Fisher in a series of fundamental papers round about 1930. In classical approach, the parameters are considered to be fixed while in the non classical or Bayesian concept, the parameters are viewed as unknown random variables. However, in many real life situations represented by life time models, the parameters cannot be treated as constant throughout the life testing period ([23]; [28]; [36]) and hence the need for Bayesian estimation for life time models.
Recently Bayesian estimation approach has received great attention by most researchers among them are [11] who studied Bayesian estimation for the extreme value distribution using progressive censored data and asymmetric loss. [10] considered Bayesian Survival Estimator for Weibull distribution with censored data. [19] discussed the Bayesian analysis of the scale parameter of inverse Gaussian distribution using different priors and loss function. [14] obtained the shape parameter of Generalized Power Distribution (GPD) via Bayesian approach under the non-informative (uniform) and informative (gamma) priors using the squared error loss function. [15] estimated the scale parameter of Nakagami distribution using Bayesian approach. The Bayesian estimate of the scale parameter of Nakagami distribution under uniform prior, inverse exponential and levy prior distributions using squared error, quadratic and precautionary loss functions were also obtained by [16] and again [24] made a Comparison between Maximum Likelihood and Bayesian Estimation Methods for a Shape Parameter of the Weibull-Exponential Distribution under uniform and

Jeffrey's priors and found that Bayesian method under uniform prior is better using quadratic loss function.

The main objective of this paper is to introduce a statistical comparison between the Bayesian and Maximum likelihood estimation procedures for estimating the shape parameter of *WFrD*. The layout of the paper is as follow. In Section 2, we take a look at the materials and methods used which include the priors and the different loss functions. In Section 3, we obtained Maximum likelihood estimates of the shape parameter in question. Also, we estimate the shape parameter of the *WFrD* under uniform and Jeffrey's priors in section 4 and section 5 respectively using three different loss functions. The posterior risks of the estimators obtained under the two priors using the three different functions were derived in section 6. Finally, a comparison between Bayes and Maximum likelihood estimates have been made using simulation study in Section 7 with Some concluding remarks given in Section 8.

2. Materials and Methods

3.

2.1 Priors and Loss Functions

The Bayesian inference requires appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear cut way from which one can conclude that one prior is better than the other. Nevertheless, very often priors are chosen according to one's subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s); otherwise, it is preferable to use non-informative prior(s). In this paper we consider two non-informative priors: the uniform and Jeffreys' prior.

To obtain the posterior distribution of the shape parameter once the data has been observed, we apply bayes' Theorem which is stated in the following form:

$$p\left(\alpha \mid \underline{X}\right)=\frac{L\left(\alpha \mid \underline{X}\right)p\left(\alpha\right)}{\int_{0}^{\infty}L\left(\alpha \mid \underline{X}\right)p\left(\alpha\right)d\alpha} \qquad (2.1)$$

where $p\left(\alpha\right)$ and $L\left(\alpha \mid \underline{X}\right)$ are the prior distribution and the Likelihood function respectively.

The uniform prior as a non-informative prior relating to the shape parameter α is defined as:

$$p\left(\alpha\right)\propto 1; 0<\alpha<\infty \qquad (2.2)$$

The posterior distribution of the shape parameter α for a given data under uniform prior is obtained from equation (2.1) using integration by substitution method as

$$p\left(\alpha \mid \underline{X}\right)=\frac{\alpha^{n}\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^{(n+1)} e^{-\alpha\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}}{\Gamma\left(n+1\right)}$$

(2.3)

Also, the Jeffrey's prior as a non-informative prior relating to the shape parameter α of the *WFrD* distribution is defined as:

$$p(\alpha) \propto \frac{1}{\alpha}; 0 < \alpha < \infty \tag{2.4}$$

The posterior distribution of the shape parameter α for a given data under Jeffrey prior is obtained from equation (2.1) using integration by substitution method as

$$p(\alpha \mid \underline{X}) = \frac{\alpha^{n-1}\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^{n} e^{-\alpha\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}}{\Gamma(n)} \tag{2.5}$$

In statistics and decision theory, a loss function is a function that maps an event into a real number intuitively representing some cost associated with the event. Typically it is used for parameter estimation and that event in question is some function of the difference between estimated and true values for an instance of data. A Loss function, $L(\alpha, \alpha_{SELF})$ is that which describes the losses incurred by making an estimate $\hat{\alpha}$ of the true value of the parameter is α. A number of symmetric and asymmetric loss functions have been shown to be functional in so many studies including; [13], [33], [12], [34], [9], [7], [4], [5], [6], [8], [21] and [20] and so forth.

With the above priors and prior distributions, we will use three loss functions to estimate the shape parameter of the *WFrD* and these loss functions are defined as follows:

(a) Squared Error Loss Function (*SELF*)

The squared error loss function relating to the scale parameter α is defined according to [16] as

$$L(\alpha, \alpha_{SELF}) = (\alpha - \alpha_{SELF})^2 \tag{2.6}$$

where α_{SELF} is the estimator of the parameter α under *SELF*.

(b) Quadratic Loss Function (QLF)

The quadratic loss function is defined from [15] as

$$L(\alpha, \alpha_{QLF}) = \left(\frac{\alpha - \alpha_{QLF}}{\alpha}\right)^2 \tag{2.7}$$

where α_{QLF} is the estimator of the parameter α under *QLF*.

(c) Precautionary Loss Function (*PLF*)

The precautionary loss function (*PLF*) according to [16] is an asymmetric loss function and is defined as

$$L(\alpha_{PLF}, \alpha) = \frac{(\alpha_{PLF} - \alpha)^2}{\alpha} \tag{2.8}$$

where α_{PLF} is the estimator of the parameter α under *PLF*.

4. Maximum Likelihood Estimation

Here we present the estimation of the shape parameter of the Weibull-Fréchet distribution (*WFrD*) using the method of maximum likelihood estimation. Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample from the *WFrD* with unknown parameter vector $\xi = (\alpha, \beta, \theta, \lambda)^T$. The total log-likelihood function for ξ is obtained from *f(x)* as follows:

$$L(X_1, X_2, \ldots\ldots, X_n / \alpha, \beta, \theta, \lambda) = (\alpha\beta\lambda\theta^{\lambda})^n \sum_{i=1}^{n} x_i^{-\lambda-1} e^{-\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\lambda}} \sum_{i=1}^{n}\left(1-e^{-\left(\frac{\theta}{x_i}\right)^{\lambda}}\right)^{-\beta-1} \exp\left\{-\alpha\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right\}$$

(3.1)

The likelihood function for the shape parameter, α, is given by;

$$L(X_1, X_2, \ldots\ldots, X_n / \alpha) = (\alpha)^n \exp\left\{-\alpha\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right\} \quad (3.2)$$

Let the log-likelihood function, $l = \log L(\alpha \mid \underline{X})$, therefore

$$l = n\log\alpha - \alpha\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta} \quad (3.3)$$

Differentiating l partially with respect to α, the shape parameter and solving for $\hat{\alpha}$ gives;

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}$$

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}} \quad (3.4)$$

Hence, equation (3.4) is the estimator for the shape parameter of the Weibull-Frechet distribution obtained by the method of maximum Likelihood estimation.

5. Bayesian Estimation of the shape parameter of the *WFrD* under Uniform prior by using the three Different Loss Functions

Here, we estimate the shape parameter of the *WFrD* under three loss functions using the posterior distribution obtained from the uniform prior in equation (2.3).

4.1 Estimation Using Squared Error Loss Function (*SELF*)

The derivation of Bayes estimator using *SELF* under uniform prior is as given below:

$$\alpha_{SELF} = E(\alpha) = E(\alpha \mid \underline{X})$$

$$E(\alpha \mid \underline{X}) = \int_0^{\infty} \alpha\, p(\alpha \mid \underline{X})\, d\alpha \quad (4.1)$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (4.1); we have:

$$E\left(\alpha \mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^{n+1}}{\Gamma\left(n+1\right)}\int_{0}^{\infty}\alpha^{n+1}e^{-\alpha\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}d\alpha \qquad (4.2)$$

Now, using integration by substitution method in equation (4.2) and simplification, we obtained the Bayes estimator using *SELF* under uniform prior as:

$$\alpha_{SELF}=E\left(\alpha \mid \underline{X}\right)=\frac{\Gamma\left(n+2\right)}{\Gamma\left(n+1\right)\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}$$

$$\alpha_{SELF}=E\left(\alpha \mid \underline{X}\right)=\frac{n+1}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}} \qquad (4.3)$$

4.2 Estimation Using Quadratic Loss Function (*QLF*)

The derivation of Bayes estimator using *QLF* under uniform prior is given below:

$$\alpha_{QLF}=\frac{E\left(\alpha^{-1}\right)}{E\left(\alpha^{-2}\right)}=\frac{E\left(\alpha^{-1} \mid \underline{X}\right)}{E\left(\alpha^{-2} \mid \underline{X}\right)}$$

$$E\left(\alpha^{-1} \mid \underline{X}\right)=\int_{0}^{\infty}\alpha^{-1}p\left(\alpha \mid \underline{X}\right)d\alpha \qquad (4.4)$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (4.4); we have:

$$E\left(\alpha^{-1} \mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^{n+1}}{\Gamma\left(n+1\right)}\int_{0}^{\infty}\alpha^{n-1}e^{-\alpha\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}d\alpha \qquad (4.5)$$

Using integration by substitution method in equation (4.5) and simplifying, we obtained the Bayes estimator using *QLF* under uniform prior as:

$$\alpha_{QLF}=\frac{E\left(\alpha^{-1}\right)}{E\left(\alpha^{-2}\right)}=\frac{E\left(\alpha^{-1} \mid \underline{X}\right)}{E\left(\alpha^{-2} \mid \underline{X}\right)}=\frac{\Gamma\left(n\right)}{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)\Gamma\left(n-1\right)}$$

$$\alpha_{QLF}=\frac{E\left(\alpha^{-1}\right)}{E\left(\alpha^{-2}\right)}=\frac{E\left(\alpha^{-1} \mid \underline{X}\right)}{E\left(\alpha^{-2} \mid \underline{X}\right)}=\frac{n-1}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}} \qquad (4.6)$$

4.3 Estimation Using Precautionary Loss Function (*PLF*)

Similarly, the derivation of Bayes estimator under *PLF* using uniform prior is given below:

$$\alpha_{PLF}=\left\{E\left(\alpha^{2}\right)\right\}^{\frac{1}{2}}=\left\{E\left(\alpha^{2} \mid \underline{X}\right)\right\}^{\frac{1}{2}}=\sqrt{E\left(\alpha^{2} \mid \underline{X}\right)}$$

$$E\left(\alpha^{2} \mid \underline{X}\right)=\int_{0}^{\infty}\alpha^{2}p\left(\alpha \mid \underline{X}\right)d\alpha \qquad (4.7)$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (4.7); we have:

$$E\left(\alpha^2 \mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^{n+1}}{\Gamma(n+1)}\int_0^{\infty}\alpha^{n+2}e^{-\alpha\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}d\alpha \tag{4.8}$$

Again using integration by substitution method in equation (4.8) and simplifying, we obtained the Bayes estimator using *PLF* under uniform prior as:

$$\alpha_{PLF}=\left\{E\left(\alpha^2 \mid \underline{X}\right)\right\}^{\frac{1}{2}}=\sqrt{\frac{\Gamma(n+3)}{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^{2}\Gamma(n+1)}}$$

$$\alpha_{PLF}=\left\{E\left(\alpha^2 \mid \underline{X}\right)\right\}^{\frac{1}{2}}=\frac{\left[(n+2)(n+1)\right]^{0.5}}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}} \tag{4.9}$$

It is very clear that the relationship: $\lambda_{PLF}>\lambda_{SELF}>\lambda_{MLE}>\lambda_{QLF}$ holds for all parameter values and λ_{QLF} under the uniform prior is obviously the minimum.

6. Bayesian Estimation of the shape parameter of the *WFrD* under Jeffrey's prior by using the three Different Loss Functions

This section presents the estimation of the shape parameter of the *WFrD* using three loss functions and the posterior distribution obtained from Jeffrey's prior in equation (2.5).

5.1 Estimation Using Squared Error Loss Function (*SELF*)

The derivation of Bayes estimator under *SELF* using Jeffrey's prior is as given below:

$$\alpha_{SELF}=E(\alpha)=E\left(\alpha \mid \underline{X}\right)$$

$$E\left(\alpha \mid \underline{X}\right)=\int_0^{\infty}\alpha\, p\left(\alpha \mid \underline{X}\right)d\alpha \tag{5.1}$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (5.1); we have:

$$E\left(\alpha \mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^{n}}{\Gamma(n)}\int_0^{\infty}\alpha^{n}e^{-\alpha\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}d\alpha \tag{5.2}$$

Using integration by substitution method in equation (5.3) and simplifying, we obtained the Bayes estimator using *SELF* under Jeffrey prior as:

$$\alpha_{SELF}=E\left(\alpha \mid \underline{X}\right)=\frac{\Gamma(n+1)}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\Gamma(n)}$$

$$\alpha_{SELF}=E\left(\alpha \mid \underline{X}\right)=\frac{n}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}} \tag{5.3}$$

5.2 Estimation Using Quadratic Loss Function (*QLF*)

Also, the derivation of Bayes estimator under Jeffrey's prior using *QLF* is given below:

$$\alpha_{QLF} = \frac{E\left(\alpha^{-1}\right)}{E\left(\alpha^{-2}\right)} = \frac{E\left(\alpha^{-1} \mid \underline{X}\right)}{E\left(\alpha^{-2} \mid \underline{X}\right)}$$

$$E\left(\alpha^{-1} \mid \underline{X}\right) = \int_0^\infty \alpha^{-1} p\left(\alpha \mid \underline{X}\right) d\alpha \qquad (5.4)$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (5.4); we have:

$$E\left(\alpha^{-1} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}} - 1\right)^{-\beta}\right)^{n}}{\Gamma(n)} \int_0^\infty \alpha^{n-2} e^{-\alpha \sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}} - 1\right)^{-\beta}} d\alpha \qquad (5.5)$$

Using integration by substitution method in equation (5.5) and simplifying, we obtained the Bayes estimator using *QLF* under Jeffrey prior as:

$$\alpha_{QLF} = \frac{E\left(\alpha^{-1}\right)}{E\left(\alpha^{-2}\right)} = \frac{E\left(\alpha^{-1} \mid \underline{X}\right)}{E\left(\alpha^{-2} \mid \underline{X}\right)} = \frac{\Gamma(n-1)}{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}} - 1\right)^{-\beta}\right)\Gamma(n-2)}$$

$$\alpha_{QLF} = \frac{E\left(\alpha^{-1}\right)}{E\left(\alpha^{-2}\right)} = \frac{E\left(\alpha^{-1} \mid \underline{X}\right)}{E\left(\alpha^{-2} \mid \underline{X}\right)} = \frac{n-2}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}} - 1\right)^{-\beta}} \qquad (5.6)$$

5.3 Estimation Using Precautionary Loss Function (*PLF*)

Similarly, the derivation of Bayes estimator under *PLF* using Jeffrey's prior is given below:

$$\alpha_{PLF} = \left\{E\left(\alpha^{2}\right)\right\}^{\frac{1}{2}} = \left\{E\left(\alpha^{2} \mid \underline{X}\right)\right\}^{\frac{1}{2}} = \sqrt{E\left(\alpha^{2} \mid \underline{X}\right)}$$

$$E\left(\alpha^{2} \mid \underline{X}\right) = \int_0^\infty \alpha^{2} p\left(\alpha \mid \underline{X}\right) d\alpha \qquad (5.7)$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (5.7); we have:

$$E\left(\alpha^{2} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}} - 1\right)^{-\beta}\right)^{n}}{\Gamma(n)} \int_0^\infty \alpha^{n+1} e^{-\alpha \sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}} - 1\right)^{-\beta}} d\alpha \qquad (5.8)$$

Using integration by substitution method in equation (5.8) and simplifying, we obtained the Bayes estimator using *PLF* under Jeffrey prior as:

$$\alpha_{PLF} = \left\{E\left(\alpha^{2} \mid \underline{X}\right)\right\}^{\frac{1}{2}} = \sqrt{\frac{\Gamma(n+2)}{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}} - 1\right)^{-\beta}\right)^{2}\Gamma(n)}}$$

$$\alpha_{PLF} = \left\{E\left(\alpha^2 \mid \underline{X}\right)\right\}^{\frac{1}{2}} = \frac{\left(\left(n+1\right)\left(n\right)\right)^{\frac{1}{2}}}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}} \tag{5.9}$$

It is also clear that λ_{MLE} is the same as λ_{SELF} under Jeffrey's prior and the relationship: $\lambda_{PLF} > \lambda_{SELF} > \lambda_{MLE} > \lambda_{QLF}$ holds for all parameter values and λ_{QLF} under the Jeffrey's prior appears to be the minimum.

7. Posterior Risks under the priors using the Different Loss Functions

The posterior risks of the Bayes estimators under the three loss functions from both uniform and Jeffrey's prior are obtained as follows:

6.1 Posterior Risks under the Uniform Prior

Using Squared Error Loss Function (*SELF*)

Using the Squared error loss function (SELF), the posterior risk, $p\left(\lambda_{SELF}\right)$ is defined from [16] as:

$$P\left(\alpha_{SELF}\right) = E\left(\alpha^2 \mid \underline{X}\right) - \left\{E\left(\alpha \mid \underline{X}\right)\right\}^2 \tag{6.1}$$

And it is obtained as

$$P\left(\alpha_{SELF}\right) = \frac{\left(n+2\right)\left(n+1\right) - \left(\left(n+1\right)\right)^2}{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^2} \tag{6.2}$$

Using Quadratic Loss Function (*QLF*)

Using the Quadratic loss function (*QLF*), the posterior risk, $p\left(\lambda_{QLF}\right)$ is defined from [16] as:

$$P\left(\alpha_{QLF}\right) = 1 - \frac{\left\{E\left(\alpha^{-1} \mid \underline{X}\right)\right\}^2}{E\left(\alpha^{-2} \mid \underline{X}\right)} \tag{6.3}$$

Therefore, the posterior risk under uniform prior using the Quadratic loss function is given as:

$$P\left(\alpha_{QLF}\right) = \frac{1}{n} \tag{6.4}$$

Precautionary Loss Function (*PLF*)

Using the Precautionary loss function (*PLF*), the posterior risk, $p\left(\lambda_{PLF}\right)$ is defined from [16] as:

$$P\left(\alpha_{PLF}\right) = 2\left\{\alpha_{PLF} - E\left(\alpha \mid \underline{X}\right)\right\} \tag{6.5}$$

And calculated to be:

$$P\left(\alpha_{PLF}\right) = 2\left\{\frac{\left\{\left(n+2\right)\left(n+1\right)\right\}^{\frac{1}{2}} - \left(n+1\right)}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}\right\} \tag{6.6}$$

6.2 Posterior Risks under Jeffrey's Prior

The posterior risks of the Bayes estimators under the three loss functions from the Jeffrey's prior are as follows:

Using Squared Error Loss Function (*SELF*)

Using the Squared error loss function (*SELF*), the posterior risk, $p\left(\lambda_{SELF}\right)$ under Jeffrey's prior is defined from [16] as:

$$P\left(\alpha_{SELF}\right)=E\left(\alpha^{2} \mid \underline{X}\right)-\left\{E\left(\alpha \mid \underline{X}\right)\right\}^{2} \tag{6.7}$$

Therefore, the posterior risk under Jeffrey's prior using the squared error loss function is:

$$P\left(\alpha_{SELF}\right)=\frac{n}{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^{2}} \tag{6.8}$$

Using Quadratic Loss Function (QLF)

Using the Quadratic loss function (*QLF*), the posterior risk, $p\left(\lambda_{QLF}\right)$ under Jeffrey's prior is defined from [16] as:

$$P\left(\alpha_{QLF}\right)=1-\frac{\left\{E\left(\alpha^{-1} \mid \underline{X}\right)\right\}^{2}}{E\left(\alpha^{-2} \mid \underline{X}\right)} \tag{6.9}$$

Hence, it is obtained as:

$$P\left(\alpha_{QLF}\right)=\frac{1}{n-1} \tag{6.10}$$

Using Precautionary Loss Function (PLF)

Using the Precautionary loss function (*PLF*), the posterior risk, $p\left(\alpha_{PLF}\right)$ is defined as:

$$P\left(\alpha_{PLF}\right)=2\left\{\alpha_{PLF}-E\left(\alpha \mid \underline{X}\right)\right\} \tag{6.11}$$

Hence, obtained as:

$$P\left(\alpha_{PLF}\right)=2\left\{\frac{\left\{n(n+1)\right\}^{\frac{1}{2}}-n}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}\right\} \tag{6.12}$$

Table 6.1: A Summary of the expressions for *MLE*, Bayes Estimators and Posterior Risks under uniform prior and Jeffrey's Prior is as follows:

PRIORS	*MLE*	*SELF*	*QLF*	*PLF*
Estimators				
UNIFORM	$\frac{n}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}$	$\frac{n+1}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}$	$\frac{n-1}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}$	$\frac{\left[(n+2)(n+1)\right]^{0.5}}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}$

JEFFREY'S	$\dfrac{n}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}$	$\dfrac{n}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}$	$\dfrac{n-2}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}$	$\dfrac{\left((n+1)(n)\right)^{\frac{1}{2}}}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}$
Posterior Risks				
UNIFORM		$\dfrac{(n+2)(n+1)-\left((n+1)\right)^2}{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^2}$	$\dfrac{1}{n}$	$2\left\{\dfrac{\left\{(n+2)(n+1)\right\}^{\frac{1}{2}}-(n+1)}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}\right\}$
JEFFREY'S		$\dfrac{n}{\left(\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}\right)^2}$	$\dfrac{1}{n-1}$	$2\left\{\dfrac{\left\{n(n+1)\right\}^{\frac{1}{2}}-n}{\sum_{i=1}^{n}\left(e^{\left(\frac{\theta}{x_i}\right)^{\lambda}}-1\right)^{-\beta}}\right\}$

8. Simulation and Comparison

We used a package in R software to generate random sample of size n = (25, 45, 85, 120) from *WFrD* by using $\alpha=1.0$, $\beta=0.5$, $\theta=1.0$ and $\lambda=1.5$; $\alpha=1.0$, $\beta=2.5$, $\theta=0.5$ and $\lambda=0.5$ and $\alpha=1.0$, $\beta=1.0$, $\theta=2.5$ and $\lambda=0.5$. The following tables present the results of our simulation study by listing the estimates of the shape parameter under the appropriate estimation methods such as the Maximum Likelihood Estimation (*MLE*), Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*) and Precautionary Loss Function (*PLF*) under both Uniform and Jeffrey prior.

Table 7.1: Estimators/Estimates, their Biases and Mean Squared Errors based on the replications and sample sizes where $\alpha=1.0$, $\beta=0.5$, $\theta=1.0$ and $\lambda=1.5$.

Sample sizes	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**		
			SELF	**QLF**	**PLF**	**SELF**	**QLF**	**PLF**
20	**Estimate**	4.1239	4.3301	3.9177	4.4320	4.1239	3.7115	4.2257
	BIAS	5.3358	5.6030	5.0685	5.7351	5.3358	4.8012	5.4678
	MSE	4.3303	4.775	3.9076	5.0023	4.3303	3.5066	4.5471
	Risk		8928.4	0.05	20.3797	8503.2	0.0526	20.3680
45	**Estimate**	2.6611	2.7203	2.6020	2.7497	2.6611	2.5429	2.6905
	BIAS	1.9517	1.9951	1.9083	2.0166	1.9517	1.8649	1.9732
	MSE	5.2313	5.4665	5.0012	5.5853	5.2313	4.7765	5.3476
	Risk		160867.3	0.0222	58.8185	157370.2	0.0227	58.8115
85	**Estimate**	4.2704	4.3206	4.2202	4.3457	4.2704	4.1699	4.2955
	BIAS	5.2844	5.3465	5.2222	5.3775	5.2844	5.1599	5.3153
	MSE	3.6619	3.7486	3.5763	3.7922	3.6619	3.4916	3.7050
	Risk		217069.5	0.0118	50.0949	214545.4	0.0119	50.0932
120	**Estimate**	8.1260	8.1937	8.0583	8.2275	8.1260	7.9905	8.1598
	BIAS	9.0401	9.1155	8.9648	9.1531	9.0401	8.8894	9.0777
	MSE	1.0284	1.0456	1.0113	1.0543	1.0284	**0.**9944	1.0370

	Risk		NaN	NaN	Inf	NaN	NaN	Inf

From table 7.1, we can see that both *MLE* and *SELF* (under Jeffrey prior) have the same estimate just as found in the derivations as well as their bias and MSE irrespective of the variation in the samples indicating that the two methods have the same performance considering this shape parameter. The table clearly shows that using the QLF under both uniform and Jeffrey's prior produces the best results and hence the best approach for estimating the shape parameter of the *WFrD* irrespective of the different sample sizes.

Table 7.2: Estimates of the shape parameter, their Biases and Mean Squared Errors and the posterior risks based on the replications and sample sizes where $\alpha = 1.0$, $\beta = 2.5$, $\theta = 0.5$ and $\lambda = 0.5$.

Sample sizes	**Measures**	***MLE***	**Uniform Prior**			**Jeffrey's Prior**		
			SELF	***QLF***	***PLF***	***SELF***	***QLF***	***PLF***
20	**Estimate**	6.7477	7.0852	6.4103	7.2518	6.7477	6.0729	6.9143
	BIAS	8.6732	9.1068	8.2395	9.3211	8.6732	7.8058	8.8873
	MSE	5.1344	5.6607	4.6338	5.9302	5.1344	4.3588	5.3911
	Risk		2390384	0.05	333.46	2276556	0.0526	333.27
45	**Estimate**	5.7931	5.9219	5.6644	5.9859	5.7931	5.5357	5.8571
	BIAS	2.9610	3.0268	2.8952	3.0595	2.9610	2.8294	2.9937
	MSE	4.7391	4.9520	4.5308	5.0597	4.7391	4.1272	4.8444
	Risk		7623573589	0.0222	12804.4	7457843728	0.0227	12802.87
85	**Estimate**	1.6114	1.6303	1.5924	1.6398	1.6114	1.5735	1.6208
	BIAS	2.3176	2.3449	2.2903	2.3585	2.3176	2.2631	2.3312
	MSE	5.3708	5.4979	5.2451	5.5618	5.3708	5.1210	5.4339
	Risk		30907082847	0.0118	18902.65	30547698162	0.0119	18902.01
120	**Estimate**	6.9325	6.9902	6.8747	7.0190	6.9325	6.8169	6.9613
	BIAS	3.2719	3.2992	3.2447	3.3128	3.2719	3.2174	3.2855
	MSE	1.0704	1.0884	1.0527	1.0973	1.0704	1.0351	1.0794
	Risk		NaN	NaN	Inf	NaN	NaN	Inf

Table 7.2 also gives a similar pattern of the result found in table 7.1 with similar estimates, biases and MSE for the *MLE* and *SELF* (under Jeffrey's prior) with *QLF* (under Jeffrey's prior) having the best performance (under Jeffrey's prior) as well as the *QLF* under uniform prior. Again these performances are found to be consistent irrespective of the different sample sizes and the parameter values used.

Table 7.3: Estimates of the shape parameter, their Biases and Mean Squared Errors and the posterior risks based on the replications and sample sizes where $\lambda = 1.0$, $\beta = 1.0$, $\theta = 2.5$ and $\lambda = 0.5$.

Sample sizes	**Measures**	***MLE***	**Uniform Prior**			**Jeffrey's Prior**		
			SELF	***QLF***	***PLF***	***SELF***	***QLF***	***PLF***
20	**Estimate**	1.1478	1.2052	1.0904	1.2336	1.1478	1.0330	1.1762
	BIAS	1.1347	1.1914	1.0780	1.2195	1.1347	1.0212	1.1627
	MSE	1.2767	1.4076	1.1522	1.4746	1.2767	1.0341	1.3406

	Risk		6916977	0.05	567.24	6587597	0.0526	566.92
45	**Estimate**	2.1914	2.2400	2.1426	2.2643	2.1914	2.0940	2.2156
	BIAS	1.7460	1.7848	1.7072	1.8041	1.7460	1.6684	1.7653
	MSE	2.9566	3.0895	2.8267	3.1566	2.9566	2.6996	3.0223
	Risk		1.09083e+13	0.0222	484349	1.067117e+13	0.0227	484291.4
85	**Estimate**	1.4828	1.5002	1.4653	1.5089	1.4828	1.4479	1.4915
	BIAS	3.0022	3.0376	2.9669	3.0552	3.0022	2.9316	3.0198
	MSE	9.0134	9.2267	8.8026	9.3340	9.0134	8.5942	9.1194
	Risk		26169876366	0.0118	17393.8	25865575478	0.0119	17393.22
120	**Estimate**	1.3414	1.3526	1.3302	1.3581	1.3414	1.3190	13470
	BIAS	4.2384	4.2738	4.2031	4.2914	4.2384	4.1678	4.2560
	MSE	1.7964	1.8265	1.7666	1.8416	1.7964	1.7371	1.8114
	Risk		NaN	NaN	Inf	NaN	NaN	Inf

The above table (Table 7.3**)** also shows that uniform and Jeffrey's priors with QLF resulting in better estimates for the shape parameter however there are some variations in the pattern of the measures or values for bias and MSE which are as a result of the increase in the value of the one and only scale parameter, $\theta = 2.5$,and hence we say that increasing the value of the scale parameter, θ affects the nature of our performance measures (increasing MSE instead of decreasing) though not the entire performance of the estimators and so looking at all the results presented in the tables, we can conclude that Bayes estimates using Quadratic loss function under Jeffrey's and uniform priors are associated with minimum risks, biases and *MSEs* and are better when compared to those obtained from *MLE*, *PLF* and *SELF* under Jeffrey's and uniform priors irrespective of the parameter values and the allocated sample sizes of n=20, 45, 85 and 120.

9. Summary and Conclusions

In this paper, we obtain Bayesian estimators of the shape parameter of *WFrD*. The Posterior distributions of this parameter are derived by using Uniform and Jeffrey's priors. Bayes estimators and their risks have been obtained by using three different loss functions under the two prior distributions. The three loss functions taken up are Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*) and Precautionary Loss Function (*PLF*). The performance of these estimators is assessed on the basis of their relative posterior risks, Biases and Mean Square Errors. The performance of the different estimators has been evaluated under a detailed simulation study. The study proposed that in order to estimate this shape parameter of the *WFrD*, the use of Quadratic loss function under Jeffrey's prior and secondly uniform prior can be preferred to produce the best results irrespective of the values of the parameters and the different sample sizes.

REFERENCES

[1] Afify AZ, Hamedani GG, Ghosh I, & Mead ME. The transmuted Marshall- Olkin Fréchet distribution: properties and applications. Int. J. of Stat. Prob., 2015; 4, 132-184.

[2] Afify AZ, Nofal ZM, & Butt NS. Transmuted complementaryWeibull geometric distribution. Pak. J. of Stat. & Operat. Res., 2014; 10, 435-454.

[3] Afify MZ, Yousof HM, Cordeiro GM, Ortega EMM, & Nofal ZM. The Weibull Frechet Distribution and Its Applications. J. of Appl. Stat., 2016; 1-22.

[4] Ahmad K, Ahmad SP, & Ahmed A. Some important characterizing properties, information measures and estimations of weibull distribution, Int. J. of Mod. Math. Sci., 2014; vol. 12, no. 2, pp. 88–97

[5] Ahmad K, Ahmad SP, & Ahmed A. Bayesian analysis of generalized gamma distribution using R software, J. of Stat. Appl. & Prob., 2015; vol. 4, no. 2, pp. 323–335.

[6] Ahmad K, Ahmad SP, & Ahmed A. On parameter estimation of erlang distribution using bayesian method under different loss functions, in Proceedings of Int. Conf. on Adv. in Comp. Comm. and Elect. Eng., 2015; pp. 200–206, University of Kashmir.

[7] Ahmad K, Ahmad SP, & Ahmed A. Bayesian analysis of generalized gamma distribution using R software. J. of Stat. Appl. Prob., 2015; 4(2): 323–335.

[8] Ahmad K, Ahmad SP, & Ahmed A. Classical and Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. J. of Prob. Stati., 2016; Volume 2016, Article ID 7581918, 8 pages http://dx.doi.org/10.1155/2016/7581918

[9] Ahmad SP, & Ahmad K. Bayesian analysis of weibull distribution using R software, Aust. J. of B. Appl. Sci., 2013; vol. 7, no. 9, pp. 156–164.

[10] Ahmed AOM, Ibrahim NA, Arasan J. & Adam MB. Extension of Jeffreys' prior estimate for weibull censored data using Lindley's approximation, Aust. J. of B Appl. Sci., 2011; 5(12): 884–889.

[11] Al-Aboud FM. Bayesian estimations for the extreme value distribution using progressive censored data and asymmetric loss, Int. Math. Forum, 2009; 4(33): 1603–1622.

[12] Al-Athari FM. Parameter estimation for the double-pareto distribution, J. of Maths. Stat., 2011; 7(4): 289–294.

[13] Aliyu Y. & Yahaya A. Bayesian estimation of the shape parameter of generalized Rayleigh distribution under non-informative prior. Int. J. of Adv. Stat Prob., 2016; 4 (1):1-10

[14] Almutairi AO. & Heng CL. Bayesian Estimate for Shape Parameter from Generalized Power Function Distribution. Maths. Theo. Mod.,2012; II (12).

[15] Azam Z. & Ahmad SA. Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. Pak. J. of Stat. Operat. Res., 2014; 10(2): 217-228

[16] Azam Z, & Ahmad SA. Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. *Int. J. of Adv. Sci. Tech.,* 2014; *65*: 71-80.

[17] Barreto-Souza WM, Cordeiro GM, & Simas AB. Some results for beta Fréchet distribution. Comm. in Stat. Theo. Meths., 2011; 40: 798-811.

[18] Elbatal I, Asha G, & Raja V. Transmuted exponentiated Fréchet distribution: properties and applications. J. of Stat. Appl. Prob., 2014; 3: 379-394.

[19] Feroze N. Estimation of scale parameter of inverse gausian distribution under a Bayesian framework using different loss functions. Sci. J. of Rev., 2012; 1(3); 39-52.

[20] Gupta I. Estimation of Parameter And Reliability Function of Exponentiated Inverted Weibull Distribution using Classical and Bayesian Approach. Int. J. of Rec. Sci. Res., 2017; 8(7): 18117-18819.

[21] Gupta PK, & Singh AK. Classical and Bayesian estimation of Weibull distribution in presence of outliers. Cogent Mathematics, 2017; 4: 1300975.

[22] Harlow DG. Applications of the Fréchet distribution function. Int. J. of Mat. Prod. Tech., 2002; 17: 482-495.

[23] Ibrahim JG, Chen MH, & Sinha D. *Bayesian survival analysis*. New York, NY: 2001; Springer-Verlag.

[24] Ieren TG, & Oguntunde PE. A Comparison between Maximum Likelihood and Bayesian Estimation Methods for a Shape Parameter of the Weibull-Exponential Distribution. *Asian J. Prob. Stat., 2018; 1(1): 1-12.*

[25] Kotz S, & Nadarajah S. Extreme value distributions: theory and applications. Imperial College Press, 2000; London.

[26] Krishna E, Jose KK, Alice T, & Ristic MM. The Marshall-Olkin Fréchet distribution. Com. Stat. Theo. Meths., 2013; 42: 4091-4107.

[27] Mahmoud MR, & Mandouh RM. On the transmuted Fréchet distribution. J. of Appl. Sci. Res., 2013; 9: 5553-5561.

[28] Martz, HF, & Waller, RA. Bayesian reliability analysis. 1982; New York, NY: John Wiley.

[29] Mead ME, & Abd-Eltawab A. R. A note on Kumaraswamy Fréchet distribution. Aust. J. of B. & Appl. Sci., 2014; 8: 294-300.

[30] Mubarak M. Parameter estimation based on the Fréchet progressive type II censored data with binomial removals. J. of Qual. Stat. & Rel.,2011,.

[31] Nadarajah S, & Gupta AK. The beta Fréchet distribution. Far East J. of Theo. Stat., 2004; 14: 15-24

[32] Nadarajah S, & Kotz S. The exponentiated Fréchet distribution. Int. Stat. Elect. J., 2003; 1-7.

[33] Pandey BN, Dwividi N, & Pulastya B. Comparison between Bayesian and maximum likelihood estimation of the scale parameter in Weibull distribution with known shape under linex loss function, J. of Sci. Res., 2011; 55: 163–172.

[34] Preda, V., Eugenia, P. & Alina, C. (2010). Bayes Estimators of Modified-Weibull Distribution parameters using Lindley's approximation. WSEAS TRANSACTIONS on MATHEMATICS, 2010; 9 (7), 539-549.

[35] Silva RVD, de Andrade TA, Maciel D, Campos RP & Cordeiro GM. A new lifetime model: the gamma extended Fréchet distribution. J. of Stat. Theo. Appl., 2013; 12, 39-54.

[36] Singpurwalla ND. Reliability and risk: A Bayesian perspective. Chichester: 2006; John Wiley.

[37] Zaharim A, Najid SK, Razali AM, & Sopian K. Analysing Malaysian wind speed data using statistical distribution. *In Proceedings of the 4th IASME/WSEAS International conference on energy and environment*, 2009; Cambridge, UK.