Original Research Article

The Transmuted Odd Lindley-G Family of Distributions

ABSTRACT

We propose a new generator of univariate continuous distributions with two extra parameters called the transmuted odd-Lindley generator which extends the odd Lindely-G family introduced by Gomes-Silva et al.[1]. Some mathematical properties of the new generator such as, the ordinary and incomplete moments, generating function, stress strength model, Rényi entropy, probability weighted moments and order statistics are investigated. Certain characterizations of the proposed family are estimated. We discuss the maximum likelihood estimates and the observed information matrix for the model parameters. The potentiality of the new family is illustrated by means of five applications to real data sets.

Keywords: Characterizations, Maximum Likelihood, Odd Lindley-G Family, Order Statistic, Stress Strength Model, Transmuted-G Family.

1. INTRODUCTION

In recent years, statisticians have proposed new generated families of the univariate distributions. These new generators are obtained by adding one or more extra shape parameters to the baseline distribution to obtain more flexibility in fitting data in different areas such as medical sciences, economics, finance and environmental sciences. Some of the well-known generated families are the following: Marshall-Olkin-G family by Marshall and Olkin [2], exponentiated-G by Gupta et al. [3], beta-G by Eugene et al. [4], Kumaraswamy-G by Cordeiro and de Castro [5], McDonald-G by Alexander et al. [6], logistic-G by Torabi and Montazari [7], Lomax-G by Cordeiro et al. [8], Kumaraswamy Marshall-Olkin-G by Alizadeh et al. [9], odd-Burr generalized-G by Alizadeh et al. [10], beta weibull-G by Yousof et al. [11], generalized odd generalized exponential family by Alizadeh et al. [12], beta transmuted-H family by Afify et al. [13], Topp-Leone odd log-logistic family by Brito et al.[14] and Type I general exponential class of distributions by Hamedani et al. [15], among others.

Let $h(x;\xi)$ and $H(x;\xi)$ denote the probability density function (pdf) and cumulative distribution function (cdf) of a baseline model with parameter vector ξ. Shaw and Buckley [16] introduced the transmuted-G (T-G) family of distributions with cdf and pdf given by

$$F(x;\xi) = H(x;\xi)\left[1+\lambda-\lambda H(x;\xi)\right], \ x\in R, \tag{1}$$

and

$$f(x;\xi) = h(x;\xi)\left[1+\lambda-2\lambda H(x;\xi)\right], \ x\in R \tag{2}$$

respectively, where, $|\lambda|\le 1$, is a shape parameter and ξ is the vector of parameters for the baseline cdf $H(x;\xi)$. The T-G density is a mixture of the baseline density and the exponentiated-G (Exp-G) density with power parameter two. If $\lambda=0$, then the T-G density reduces to the baseline density. Gomes-Silva et al. [1] defined the odd Lindely-G (OL-G) family of distributions with cdf and pdf given by

$$H(x;\xi) = 1-\frac{\alpha+\bar{G}(x;\xi)}{(1+\alpha)\bar{G}(x;\xi)}\exp\left\{-\alpha\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right\}, \ \alpha>0, \ x\in R, \tag{3}$$

and

$$h(x;\xi)=\frac{\alpha^2 g(x;\xi)}{(1+\alpha)\bar{G}(x;\xi)^3}\exp\left\{-\alpha\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right\},\ x\in R, \tag{4}$$

respectively, where, $G(x;\xi)$ and $g(x;\xi)$ are given cdf and pdf depend on vector parameter ξ.

The goal of this study is to introduce a new class of continuous distributions called the transmuted Odd Lindley-G (TOL-G) family in view of the T-G and OL-G families and study some of its statistical properties. The cdf and pdf of the TOL-G family are given, respectively, by

$$F(x)=\left\{1-\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right\}\left\{1+\lambda\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right\},\ x\in R, \tag{5}$$

and

$$f(x)=\frac{\alpha^2 g(x)e^{-\alpha\frac{G(x)}{\bar{G}(x)}}}{(1+\alpha)\bar{G}(x)^3}\left\{1-\lambda+2\lambda\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right\},\ x\in R. \tag{6}$$

Henceforth, a random variable with density (6) is denoted by $X\sim \text{TOL-G}(\alpha,\lambda,\xi)$. If $\lambda=0$, then TOL-G class is reduced to the OL-G family of distributions. The hazard function $\tau(x)$ for the TOL-G family is given by

$$\tau(x)=\frac{\dfrac{\alpha^2 g(x)e^{-\alpha\frac{G(x)}{\bar{G}(x)}}}{(1+\alpha)\bar{G}(x)^3}\left\{1-\lambda+2\lambda\left[\dfrac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right\}}{1-\left\{1-\left[\dfrac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right\}\left\{1+\lambda\left[\dfrac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right\}},\ x\in R. \tag{7}$$

The rest of this paper is outlined as follows:. In Section 2, linear representation of TOL-G family is discussed. Three special sub-models corresponding to TOL-G family are presented in Section 3. In Section 4, some mathematical properties of the TOL-G family are investigated. Certain characterizations of the new family are presented. in Section 5. In Section 6, the maximum likelihood estimates are derived for the parameters of TOL-G family in complete and censored samples. A simulation study is conducted in Section 7. In Section 8, five applications for TOL-G are presented. Some concluding remarks are given in the last Section.

2. USEFUL EXPANSIONS

In this section, we introduce a useful representation for the TOL-G pdf and cdf.

The pdf given in (6) can be written as

$$f(x)=\frac{(1-\lambda)\alpha^2 g(x)e^{-\alpha\frac{G(x)}{\bar{G}(x)}}}{(1+\alpha)\bar{G}(x)^3}+\frac{2\lambda\alpha^3 g(x)e^{-2\alpha\frac{G(x)}{\bar{G}(x)}}}{(1+\alpha)^2\bar{G}(x)^4}+\frac{2\lambda\alpha^2 g(x)e^{-2\alpha\frac{G(x)}{\bar{G}(x)}}}{(1+\alpha)^2\bar{G}(x)^3}$$

Using generalized binomial and Taylor expansion in the above equation, we obtain

$$f(x)=\sum_{j=0}^{\infty}\frac{(-1)^j(1-\lambda)\alpha^{j+2}g(x)G(x)^j}{j!\,(1+\alpha)\bar{G}(x)^{j+3}}+\sum_{j=0}^{\infty}\frac{(-1)^j 2^{j+1}\lambda\alpha^{j+3}g(x)G(x)^j}{j!\,(1+\alpha)^2\bar{G}(x)^{j+4}}$$

$$+\sum_{j=0}^{\infty}\frac{(-1)^j 2^{j+1}\lambda\alpha^{j+2}g(x)G(x)^j}{j!\,(1+\alpha)^2\bar{G}(x)^{j+3}}$$

$$f(x)=\sum_{j,i=0}^{\infty}\frac{(-1)^{j+i}\binom{-j-3}{i}(1-\lambda)\alpha^{j+2}g(x)G(x)^{j+i}}{j!\,(1+\alpha)}+\sum_{j,i=0}^{\infty}\frac{(-1)^{j+i}\binom{-j-4}{i}2^{j+1}\lambda\alpha^{j+3}g(x)G(x)^{j+i}}{j!\,(1+\alpha)^2}$$

$$+\sum_{j,i=0}^{\infty}\frac{(-1)^{j+i}\binom{-j-3}{i}2^{j+1}\lambda\alpha^{j+2}g(x)G(x)^{j+i}}{j!\,(1+\alpha)^2}$$

or

$$f(x)=\sum_{j,i=0}^{\infty}\pi_{j,i}\,h_{j+i+1}(x), \tag{8}$$

where

$$\pi_{j,i}=\frac{(-1)^{j+i}\alpha^{j+2}}{j!(j+i+1)(1+\alpha)^2}\left\{\left[(1-\lambda)(1+\alpha)+2^{j+1}\lambda\right]\binom{-j-3}{i}+2^{j+1}\lambda\alpha\binom{-j-4}{i}\right\},$$

and $h_{j+i+1}(x)=(j+i+1)g(x)G(x)^{j+i}$ is the exponentiated-G distribution with power parameter $j+i+1$.

Integrating (8) with respect to x, we have

$$F(x)=\sum_{j,i=0}^{\infty}\pi_{j,i}\,H_{j+i+1}(x), \tag{9}$$

where, $H_{j+i+1}(x)=G(x)^{j+i+1}$.

3. THE SUB-MODELS OF TOL-G

In this section, we introduce three special sub-models of the TOL-G family.

3.1 The TOL-Kumaraswamy (TOLKw) Model

Suppose the cdf and pdf of the Kumaraswamy distribution are the following $G(x)=1-\left(1-x^b\right)^a,\ 0\le x\le 1,$ and $g(x)=abx^{b-1}(1-x^b)^{a-1},\ 0<x<1,\ a,b>0,$ respectively. Then, the cdf and pdf of TOLKw distribution are given, respectively, by

$$f(x)=\left(\frac{ab\alpha^2}{1+\alpha}\right)x^{b-1}\left(1-x^b\right)^{-(2a+1)}e^{-\alpha\left(\frac{1-\left(1-x^b\right)^a}{\left(1-x^b\right)^a}\right)}\left\{1-\lambda+2\lambda\left[\frac{\alpha+\left(1-x^b\right)^a}{(1+\alpha)\left(1-x^b\right)^a}\right]e^{-\alpha\left(\frac{1-\left(1-x^b\right)^a}{\left(1-x^b\right)^a}\right)}\right\},\ 0<x<1,$$

and

$$F(x)=(1+\lambda)\left\{1-\left[\frac{\alpha+\left(1-x^b\right)^a}{(1+\alpha)\left(1-x^b\right)^a}\right]e^{-\alpha\left(\frac{1-\left(1-x^b\right)^a}{\left(1-x^b\right)^a}\right)}\right\}-\lambda\left\{1-\left[\frac{\alpha+\left(1-x^b\right)^a}{(1+\alpha)\left(1-x^b\right)^a}\right]e^{-\alpha\left(\frac{1-\left(1-x^b\right)^a}{\left(1-x^b\right)^a}\right)}\right\}^2,\ 0\le x\le 1.$$

The plots of the density and hazard functions are displayed in Figure 1.

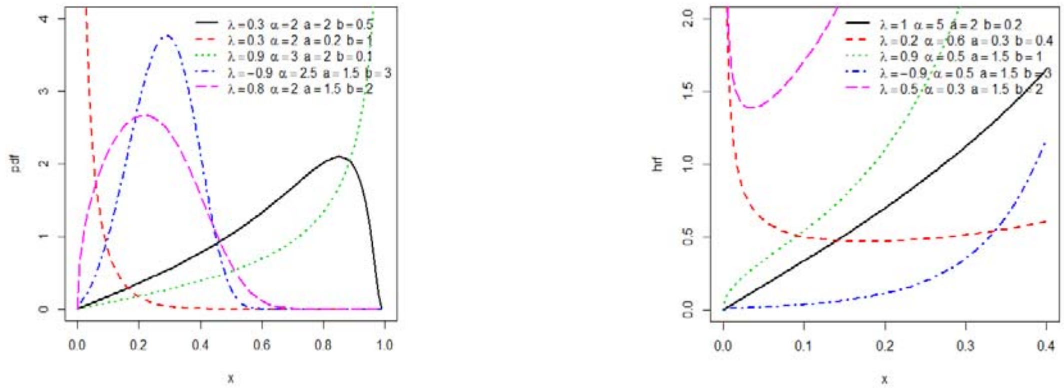

Fig. 1: Plots of the TOLKw pdf and hrf for selected parameter values.

3.2 The TOL-Lomax (TOLLx) Model

Consider the cdf and pdf of the Lomax distribution $G(x)=1-(1+\beta x)^{-\theta},\ x\geq 0,$ and $g(x)=\theta\beta(1+\beta x)^{-(\theta+1)}, x>0,\ \theta,\beta>0,$ respectively. Then, the cdf and pdf of TOLLx are given, respectively, by

$$f(x)=\left(\frac{\theta\beta\alpha^{2}}{1+\alpha}\right)(1+\beta x)^{2\theta-1}e^{-\alpha\left[\frac{1-(1+\beta x)^{-\theta}}{(1+\beta x)^{-\theta}}\right]}\left\{1-\lambda+2\lambda\left[\frac{\alpha+(1+\beta x)^{-\theta}}{(1+\alpha)(1+\beta x)^{-\theta}}\right]e^{-\alpha\left[\frac{1-(1+\beta x)^{-\theta}}{(1+\beta x)^{-\theta}}\right]}\right\},\ x>0,$$

and

$$F(x)=(1+\lambda)\left\{1-\left[\frac{\alpha+(1+\beta x)^{-\theta}}{(1+\alpha)(1+\beta x)^{-\theta}}\right]e^{-\alpha\left[\frac{1-(1+\beta x)^{-\theta}}{(1+\beta x)^{-\theta}}\right]}\right\}-\lambda\left\{1-\left[\frac{\alpha+(1+\beta x)^{-\theta}}{(1+\alpha)(1+\beta x)^{-\theta}}\right]e^{-\alpha\left[\frac{1-(1+\beta x)^{-\theta}}{(1+\beta x)^{-\theta}}\right]}\right\}^{2},\ x\geq 0.$$

The plots of the density and hazard functions are given in Figure 2.

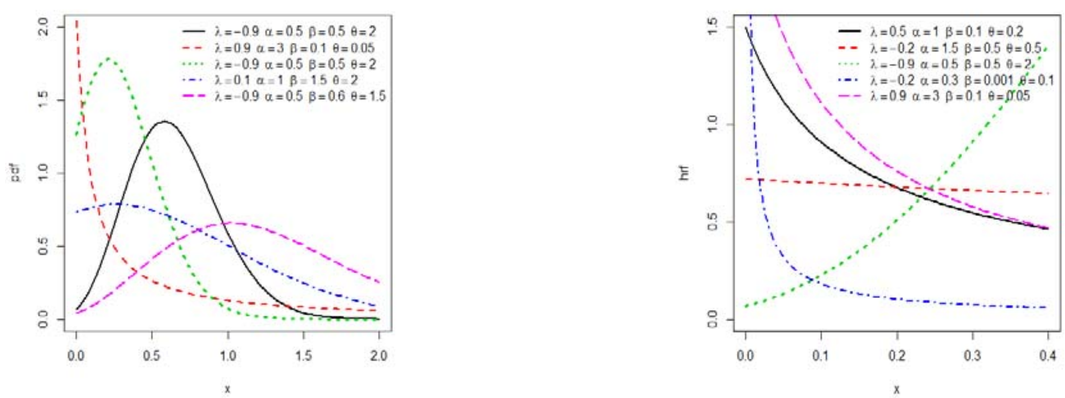

Fig. 2: Plots of the TOLLx pdf and hrf for selected parameter values.

3.3 The TOL-Frechet (TOLFr) Model

The cdf and pdf of the Frechet distribution are $G(x) = e^{-(a/x)^b},\ x \geq 0,$ and $g(x) = ba^b x^{-(b+1)} e^{-(a/x)^b},\ x > 0,\ a,\ b > 0,$ respectively. Then, the cdf and pdf of TOLFr are given, respectively, by

$$f(x) = \left(\frac{b\alpha^2 a^b x^{-(b+1)}}{(1+\alpha)\left(1-e^{-(a/x)^b}\right)^3} \right) e^{-\left\{ \left(\frac{a}{x}\right)^b + \alpha \left[\frac{e^{-(a/x)^b}}{1-e^{-(a/x)^b}} \right] \right\}} \left\{ 1 - \lambda + 2\lambda \left[\frac{1+\alpha+e^{-(a/x)^b}}{(1+\alpha)e^{-(a/x)^b}} \right] e^{-\alpha\left[\frac{e^{-(a/x)^b}}{1-e^{-(a/x)^b}} \right]} \right\},\ x > 0,$$

and

$$F(x) = (1+\lambda)\left\{ 1 - \left[\frac{1+\alpha-e^{-(a/x)^b}}{(1+\alpha)\left(1-e^{-(a/x)^b}\right)} \right] e^{-\alpha\left[\frac{e^{-(a/x)^b}}{1-e^{-(a/x)^b}} \right]} \right\} - \lambda \left\{ 1 - \left[\frac{1+\alpha-e^{-(a/x)^b}}{(1+\alpha)\left(1-e^{-(a/x)^b}\right)} \right] e^{-\alpha\left[\frac{e^{-(a/x)^b}}{1-e^{-(a/x)^b}} \right]} \right\}^2,\ x \geq 0.$$

The plots of the density and hazard functions are given in Figure 3.

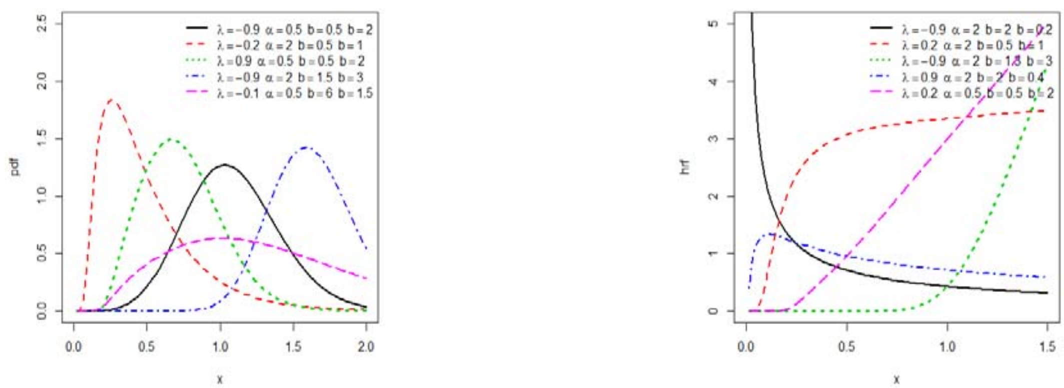

Fig. 3: Plots of the TOLFr pdf and hrf for selected parameter values.

4. MATHEMATICAL PROPERTIES

This section deals with some mathematical properties of the TOL-G family such as: quantile function, ordinary and incomplete moments, generating function, Rényi entropy, probability weighted moment, Lorenz and Bonferroni curves, stress strength model and order statistics.

4.1. Quantile Function

The quantile function of the TOL-G family, say $Q(u) = F^{-1}(u)$ for $u \in (0,1),\ \lambda \neq 0$ and $\alpha \neq 0$ is the solution of the non-linear equation

$$Q(u) = G^{-1}\left\{ 1 + \alpha \left[1 + W_{-1}\left\{ (1+\alpha)e^{-(1+\alpha)} \left[1 - \left(\frac{1+\lambda-\sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda} \right) \right] \right\}^{-1} \right] \right\}, \tag{10}$$

where $W_{-1}(.)$ denotes the negative branch of the Lambert W function.

4.2. Ordinary, Incomplete Moments and Generating Function

Let X be a random variable with TOL-G distribution, then the ordinary moment, say μ_r', is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} X^r f(x)\,dx$$

$$= \sum_{j,i=0}^{\infty} \pi_{j,i}^{*}\,\psi_{r,j+i}, \tag{11}$$

where, $\pi_{j,i}^{*} = (j+i+1)\pi_{j,i}$ and $\psi_{r,j+i} = \int_{-\infty}^{\infty} x^r g(x) G(x)^{j+i}\,dx$ is the probability weighted moment of the baseline distribution. The nth central moment of the TOL-G distribution, say μ_n, can be obtained from

$$\mu_n = \sum_{r=0}^{n} \binom{n}{r} \left(-\mu_1'\right)^{n-r} E(x^r)$$

$$= \sum_{r=0}^{n} \sum_{j,i=0}^{\infty} \binom{n}{r} \left(-\mu_1'\right)^{n-r} \pi_{j,i}^{*}\,\psi_{r,j+1}. \tag{12}$$

The cumulants of X, denoted by, κ_n, is

$$\kappa_n = \mu_n' - \sum_{r=0}^{n-1} \binom{n-1}{r-1} \kappa_r\, \mu_{n-r}' \tag{13}$$

where, $\kappa_1 = \mu_1'$, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + \mu_1'^3$, etc. The rth incomplete moment of X, denoted by $\varphi_s(t)$, is

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)\,dx$$

$$= \sum_{j,i=0}^{\infty} \pi_{j,i}^{*}\,\ell_{s,j+i}, \tag{14}$$

where, $\ell_{s,j+i} = \int_{-\infty}^{t} x^s g(x) G(x)^{j+i}\,dx.$

The moment generating function, say $M_x(t)$, of the TOL-G distribution is

$$M_x(t) = E(e^{tx}) = \sum_{r,j,i=0}^{\infty} \frac{t^r}{r!} \pi_{j,i}^{*}\,\psi_{r,j+i}. \tag{15}$$

Similarly, the probability generating function say, $M_{[x]}(t)$, of the TOL-G distribution is given by

$$M_{[x]}(t) = E(t^x) = \sum_{r,j,i=0}^{\infty} \frac{(\ln t)^r}{r!} \pi_{j,i}^{*}\,\psi_{r,j+i}. \tag{16}$$

4.3. Probability Weighted Moments

The PWM criterion can be constructed for estimating the model parameters of that distribution whose inverse form cannot be expressed in an explicit form. The PWM are expectation of certain functions of a random variable and they can be defined for any random variable whose raw moments exist. The $(r+s)$th PWM of X with TOL-G distribution, say $M_{r,s}$, is given by

$$M_{r,s} = E\left(X^r F(x)^s\right) = \int_{-\infty}^{\infty} X^r F(x)^s f(x)\,dx,$$

From (5) and (6), we can obtain

$$f(x)F(x)^s = \sum_{j,k,h,\ell,z=0}^{\infty} \eta_{j,k,h,\ell,z}\, g(x)G(x)^{h+z},$$

where,

$$\eta_{j,k,h,\ell,z} = \frac{(-1)^{j+k+\ell+z}\binom{s}{j}\binom{s+j}{k}\binom{k}{\ell}\lambda^j(1+\lambda)^{s-j}\alpha^{h+k-\ell+2}}{h!\,(1+\alpha)^{k+1}}\left[\binom{\ell-k-h-3}{z}+\frac{2\lambda(1-\lambda)(k+2)^h}{(1+\alpha)}\binom{\ell-k-h-4}{z}\right].$$

Therefore, we have

$$M_{r,s} = \sum_{j,k,h,\ell,z=0}^{\infty} \eta_{j,k,h,\ell,z}\,\psi_{r,h+z}. \tag{17}$$

4.4. Rényi Entropy

The concept of entropy has been applied in different areas such as statistics, queuing theory and reliability estimation. The Rényi entropy is defined as

$$I_R(X) = (1-\mu)^{-1}\log\int_{-\infty}^{\infty} f(x)^{\mu}\,dx,\ \mu>0,\ \mu\neq 0.$$

From (6), we obtain

$$f(x)^{\mu} = \sum_{j,i,\ell,h=0}^{\infty} \Omega_{j,i,\ell,h}\, g(x)^{\mu} G(x)^{i+h},$$

where, $\Omega_{j,i,\ell,h} = (-1)^{i+h}(i!)^{-1}(1+\alpha)^{-(\mu+j)}\binom{\mu}{j}\binom{j}{\ell}\binom{\ell-3\mu-j-i}{h}2^j\lambda^j\alpha^{2\mu+i+j-\ell}(j+\mu)^i(1-\lambda)^{\mu-j}$.

Consequently, the Rényi entropy for the TOL-G family is given by

$$I_R(X) = (1-\mu)^{-1}\log\left(\sum_{j,i,\ell,h=0}^{\infty}\Omega_{j,i,\ell,h}\int_{-\infty}^{\infty} g(x)^{\mu}G(x)^{i+h}\,dx\right). \tag{18}$$

4.5. Lorenz and Bonferroni Curves

The Lorenz and Bonferroni curves have been used in different areas such as reliability, economics, demography, insurance and medicine. The Lorenz $L_F(x)$ and Bonferroni $B(F(x))$ curves are defined respectively as follows:

$$L_F(x) = \frac{1}{E(x)}\int_0^x t\,f(t)\,dt, \qquad B(F(x)) = \frac{1}{F(x)E(x)}\int_0^x t\,f(t)\,dt = \frac{L_F(x)}{F(x)}.$$

Therefore, these quantities for the TOL-G distribution are given below

$$L_F(x) = \frac{\sum_{j,i=0}^{\infty}\pi^*_{j,i}\,\ell_{1,j+i}}{\sum_{j,i=0}^{\infty}\pi^*_{j,i}\,\psi_{1,j+i}}, \tag{19}$$

and

$$B(F(x)) = \frac{\sum_{j,i=0}^{\infty}\pi^*_{j,i}\,\ell_{1,j+i}}{F(x)\sum_{j,i=0}^{\infty}\pi^*_{j,i}\,\psi_{1,j+i}}. \tag{20}$$

4.6. Stress Strength Model

The stress strength model is a common criterion used in different applications in physics and engineering such as strength failure and system collapse. Let X_1 and X_2 be two independent random variables with $\mathrm{TOL}(\alpha_1,\lambda_1,\xi)$ and $\mathrm{TOL}(\alpha_2,\lambda_2,\xi)$ distributions. Then, the stress strength model is given by

$$R=\Pr(X_2<X_1)=\int_0^\infty f_1(\alpha_1,\lambda_1;\xi)F_2(\alpha_2,\lambda_2;\xi)\,dx.$$

Using (5) and (6), we have

$$f_1(\alpha_1,\lambda_1;\xi)F_2(\alpha_2,\lambda_2;\xi)=\sum_{k,h=0}^{\infty}\varepsilon_{k,h}\,g(x)G(x)^{k+h},$$

where,

$$\varepsilon_{k,h}=(-1)^{k+h}(k!)^{-1}(1+\alpha_1)^{-1}\alpha_1^2\left(\rho_1-\rho_2+\rho_3-\rho_4\right),$$

$$\rho_1=(1-\lambda_1)(1+\lambda_2)(1+\alpha_2)^{-1}\left\{\left[\alpha_1^k(1+\alpha_2)-\alpha_2^k\right]\binom{-k-3}{h}-\alpha_2^{k+1}\binom{-k-4}{h}\right\},$$

$$\rho_2=\sum_{w=0}^{2}\sum_{s=0}^{w}(-1)^w(k!)^{-1}(1+\alpha_2)^{-w}\binom{2}{w}\binom{w}{s}\binom{s-k-w-3}{h}(1-\lambda_1)\lambda_2(\alpha_1+\alpha_2 w)^k,$$

$$\rho_3=2\lambda_1(1+\lambda_2)\left[(1+\alpha_1)(1+\alpha_2)\right]^{-1}\left(\mathrm{E}_1+\mathrm{E}_2-\mathrm{E}_3\right),$$

$$\mathrm{E}_1=\left[(2\alpha_1)^k(1+\alpha_2)-(2\alpha_1+\alpha_2)^k\right]\binom{-k-3}{h},$$

$$\mathrm{E}_2=\left[\alpha_1^{k+1}(1+\alpha_2)-(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)^k\right]\binom{-k-4}{h},$$

$$\mathrm{E}_3=\left[\alpha_1\alpha_2(2\alpha_1+\alpha_2)^k\right]\binom{-k-5}{h},$$

and

$$\rho_4=2\sum_{w=0}^{2}\sum_{s=0}^{w}(-1)^w\left[(1+\alpha_1)(1+\alpha_2)^w\right]^{-1}\lambda_1\lambda_2\alpha_2^{w-s}(2\alpha_1+\alpha_2)^k\binom{2}{w}\binom{w}{s}\left[\alpha_1\binom{s-w-4}{h}+\binom{s-w-3}{h}\right].$$

Therefore, the stress strength model is given below

$$R=\sum_{k,h=0}^{\infty}\varepsilon_{k,h}^{*},\tag{21}$$

where, $\varepsilon_{k,h}^{*}=(k+h+1)^{-1}\varepsilon_{k,h}$.

4.7. Order Statistics

Order statistics play an important role in probability and statistics. Let $X_{1,n}\le X_{2:n},\ldots\le X_{n:n}$ be the ordered sample from a continuous population with pdf $f(x)$ and cdf $F(x)$. The pdf of $X_{k:n}$, the kth order statistic is given by

$$f_{X_{k:n}}(x)=\frac{1}{\beta(k,n-k+1)}\sum_{w=0}^{n-k}(-1)^w\binom{n-k}{w}f(x)F(x)^{k+w-1},$$

where, $\beta(.,.)$ is the beta function. Substitution from (5) and (6) in the above equation and after some algebra, we arrive at

$$f_{X_{k:n}}(x)=\sum_{w=0}^{n-k}\sum_{j,i,s,h,m=0}^{\infty}\mathrm{T}_{j,i,s,h,m}\,h_{s+m+1},\tag{22}$$

where,

$$\mathrm{T}_{j,i,s,h,m}=\frac{(-1)^{w+j+i+s+m}\lambda^{j}(1+\lambda)^{k+w-j-1}\alpha^{s+i-h+1}}{(s+m+1)\ s!\ \beta(k,n-k+1)\ (1+\alpha)^{i+1}}\binom{n-k}{w}\binom{k+w-1}{j}\binom{k+w+j-1}{i}$$
$$\times\left[\alpha(1-\lambda)(i+1)^{s}\binom{i}{h}\binom{h-s-i-3}{m}+\frac{2\lambda(i+2)^{s}}{1+\alpha}\binom{i+1}{h}\binom{h-s-i-4}{m}\right].$$

Furthermore, the rth moment of the kth order statistic for TOL-G family is given by

$$E\left(X_{k:n}^{r}\right)=\sum_{w=0}^{n-k}\sum_{j,i,s,h,m=0}^{\infty}\mathrm{T}^{*}_{j,i,s,h,m}\,\psi_{r,s+m},\tag{23}$$

where, $\mathrm{T}^{*}_{j,i,s,h,m}=(s+m+1)\mathrm{T}_{j,i,s,h,m}$. .

5. CHARACTERIZATIONS RESULTS

This section is devoted to the characterizations of the TOL-G distribution in different directions: *(i)* based on the ratio of two truncated moments; *(ii)* in terms of the hazard function; *(iii)* in terms of the reverse hazard function. Note that *(i)* can be employed also when the cdf does not have a closed form. We would also like to mention that due to the nature of TOL-G distribution, our characterizations may be the only possible ones. We present our characterizations (i)-(iii) in three subsections.

5.1. Characterizations based on two truncated moments

This subsection is devoted to the characterizations of TOL-G distribution based on the ratio of two truncated moments. Our first characterization employs a theorem due to Glanzel [17], see Theorem 1 of Appendix A. The result, however, holds also when the interval H is not closed, since the condition of the Theorem is on the interior of H.

Proposition 5.1. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let, $q_1=\bar{G}(x)\left\{1-\lambda+2\lambda\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right\}^{-1}$ and $q_2(x)=q_1(x)e^{-\alpha\frac{G(x)}{\bar{G}(x)}}$ for $x\in\mathbb{R}$. The random variable X has pdf (6) if and only if the function η defined in Theorem 1 is of the form

$$\eta(x)=\frac{1}{2}e^{-\alpha\frac{G(x)}{\bar{G}(x)}},\ x\in\mathbb{R}.$$

Proof. Suppose the random variable X has pdf (6), then

$$\left(1-F(x)\right)E\left[q_1(X)\middle|X\geq x\right]=\frac{\alpha}{1+\alpha}e^{-\alpha\frac{G(x)}{\bar{G}(x)}},\ x\in\mathbb{R},$$

and

$$\left(1-F(x)\right)E\left[q_2(X)\middle|X\geq x\right]=\frac{\alpha}{2(1+\alpha)}e^{-\alpha\frac{G(x)}{\bar{G}(x)}},\ x\in\mathbb{R}.$$

Further,

$$\eta(x)q_1(x)-q_2(x)=-\frac{q_1(x)}{2}e^{-\alpha\frac{G(x)}{\bar{G}(x)}}<0\ \text{for}\ x\in\mathbb{R}.$$

Conversely, if η is of the above form, then

$$s'(x)=\frac{\eta'(x)q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\alpha\,g(x)\,\bar{G}(x)^{-2},\;\; x\in\mathbb{R},$$

and consequently

$$s(x)=\alpha\,\bar{G}(x)^{-1},\;\; x\in\mathbb{R}.$$

Now, according to Theorem 1, X has density (6).

Corollary 5.1. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 5.1. The random variable X has pdf (6) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\alpha\,g(x)\,\bar{G}(x)^{-2},\;\; x\in\mathbb{R}.$$

Corollary 5.2. The general solution of the differential equation in Corollary 5.1 is

$$\eta(x)=e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\left[-\int \alpha\, g(x)\,\bar{G}(x)^{-2}e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 5.1 with $D=0$. Clearly, there are other triplets (q_1, q_2, η) which satisfy conditions of Theorem 1.

5.2 Characterization in terms of hazard function

The hazard function, h_F, of a twice differentiable distribution function, F, satisfies the following first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h_F'(x)}{h_F(x)}-h_F(x).$$

It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection we present non-trivial characterizations of TOL-G distribution for two cases: $\lambda=0$ and $\lambda=1$ in terms of the hazard function.

Proposition 5.2. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable. The random variable X has pdf (6) if and only if its hazard function $h_F(x)$ satisfies the following differential equation

$$h_F'(x)+\frac{\alpha g(x)}{\bar{G}(x)^2}h_F(x)=\frac{\alpha^2}{(1+\alpha)}e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\frac{d}{dx}\left\{\frac{g(x)\left\{1-\lambda+2\lambda\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right\}}{\bar{G}(x)^3\left\{1-\left(1-\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right)\left(1+\lambda\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right)\right\}}\right\},\;\; x\in\mathbb{R}.$$

Proof. If X has pdf (6), then clearly the above differential equation holds. If the differential equation holds, then

$$\frac{d}{dx}\left\{e^{\alpha\frac{G(x)}{\bar{G}(x)}}h_F(x)\right\}=\frac{\alpha^2}{(1+\alpha)}\frac{d}{dx}\left\{\frac{g(x)\left\{1-\lambda+2\lambda\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right\}}{\bar{G}(x)^3\left\{1-\left(1-\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right)\left(1+\lambda\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right)\right\}}\right\},$$

from which we arrive at the hazard function (7).

5.3 Characterizations in terms of the reverse hazard function

The reverse hazard function r_F of a twice differentiable distribution function, F, is defined as

$$r_F(x)=\frac{f(x)}{F(x)},\ x\in \text{ support of } F.$$

In this subsection we present a characterization of TOL-G distribution in terms of the reverse hazard function.

Proposition 5.3. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable. The random variable X has pdf (6) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x)+\frac{\alpha\, g(x)}{\bar{G}(x)^2}r_F(x)==\frac{\alpha^2}{(1+\alpha)}e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\frac{d}{dx}\left\{\frac{g(x)\left\{1-\lambda+2\lambda\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right\}}{\bar{G}(x)^3\left\{1-\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\left(1+\lambda\left[\frac{\alpha+\bar{G}(x)}{(1+\alpha)\bar{G}(x)}\right]e^{-\alpha\frac{G(x)}{\bar{G}(x)}}\right)\right\}}\right\},\ x\in\mathbb{R}.$$

Proof. Is similar to that of Proposition 5.2.

6. MAXIMUM LIKELIHOOD ESTIMATION

This section discusses the maximum likelihood estimates (MLEs) of the parameters of the TOL-G family for complete and censored samples.

6.1. Maximum Likelihood Estimation in Complete Samples

Let $x_1,x_2,...,x_n$ be the observed values of a random sample from TOL-G family with set of parameters $\Theta=(\alpha,\lambda,\xi)^T$, then the corresponding log-likelihood function is given by

$$\ell=2n\log(\alpha)-n\log(1+\alpha)+\sum_{i=1}^{n}\log\left(g(x_i,\xi)\right)-3\sum_{i=1}^{n}\log\left(\bar{G}(x_i,\xi)\right)-\alpha\sum_{i=1}^{n}\left(\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}\right)$$
$$+\sum_{i=1}^{n}\log\left\{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}. \quad (24)$$

The components of the score vector $\nabla\ell=\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial\xi}\right)$ are

$$\frac{\partial\ell}{\partial\alpha}=\frac{n(\alpha+2)}{\alpha(1+\alpha)}-\sum_{i=1}^{n}\left(\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}\right)-\frac{2\lambda\alpha}{(1+\alpha)^2}\sum_{i=1}^{n}\left\{\frac{G(x_i,\xi)\bar{G}(x_i,\xi)^{-2}\left(1+\alpha+\bar{G}(x_i,\xi)\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}}{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}}\right\}, \quad (25)$$

$$\frac{\partial\ell}{\partial\lambda}=\sum_{i=1}^{n}\left\{\frac{2\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}-1}{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}}\right\}, \tag{26}$$

and

$$\frac{\partial\ell}{\partial\xi}=\sum_{i=1}^{n}\left(\frac{g'(x_i,\xi)}{g(x_i,\xi)}\right)+3\sum_{i=1}^{n}\left(\frac{G'(x_i,\xi)}{\bar{G}(x_i,\xi)}\right)-\alpha\sum_{i=1}^{n}\left(\frac{G'(x_i,\xi)}{\bar{G}(x_i,\xi)^2}\right)$$
$$+\frac{2\lambda}{(1+\alpha)^2}\sum_{i=1}^{n}\left\{\frac{G'(x_i,\xi)\bar{G}(x_i,\xi)^{-3}\left[\alpha^2\left(1+\alpha+\bar{G}(x_i,\xi)\right)+(2\alpha-1)\bar{G}(x_i,\xi)\right]e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}}{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}}\right\}, \tag{27}$$

where, $g'(x_i,\xi)=\partial g(x_i,\xi)/\partial\xi$ and $G'(x_i,\xi)=\partial G(x_i,\xi)/\partial\xi$.

The MLEs, say $\hat{\Theta}=(\hat{\alpha},\hat{\lambda},\hat{\xi})$ of $\Theta=(\alpha,\lambda,\xi)^T$ can be obtained by equating the system of nonlinear equations (25) through (27) to zero and solving them simultaneously. Clearly, if analytical solutions are not possible we use certain software Package. For the purposes of interval estimation and testing hypotheses for the vector of parameters $\Theta=(\alpha,\lambda,\xi)^T$, we derive the $(q+3)\times(q+3)$ observed information matrix $J(\Theta)=\{J_{wv}\}$ (for $w,v=\alpha,\lambda,\xi$) to be

$$J(\Theta)=\begin{bmatrix} J_{\alpha\alpha} & J_{\alpha\lambda} & J_{\alpha\xi} \\ J_{\lambda\alpha} & J_{\lambda\lambda} & J_{\lambda\xi} \\ J_{\xi\alpha} & J_{\xi\lambda} & J_{\xi\xi} \end{bmatrix}$$

whose elements are given in Appendix B.

6.2. Maximum Likelihood Estimation in Censored Samples

If the lifetime of the first r failed items $x_1,x_2,...,x_r$ have been observed, then the likelihood function under type-II censoring is given by

$$L(x_i,\xi)=A\left(\prod_{i=1}^{r}f(x_i,\xi)\right)\times\{1-F(x_0,\xi)\}^{n-r}, \tag{28}$$

where, $x=(x_1,x_2,...,x_r)^T$, $\Theta=(\alpha,\lambda,\xi)^T$ and A is a constant. Using (5) and (6) in (28), the log-likelihood function for the TOL-G family in censored samples is given by

$$\ell=2r\log(\alpha)-r\log(1+\alpha)+\sum_{i=1}^{r}\log\left(g(x_i,\xi)\right)-3\sum_{i=1}^{r}\log\left(\bar{G}(x_i,\xi)\right)-\alpha\sum_{i=1}^{r}\left(\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}\right)$$
$$+\sum_{i=1}^{r}\log\left\{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}$$
$$+(n-r)\log\left\{1-\left[1-\left(\frac{\alpha+\bar{G}(x_{(0)},\xi)}{(1+\alpha)\bar{G}(x_{(0)},\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right]\left\{1+\lambda\left(\frac{\alpha+\bar{G}(x_{(0)},\xi)}{(1+\alpha)\bar{G}(x_{(0)},\xi)}\right)e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}\right\}\right\}. \tag{29}$$

The components of the score vector $\nabla\ell=\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial\xi}\right)$ are

$$\frac{\partial \ell}{\partial \alpha}=\frac{r(\alpha+2)}{\alpha(1+\alpha)}-\sum_{i=1}^{r}\left(\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}\right)-\frac{2\lambda\alpha}{(1+\alpha)^2}\sum_{i=1}^{r}\left\{\frac{G(x_i,\xi)\bar{G}(x_i,\xi)^{-2}\left(1+\alpha+\bar{G}(x_i,\xi)\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}}{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}}\right\}$$

$$-\frac{\alpha(n-r)(1+\lambda)}{(1+\alpha)^2}\left\{\frac{G(x_{(0)},\xi)\bar{G}(x_{(0)},\xi)^{-2}\left(1+\alpha+\bar{G}(x_{(0)},\xi)\right)e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}}{1-\left\{1-\left(\frac{\alpha+\bar{G}(x_{(0)},\xi)}{(1+\alpha)\bar{G}(x_{(0)},\xi)}\right)e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}\right\}\left\{1+\lambda\left(\frac{\alpha+\bar{G}(x_{(0)},\xi)}{(1+\alpha)\bar{G}(x_{(0)},\xi)}\right)e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}\right\}}\right\}, \qquad (30)$$

$$\frac{\partial \ell}{\partial \lambda}=\sum_{i=1}^{r}\left\{\frac{2\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}-1}{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}}\right\}$$

$$-(n-r)\left\{\frac{\left(\frac{\alpha+\bar{G}(x_{(0)},\xi)}{(1+\alpha)\bar{G}(x_{(0)},\xi)}\right)e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}\left\{1-\left(\frac{\alpha+\bar{G}(x_{(0)},\xi)}{(1+\alpha)\bar{G}(x_{(0)},\xi)}\right)e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}\right\}}{1-\left\{1-\left(\frac{\alpha+\bar{G}(x_{(0)},\xi)}{(1+\alpha)\bar{G}(x_{(0)},\xi)}\right)e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}\right\}\left\{1+\lambda\left(\frac{\alpha+\bar{G}(x_{(0)},\xi)}{(1+\alpha)\bar{G}(x_{(0)},\xi)}\right)e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}\right\}}\right\}, \qquad (31)$$

and

$$\frac{\partial \ell}{\partial \xi}=\sum_{i=1}^{r}\left(\frac{g'(x_i,\xi)}{g(x_i,\xi)}\right)+3\sum_{i=1}^{r}\left(\frac{G'(x_i,\xi)}{\bar{G}(x_i,\xi)}\right)-\alpha\sum_{i=1}^{n}\left(\frac{G'(x_i,\xi)}{\bar{G}(x_i,\xi)^2}\right)$$

$$+\frac{2\lambda}{(1+\alpha)^2}\sum_{i=1}^{n}\left\{\frac{G'(x_i,\xi)\bar{G}(x_i,\xi)^{-3}\left[\alpha^2\left(1+\alpha+\bar{G}(x_i,\xi)\right)+(2\alpha-1)\bar{G}(x_i,\xi)\right]e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}}{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}}\right\}$$

$$+\frac{(n-r)(1+\lambda)}{(1+\alpha)^2}\sum_{i=1}^{n}\left\{\frac{G'(x_{(0)},\xi)\bar{G}(x_{(0)},\xi)^{-3}\left[\alpha^2\left(1+\alpha+\bar{G}(x_{(0)},\xi)\right)+(2\alpha-1)\bar{G}(x_{(0)},\xi)\right]e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}}{1-\left\{1-\left(\frac{\alpha+\bar{G}(x_{(0)},\xi)}{(1+\alpha)\bar{G}(x_{(0)},\xi)}\right)e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}\right\}\left\{1+\lambda\left(\frac{\alpha+\bar{G}(x_{(0)},\xi)}{(1+\alpha)\bar{G}(x_{(0)},\xi)}\right)e^{-\alpha\frac{G(x_{(0)},\xi)}{\bar{G}(x_{(0)},\xi)}}\right\}}\right\}. \qquad (32)$$

The MLEs, say $\hat{\Theta}=(\hat{\alpha},\hat{\lambda},\hat{\xi})$ of $\Theta=(\alpha,\lambda,\xi)^T$ in censored samples can be obtained by setting the system of nonlinear equations (30) through (32) to zero and solving them simultaneously.

7. SIMULATION STUDY

In this section we evaluate the performance of the MLEs of the model parameter for the TOLLx distribution using Monte Carlo simulation varying the sample size and for selected parameter values. The simulation is repeated 1000 times each with sample size $n=20,50,150,300$ and 500. The parametric values are; first group: $\lambda = 0.70,\quad \alpha = 2.50,\ \beta = 1.20,\ \theta = 2.00$ and for second group $\lambda = 1.00,\quad \alpha = 5.00,\ \beta = 1.10,\ \theta = 2.00$. The MLEs are obtained by maximizing the log-likelihood function in (24) using optim routine in R software.

Tables (1) and (2) provide the maximum likelihood estimates (MLEs), average bias (Bias), mean square errors (MSE), coverage probability (CP) for the parameters $\lambda,\ \alpha,\ \beta,$ and θ under different sample sizes. From Tables (1) and (2), we observe that Biases and MSEs decrease as sample size increases, MLEs tends close to the original values. The CP of the confidence intervals are quite close to the nominal level of 95 % so the MLEs and their asymptotic results can be used for estimating and constructing confidence intervals.

Table (1):MLEs, Bias, MSE and CP for first group.

n	parameters	MLEs	Bias	MSE	CP
20	λ	0.4512	0.0498	0.2400	0.9100
	α	3.8171	1.7070	6.1006	0.7388
	β	1.6254	0.1151	0.2431	0.9512
	θ	2.9229	0.4229	0.8144	0.9801
50	λ	0.4900	0.0457	0.2340	0.9207
	α	3.1816	1.1136	5.6348	0.8950
	β	1.4112	0.1112	0.1993	0.9808
	θ	2.9253	0.3253	0.7217	0.9990
150	λ	0.5962	0.0312	0.2102	0.9477
	α	3.1261	0.8261	2.1361	0.8993
	β	1.3830	0.1070	0.1489	0.9604
	θ	2.3619	0.2381	0.5219	0.9447
300	λ	0.6397	0.0237	0.1867	0.9705
	α	2.9230	0.5550	1.0371	0.9210
	β	1.3924	0.0186	0.0310	0.9509
	θ	2.1241	0.1245	0.4598	0.8737
500	λ	0.7059	0.0114	0.1001	0.9501
	α	2.4888	0.1078	0.5571	0.9409
	β	1.2610	0.0105	0.0181	0.9511
	θ	1.9997	0.1104	0.3403	0.9409

Table (2):MLEs, Bias, MSE and CP for second group.

n	parameters	MLEs	Bias	MSE	CP
20	λ	0.3410	0.1990	0.5500	0.9925
	α	8.1291	2.1291	5.4283	0.8810
	β	1.6611	0.9610	2.1992	0.9197
	θ	4.2878	0.3278	1.9908	0.6500
50	λ	0.4234	0.1266	0.4557	0.8995
	α	7.9731	2.0502	4.0956	0.8798
	β	1.4015	0.8075	1.1288	0.9508
	θ	3.0301	0.2009	1.8106	0.7054
150	λ	0.5191	0.0105	0.3061	0.9765
	α	6.0902	1.0112	3.0413	0.9011
	β	1.3929	0.5058	0.8697	0.9318
	θ	2.9769	0.1231	1.7890	0.8491
300	λ	0.7803	0.1191	2.2083	0.9891
	α	5.7405	0.7425	3.4347	0.9113
	β	1.1831	0.3039	0.5474	0.9204
	θ	1.9930	0.3161	0.5604	0.8903
500	λ	0.9994	0.0023	1.1021	0.9501
	α	5.1101	0.3607	0.8446	0.9502
	β	1.1142	-0.0090	0.2366	0.9493
	θ	2.0191	0.0158	0.2033	0.9530

8. APPLICATIONS

In this section, we introduce five application to real data to show the applicability of the TOL-G family in complete and censored samples. We focus on the TOLLx distribution introduced in Subsection 3.2.

8.1. Complete Data Sets

In this subsection, we provide four application for TOLLx distribution in complete (uncensored) data sets. The first data set from Ratan [18] and it contain 50 observations on burr (in the unit of millimeter), the diameter is 12 mm and the sheet thickness is 3.15 mm. The data are given as follows: *0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32,0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24,0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.*

The second data set are the quarterly earnings per Johnson and Johnson Share (1960 to 1980) Source R package. The data are: *0.71, 0.63, 0.85, 0.44, 0.61, 0.69, 0.92, 0.55, 0.72, 0.77, 0.92, 0.60, 0.83, 0.80, 1.00, 0.77, 0.92,1.00,1.24, 1.00, 1.16, 1.30, 1.45, 1.25, 1.26, 1.38, 1.86, 1.56, 1.53, 1.59, 1.83, 1.86, 1.53, 2.07, 2.34, 2.25, 2.16, 2.43, 2.70, 2.25, 2.79, 3.42, 3.69, 3.60, 3.60, 4.32, 4.32, 4.05, 4.86, 5.04, 5.04, 4.41, 5.58, 5.85, 6.5, 5.31, 6.03, 6.39, 6.93, 5.85, 6.93, 7.74, 7.83, 6.12, 7.74, 8.91, 8.28, 6.84, 9.54, 10.26, 9.54, 8.73, 11.88, 12.06, 12.15, 8.91, 14.04, 12.96, 14.85.*

The third data corresponding to intervals in days between 109 successive coal-mining disasters in Great Britain, for the period (1875-1951) published by Maguire et al. [19]. The sorted data are given as follows: *1, 4, 4, 7, 11, 13, 15, 15, 17, 18, 19, 19, 20, 20, 22, 23, 28, 29, 31, 32, 36, 37, 47, 48, 49, 50, 54, 54, 55, 59, 59, 61, 61, 66, 72, 72, 75, 78, 78, 81, 93, 96, 99, 108, 113, 114, 120, 120, 120, 123, 124, 129, 131, 137, 145, 151, 156, 171, 176, 182, 188, 189, 195, 203, 208, 215, 217, 217, 217, 224, 228, 233, 255, 271, 275, 275, 275, 286, 291, 312, 312, 312, 315, 326, 326, 329, 330, 336, 338, 345, 348, 354, 361, 364, 369, 378, 390, 457, 467, 498, 517, 566, 644, 745, 871, 1312, 1357, 1613, 1630.*

The fourth data set consists of 50 observations, hole diameter and sheet thickness are 9 mm and 2 mm respectively from Ratan [18]. Hole diameter readings are taken on jobs with respect to one hole, selected and fixed as per a predetermined orientation. The data are: *0.06, 0.12,0.14, 0.04,0.14,0.16, 0.08, 0.26, 0.32, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.22, 0.16,0.12, 0.24, 0.06 ,0.02, 0.18, 0.22, 0.14, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.22, 0.14, 0.06, 0.04, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.04, 0.14, 0.26, 0.18,0.16.*

The MLEs are computed using Quasi-Newton Code for Bound Constrained Optimization (L-BFGS-B) and the log-likelihood function evaluated. The goodness-of-fit measures, Anderson-Darling (A^*), Cramér–von Mises (W^*) are computed. The lower the values of these criteria, the better the fit. The value for the Kolmogorov Smirnov (KS) statistic and its p-value are also provided.
We compare the TOLLx distribution with those of the Lomax (Lx), beta Lomax (BLx) (Lemonte and Cordeiro [20]), exponentiated Lomax (ELx) (El-Bassiouny et al.[21]) Kumaraswamy Lomax (KwLx) (Lemonte and Cordeiro, [20], Weibull Lomax (WLx) (Tahir et al., [22]) The MLEs and some statistics of the models for all data sets are presented in Tables (3-10).

Table 3: The MLEs for the first data set.

Distribution	Estimates with standard error in parenthesis					
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
TOLLx	0.0847 (0.9966)	0.4622 (0.3952)	2.2790 (8.6509)	19.1411 (66.8045)	---	---
WLx	---	35.0886 (138.6751)	---	9.4534 (37.7751)	1.6103 (0.2010	0.0462 (1.0384)
KwLx	---	183.1897 (79.1424)	---	793.5680 (231.0023)	2.1456 (0.2676)	925.9396 (315.5829)
BLx	---	163.6522 (151.2978)	---	24.2602 (127.8382)	3.0318 (0.5768)	103.7089 (33.2213)
ELx	---	1207.8458 (823.9346)	3.1707 (0.7087)	106.2892 (76.5806)	---	---
Lx	---	573.9920 (237.4615)	---	93.6344 (38.2760)	---	---

Table 4: Some statistics for the models fitted to the first data set.

Distribution	Statistics				
	A*	W*	L	KS	P-value
TOLLx	0.4205	0.0716	-57.0434	0.0769	0.7493
WLx	0.5101	0.0798	-56.0772	0.0876	0.7119
KwLx	0.6640	0.1085	-55.7727	0.1127	0.5489
BLx	1.0915	0.1819	-53.3633	0.1541	0.1860
ELx	1.2651	0.2124	-52.2737	0.1652	0.1305
Lx	1.1005	0.1835	-40.6059	0.2806	0.0008

Table 5: The MLEs for the second data set.

Distribution	Estimates with standard error in parenthesis					
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
TOLLx	0.2516 (0.3996)	3.9758 (3.1002)	328.1790 (137.3812)	15.0087 (62.5546)	---	---
WLx	---	0.0967 (0.0026)	---	0.0037 (0.0006)	5.7014 (0.5065)	2.4333 (3.7434)
KwLx	---	0.9009 (0.0484)	---	171.7518 (104.1595)	1.1316 (0.1046)	63.5329 (22.4535)
BLx	---	301.7541 (156.1994)	---	5.3708 (16.5882)	1.2320 (0.1869)	14.5150 (52.2568)
ELx	---	53.6032 (20.0232)	218.4295 (84.2147)	1.257178 (0.2034)	---	---
Lx	---	180.6657 (137.1960)	---	862.8725 (478.399)	---	---

Table 6: Some statistics for the models fitted to the second data set.

Distribution	Statistics				
	A*	W*	L	KS	P-value
TOLLx	1.4195	0.2156	216.8447	0.0859	0.4299
WLx	1.4843	0.2291	213.7185	0.1183	0.1907
KwLx	1.5016	0.2361	215.0746	0.1129	0.2346
BLx	1.4785	0.2331	214.8074	0.1168	0.2018
ELx	1.4791	0.2337	214.8122	0.1164	0.2048
Lx	1.4746	0.2324	215.7926	0.0968	0.4102

Table 7: The MLEs for the third data set.

Distribution	Estimates with standard error in parenthesis					
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
TOLLx	-0.4146 (0.9354)	0.6940 (5.9101)	16.5184 (42.1278)	0.5282 (0.1923)	---	---
WLx	---	0.0778 (0.0171)	---	0.0277 (0.0538)	5.5496 (1.3507)	0.9743 (1.3423)
KwLx	---	0.0484 (0.1183)	---	311.6364 (203.2984)	1.1596 (0.1521)	77.1793 (214.4590)
BLx	---	301.0638 (144.6873)	---	0.1107 (0.7141)	1.2256 (0.1933)	23.0306 (49.4264)
ELx	---	2.4852 (0.7436)	326.1827 (150.7728)	1.2060 (0.1911)	---	---
Lx	---	4.7407 (2.4544)	---	874.6789 (538.2643)	---	---

Table 8: Some statistics for the models fitted to the third data set.

Distribution	Statistics				
	A*	W*	L	KS	P-value
TOLLx	0.4548	0.0660	698.8196	0.0669	0.7142
WLx	0.5223	0.0762	700.8432	0.0708	0.6449
KwLx	0.5970	0.1020	701.1456	0.0661	0.7284
BLx	0.6960	0.1219	701.6078	0.0749	0.5741
ELx	0.6865	0.1201	701.7234	0.0746	0.5787
Lx	0.4707	0.0703	700.7164	0.0640	0.7628

Table 9: The MLEs for the fourth data set.

Distribution	Estimates with standard error in parenthesis					
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
TOLLx	-0.1718 (0.7065)	0.6543 (0.4913)	4.2929 (1.4339)	32.6701 (13.4579)	---	---
WLx	---	35.0033 (116.6274)	---	8.7573 (29.4706)	1.5301 (0.1922)	2.3223 (0.9388)
KwLx	---	135.4032 (383.1977)	---	293.9294 (739.0565)	2.0434 (0.26663)	196.8303 (437.5716)
BLx	---	294.8861 (32.3921)	---	69.2109 (44.5040)	2.6689 (0.4853)	74.0116 (4.2323)
ELx	---	358.5664 (120.2875)	31.4201 (97.3856)	2.7340 (0.5911)	---	---
Lx	---	741.8985 (306.8088)	---	112.7277 (233.3167)	---	---

Table 10: Some statistics for the models fitted to the forth data set.

Distribution	Statistics				
	A*	W*	L	KS	P-value
TOLLx	0.6694	0.1067	-59.3151	0.1216	0.4504
WLx	0.8579	0.1468	-58.9369	0.1494	0.2145
KwLx	1.2648	0.2239	-57.0273	0.1742	0.0963
BLx	1.7772	0.3223	-54.6241	0.2097	0.0246
ELx	1.9583	0.3568	-53.6001	0.2176	0.0176
Lx	1.7879	0.3244	-52.4523	0.2859	0.0006

The values in Tables (3-10) show that the TOLLx model has the smallest values for A*, W*, KS and largest P-values among all fitted models (for the four real data sets). So, the TOLLx model could be selected as the best model.
The estimated pdfs and cdfs plots are displayed in Figures (4), (5), (6) and (7). It is clear from Figures (4-7), that the new TOLLx distribution provides the best fits to the four data sets.

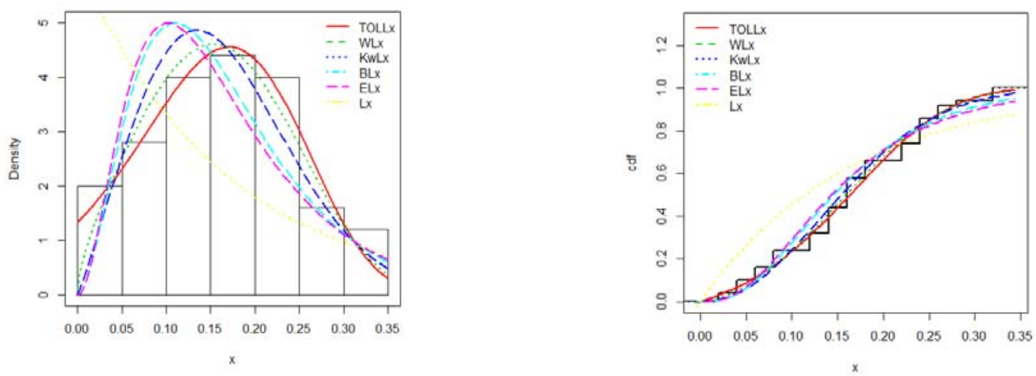

Fig. 4: Estimated pdfs and cdfs plots of the TOLLx distribution for data set 1.

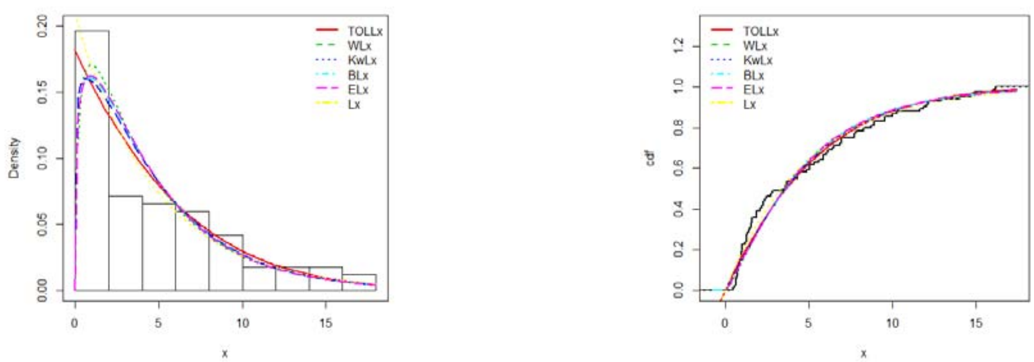

Fig. 5: Estimated pdfs and cdfs plots of the TOLLx distribution for data set 2.

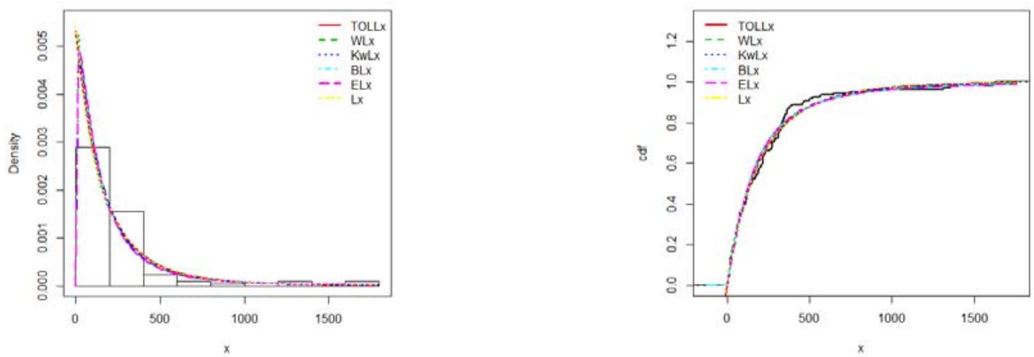

Fig. 6: Estimated pdfs and cdfs plots of the TOLLx distribution for data set 3

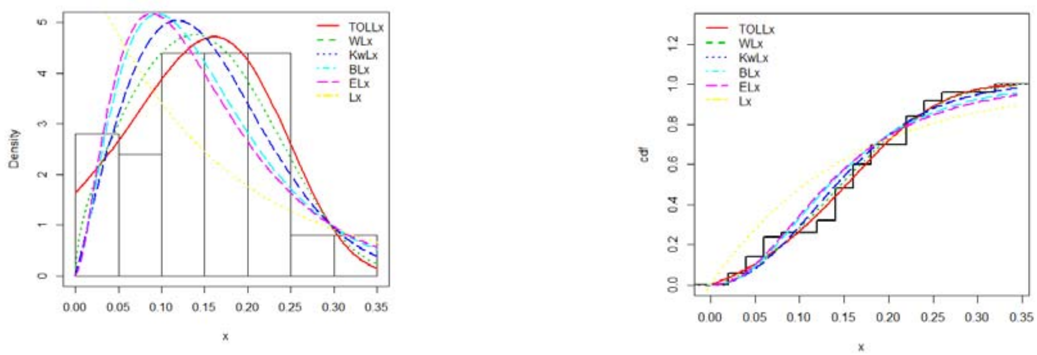

Fig. 7: Estimated pdfs and cdfs plots of the TOLLx distribution for data set 4.

8.2. Censored Data Set

In this subsection, we provide an application for TOLLx model under type-II censored data. The data consist of death times (in weeks)of patients with cancer of tongue with aneuploidy DNA profile (Lee and Wang, [23]).

The MLEs are computed using Quasi-Newton Code for Bound Constrained Optimization (L-BFGS-B) and the log-likelihood function evaluated. The statistics AIC and BIC are computed and compared the proposed and competitive models: The lower the values of these criteria, the better the fit.

We compare the TOLLx distribution with those of the Lomax (Lx), beta Lomax (BLx) (Lemonte and Cordeiro [20]), exponentiated Lomax (ELx) (El-Bassiouny et al.[21]) Kumaraswamy Lomax (KwLx) (Lemonte and Cordeiro [20]), Weibull Lomax (WLx) (Tahir et al. [22]) The MLEs and some statistics of the models for all data sets are presented in Tables (11) and (12).

Table 11: The MLEs for the fifth data set.

Distribution	Estimates with standard error in parenthesis					
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
TOLLx	0.4250 (1.1092)	0.0.8359 (0.5549)	22.7855 (15.3342)	0.4610 (0.1053)	---	---
WLx	---	0.0685 (0.0099)	---	0.4343 (0.1061)	2.0572 (0.6190)	8.1386 (0.8779)
KwLx	---	0.2868 (0.1768)	---	5.9296 (2.4534)	2.2666 (0.9474)	2.2959 (2.0349)
BLx	---	14.5901 (10.5600)	---	4.1448 (5.8967)	0.0552 (0.1141)	0.2018 (0.1819)
ELx	---	0.7443 (0.3257)	39.2425 (38.6966)	1.2541 (0.4195)	---	---
Lx	---	0.3471 (13.5801)	---	93.6344 (38.2760)	---	---

Table 12: Some statistics for the models fitted to the fifth data set.

Distribution	Statistics		
	L	AIC	BIC
TOLLx	-181.2062	370.4124	378.2174
WLx	-183.7187	375.4373	383.1423
KwLx	-183.8169	375.6337	383.4387
BLx	-183.9097	375.8194	383.6243
ELx	-182.5575	372.1150	376.9687
Lx	-185.7654	375.5309	379.4334

The values in Table 12 show that the TOLLx model has the lowest values for AIC and BIC. Then, the TOLLx distribution could be chosen as the best model within other competitive models. The estimated cdfs plots are displayed in Figure (8). It is clear from Figure 8, that the TOLLx distribution provides a better fit to the censored data as compared to other models.

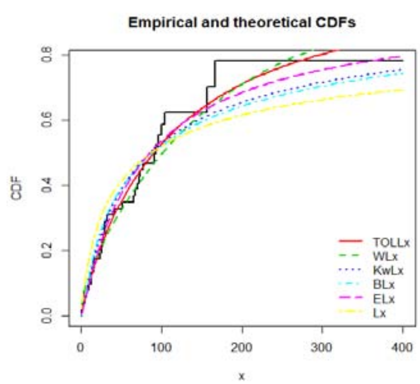

Fig. 8: Plots of estimated cdfs of the models compared in censored data set.

9. CONCLUSION

We propose a new class of continuous distributions, called the transmuted odd Lindley-G (TOL-G) family by using the OL-G family as a parent distribution in the T-G class of distributions. We study the mathematical properties of the new family such as ordinary and incomplete moments, generating function, Rényi of entropy, stress strength model, probability weighted moment and order statistics. Certain characterizations of the new family are also introduced. The method of maximum likelihood is used to estimate the model parameters in complete and censored samples. Five real data sets are used to illustrate that some sub-models corresponding to the TOL-G family can give better fit than similar models generated by well-known families.

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APPENDIX A

Theorem 1. Let (Ω, F, P) be a given probability space and let $H = [a,b]$ be an interval for some $d < b$ ($a = -\infty$, $b = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$E\left[q_2(X) \middle| X \geq x\right] = E\left[q_1(X) \middle| X \geq x\right]\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^{-1}(H), \eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)\, q_1(u) - q_2(u)}\right| \exp\left(-s(u)\right) du,$$

where the function s is a solution of the differential equation $s' = \dfrac{\eta' q_1}{\eta q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see Glanzel [24]), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution function $\{F_n\}$ such that the functions q_{1n}, q_{2n} and η_n $(n \in N)$ satisfy the conditions of Theorem 1 and let $q_{1n} \to q_1$, $q_{2n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F. Under the condition that $q_{1n}(X)$ and $q_{2n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η, where

$$\eta(x) = \frac{E\left[q_2(X) \middle| X \geq x\right]}{E\left[q_1(X) \middle| X \geq x\right]}$$

This stability theorem makes sure that the convergence of distribution function is reflected by corresponding convergence of the function q_1, q_2 and η, respectively. It guarantees, for instance, the convergence of characterization on the Wald distribution to that of the Levy-Smirrnov distribution if $\alpha \to \infty$.

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

APPENDIX B

The elements of the observed information matrix are given below

$$J_{\alpha\alpha} = \frac{-n(\alpha^2 + 2(2\alpha + 1))}{\alpha^2 (1+\alpha)^2}$$

$$-\frac{2\lambda}{(1+\alpha)^2}\sum_{i=1}^{n}G(x_i,\xi)^2\bar{G}(x_i,\xi)^{-2}e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\left\{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}^{-2}$$

$$\times\left\{\begin{array}{l}\left((1-\alpha)\bar{G}(x_i,\xi)+\alpha(1+\alpha)\right)\left[1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right]\\ +\alpha^2(1+\alpha)^{-2}\bar{G}(x_i,\xi)^{-1}\left(1+\alpha+\bar{G}(x_i,\xi)\right)^2e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\end{array}\right\}$$

$$J_{\alpha\lambda}=\frac{-2\alpha}{(1+\alpha)^2}\sum_{i=1}^{n}\left\{G(x_i,\xi)\bar{G}(x_i,\xi)^{-2}\left(1+\alpha+\bar{G}(x_i,\xi)\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\left\{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}^{-2}\right\},$$

$$J_{\alpha\xi}=-\sum_{i=1}^{n}\left[\frac{G'(x_i,\xi)}{\bar{G}(x_i,\xi)^2}\right]-\frac{2\lambda\alpha}{(1+\alpha)^2}\sum_{i=1}^{n}G'(x_i,\xi)\bar{G}(x_i,\xi)^{-4}e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\left\{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}^{-2}$$

$$\times\left\{\begin{array}{l}\left\{2\bar{G}(x_i,\xi)+\alpha\left(1-(2+\alpha)G(x_i,\xi)\right)\right\}\left\{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}\\ +2\lambda(1+\alpha)^{-2}G(x_i,\xi)\bar{G}(x_i,\xi)^{-1}\left(1+\alpha+\bar{G}(x_i,\xi)\right)\left\{\bar{G}(x_i,\xi)^2+\alpha\left(1+\alpha^2+\bar{G}(x_i,\xi)\right)\right\}e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\end{array}\right\},$$

$$J_{\lambda\lambda}=-\sum_{i=1}^{n}\left\{\left\{1-2\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}^2\left\{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}^{-2}\right\},$$

$$J_{\lambda\xi}=\left(\frac{-2}{1+\alpha}\right)\sum_{i=1}^{n}G'(x_i,\xi)\bar{G}(x_i,\xi)^{-3}e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\left\{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}^{-2}$$

$$\times\left\{\begin{array}{l}\left[\alpha\left(\alpha+\bar{G}(x_i,\xi)\right)+\bar{G}(x_i,\xi)^2\right]\left\{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}\\ +\left(\lambda(1+\alpha)^{-1}\left[\bar{G}(x_i,\xi)^2-\alpha(1+\alpha)\left(\alpha+\bar{G}(x_i,\xi)\right)\right]\right)\left\{2\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}-1\right\}\end{array}\right\},$$

$$J_{\xi\xi}=\sum_{i=1}^{n}\left\{\frac{g(x_i,\xi)g''(x_i,\xi)-g'(x_i,\xi)^2}{g(x_i,\xi)^2}\right\}+3\sum_{i=1}^{n}\left\{\frac{\bar{G}(x_i,\xi)\bar{G}''(x_i,\xi)+G'(x_i,\xi)^2}{\bar{G}(x_i,\xi)^2}\right\}-\alpha\sum_{i=1}^{n}\left\{\frac{\bar{G}(x_i,\xi)\bar{G}''(x_i,\xi)+2G'(x_i,\xi)^2}{\bar{G}(x_i,\phi)^3}\right\}$$

$$+\left(\frac{2\lambda}{(1+\alpha)^2}\right)\sum_{i=1}^{n}\bar{G}(x_i,\xi)^{-3}e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\left\{1-\lambda+2\lambda\left(\frac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\}^{-2}$$

$$\times\left\{\begin{array}{l}\left\{1-\lambda+2\lambda\left(\dfrac{\alpha+\bar{G}(x_i,\xi)}{(1+\alpha)\bar{G}(x_i,\xi)}\right)e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\right\} \\ \times\left\{\begin{array}{l}\bar{G}(x_i,\xi)^{-2}\left[\alpha^2\left(1+\alpha+\bar{G}(x_i,\xi)\right)+(2\alpha-1)\bar{G}(x_i,\xi)\right]\left[\bar{G}(x_i,\xi)\left(\bar{G}(x_i,\xi)G''(x_i,\xi)+3G'(x_i,\xi)^2\right)-\alpha\right] \\ -(1+\alpha)^2G'(x_i,\xi)^2\end{array}\right\} \\ +\left\{2\lambda(1+\alpha)^{-2}G'(x_i,\xi)\bar{G}(x_i,\xi)^{-3}e^{-\alpha\frac{G(x_i,\xi)}{\bar{G}(x_i,\xi)}}\left[\alpha^2(1+\alpha)+\bar{G}(x_i,\xi)\left(\alpha+\bar{G}(x_i,\xi)\right)\right]\left[\alpha^2\left(1+\alpha+\bar{G}(x_i,\xi)\right)+(2\alpha-1)\bar{G}(x_i,\xi)\right]\right\}\end{array}\right\}$$

where, $g''(x_i,\xi)=\partial^2 g(x_i,\xi)/\partial\xi^2$ and $G''(x_i,\xi)=\partial^2 G(x_i,\xi)/\partial\xi^2$.