Generalized Euclidean Least Square Approximation

A. S. Oke, S. M. Akintewe, A. G. Akinbande

July 3, 2018

Department of Mathematical Sciences, Faculty of Science, Adekunle Ajasin University, Akungba Akoko, Ondo State, Nigeria.
email: abayomi.oke@aaua.edu.ng, okeabayomisamuel@gmail.com. Telephone: +2348138224937

Abstract

A Generalised Euclidean Least Square Approximation (ELS) is derived in this paper. The Generalised Euclidean Least Square Approximation is derived by generalizing the interpolation of n arbitrary data set to approximate functions. Existence and uniqueness of the ELS schemes are shown by establishing the invertibility of the coefficient matrix using condensation method to reduce the matrix. The method is illustrated for some transcendental functions and the results are compared to the classical Maclaurin's series.

Keywords

2010 Mathematics Subject Classification: 34L99
Keywords: Euclidean Space; norm; interpolation; least square; approximation.

1 Introduction

In order to deal with some difficult functions, it is important to reduce such to a linear combination of some simpler functions. A way to reduce complicated functions is by approximation. It involves finding a combination of some elementary functions to a desired degree of accuracy. Interpolation is another way to achieve this approximation but it involves fitting of functions to a given set of data and finding the best function in a certain class that can be used to represent the data. Interpolation and approximation become very significant because many problems in engineering and science require fitting a curve for a given set of experimental data or reduction of some nonlinear problems for easy manipulation. Some common methods include the Least-Square, Min-Max, Taylor series and Interpolatory Approximation (like the Lagrange and Newton) and many more. Many approximation methods have been employed in mathematics but Maclaurin series stand out in that it produces an elegant mathematical formula that requires less computation except for finding the derivatives of functions. In a study by Adam [1], Euclidean spaces with their inner products was used to describe the methods of least squares adjustment as orthogonal projections on finite-dimensional subspaces and it was established that both the Euclidean and Hilbert space techniques in least squares adjustment are elegant and powerful geodetic methods. Xiao-Wen and Christopher [16] made an analysis of euclidean distances and least squares problems for a given set of vectors, he proved that given an $m \times n$ real matrix $A = [a_1, ..., a_n]$, with $D(A) \equiv \operatorname{diag}(\delta_i(A))$ $(i = 1, \cdots, n)$, $\delta_i(A)$ is the Euclidean distance from a_i to the space spanned by all other columns of A i.e.

$$\delta_i(A) = \min_{x_i} \left\| a_i - [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n]\, x_i \right\|_2 = \left\| A\overline{x}_i \right\|_2.$$

Application of interpolation with finite element are considered by Hill and Baskharone [9], Chongbin et al. [4]. Papageorgakopoulos et al. [14] made an advancement of the momentum interpolation method on non-staggered grids. Hon and Zongmin [10] developed a numerical method to quasi-interpolate the forcing term of differential equations by using radial basis functions. Least square approximations are of high significance in regression analysis for trimming outliers. Moez and Abdelkader [13] introduced a new methodology for identification of the parameters of the local

linear Takagi-Sugeno fuzzy models using weighted recursive least squares (WRLS). Chatzinakos and Zioutas [3] developed an algorithm for robust leverage to identify multiple outliers estimate based on Least Trimmed Euclidean Deviations (LTED). Youngmo and Park [17] developed the least square tracking algorithm which generalities a suitable way the standard LQR formulation on vector spaces to systems that evolve on the Euclidean group. The applications of least-squares include robotics, manipulating parts with complex shapes in unstructured dynamic environments, planning end-effect or motions around singularities, writing on a blackboard, and other motion planning problems involving kinematic constraints (Youngmo and Park [17]). Other works involving the use of interpolation in Euclidean space can be found in Karl [12], Froncioni et al. [7], Gabrielle, David and Simon [8], Zhiye [18], Kang and Park [11], Wei et al. [15], Dorst [6]. Desmet [5] made an investigation of the effects of interpolation error, using two error methods. Annaby and Hassan [2] derived the exact formulas for trigonometric sums at different nodes using Hermite interpolations whose nodes are basically zeros of Chebyshev polynomials of the first and second kinds.

In this paper, a generalized scheme for least square method called Euclidean Least Square Method (ELS) is derived by generalizing the interpolation of n arbitrary data set to approximate functions. Existence and uniqueness theorems are proved for the derived ELS schemes by establishing the invertibility of the coefficient matrix using condensation method to reduce the matrix. The method is illustrated for some transcendental functions and the results are compared to the classical Maclaurin's series. These schemes are useful for obtaining approximation to functions and also to fit a function for a set of data. The scheme avoids the difficulty that may arise in finding higher derivatives of functions.

2 Derivation of the ELS Formula

In order to derive the ELS schemes, the Euclidean norm for a given of $n+1$ data points $((x_i, y_i)\,,\ i = 0\,(1)\,n)$ will be taken and the least square method will be used to minimize the residual. It is referred to as "generalized" because the scheme assumes an arbitrary order s.

Definition 2.1. Let $x \in \mathbb{R}^n$, then we define the Euclidean norm ([3]) as

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}.$$

Define the Euclidean Least Square (ELS) polynomial $P_s(x)$ of order s as

$$P_s(x) = \sum_{m=0}^{s} a_m x^m$$

for a given of $n+1$ data points $((x_i, y_i)\,,\ i = 0\,(1)\,n)$. Assume the exact function that produced the set is $y = f(x)$ and set $y_i = f\,(x_i)$ then define the deviation R_i of $P_s\,(x)$ from y at a point x_i and take the Euclidean norm

$$R_i = y_i - \sum_{m=0}^{s} a_m x_i^m \qquad \text{and so} \qquad \|R\|^2 = \sum_{i=0}^{n}\left(y_i - \sum_{m=0}^{s} a_m x_i^m\right)^2$$

Setting $\frac{\partial \left\|R\right\|^2}{\partial a_j} = 0$ to minimize $\left\|R\right\|^2$ and rearranging to get

$$\sum_{m=0}^{s}\left(a_m \sum_{i=0}^{n} x_k^{m+j}\right) = \sum_{i=0}^{n} x_i^j y_i, \quad \Rightarrow \quad \sum_{m=0}^{s} \alpha_m \alpha_{j+m} = \beta_j, \quad j = 0, 1, 2, \cdots, s. \tag{2.1}$$

where

$$\alpha_r = \sum_{i=0}^{n} x_i^r, \quad \beta_r = \sum_{i=0}^{n} x_i^r y_i. \tag{2.2}$$

Putting

$$\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{s+1} \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_s & \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{2s} \end{pmatrix}, \; A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_s \end{pmatrix}, \; \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_s \end{pmatrix}$$

We finally have

$$\alpha A = \beta \quad \Rightarrow \; A = \alpha^{-1}\beta. \tag{2.3}$$

Clearly, the coefficient matrix α is a symmetric $(s+1)\times(s+1)$ matrix andβ is an $(s+1)$ column vector and a unique solution to equation 2.3 exists if $|\alpha| \neq 0$.

2.1 Existence and Uniqueness theorem

Theorem 2.2. *The determinant of α is the the sum of exactly $(n+1)^{(s+1)}$ determinants of $(s+1)\times(s+1)$ matrices*

Proof. Define the determinant of α as

$$|\alpha| \quad = \quad \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{s+1} \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_s & \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{2s} \end{vmatrix}$$

and on substituting equation (2.2),

$$|\alpha| \quad = \quad \begin{vmatrix} \sum_{i=0}^n x_i^0 & \sum_{i=0}^n x_i^1 & \sum_{i=0}^n x_i^2 & \cdots & \sum_{i=0}^n x_i^s \\ \sum_{i=0}^n x_i^1 & \sum_{i=0}^n x_i^2 & \sum_{i=0}^n x_i^3 & \cdots & \sum_{i=0}^n x_i^{s+1} \\ \sum_{i=0}^n x_i^2 & \sum_{i=0}^n x_i^3 & \sum_{i=0}^n x_i^4 & \cdots & \sum_{i=0}^n x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^n x_i^s & \sum_{i=0}^n x_i^{s+1} & \sum_{i=0}^n x_i^{s+2} & \cdots & \sum_{i=0}^n x_i^{2s} \end{vmatrix}$$

and by condensation method gives

$$|\alpha| = \sum_{k_0=0}^n \begin{vmatrix} x_{k_0}^0 & \sum_{i=0}^n x_i^1 & \sum_{i=0}^n x_i^2 & \cdots & \sum_{i=0}^n x_i^s \\ x_{k_0}^1 & \sum_{i=0}^n x_i^2 & \sum_{i=0}^n x_i^3 & \cdots & \sum_{i=0}^n x_i^{s+1} \\ x_{k_0}^2 & \sum_{i=0}^n x_i^3 & \sum_{i=0}^n x_i^4 & \cdots & \sum_{i=0}^n x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & \sum_{i=0}^n x_i^{s+1} & \sum_{i=0}^n x_i^{s+2} & \cdots & \sum_{i=0}^n x_i^{2s} \end{vmatrix}.$$

and thus,

$$|\alpha| = \sum_{k_0=0}^n \chi_{k_0} \; k_0 = 0, 1, \cdots, n. \quad \Rightarrow \; \chi_{k_0} = \begin{vmatrix} x_{k_0}^0 & \sum_{i=0}^n x_i^1 & \sum_{i=0}^n x_i^2 & \cdots & \sum_{i=0}^n x_i^s \\ x_{k_0}^1 & \sum_{i=0}^n x_i^2 & \sum_{i=0}^n x_i^3 & \cdots & \sum_{i=0}^n x_i^{s+1} \\ x_{k_0}^2 & \sum_{i=0}^n x_i^3 & \sum_{i=0}^n x_i^4 & \cdots & \sum_{i=0}^n x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & \sum_{i=0}^n x_i^{s+1} & \sum_{i=0}^n x_i^{s+2} & \cdots & \sum_{i=0}^n x_i^{2s} \end{vmatrix}$$

Further condensation of χ_{k_0} gives

$$\chi_{k_0} = \sum_{k_1=0}^n \begin{vmatrix} x_{k_0}^0 & x_{k_1}^1 & \sum_{i=0}^n x_i^2 & \cdots & \sum_{i=0}^n x_i^s \\ x_{k_0}^1 & x_{k_1}^2 & \sum_{i=0}^n x_i^3 & \cdots & \sum_{i=0}^n x_i^{s+1} \\ x_{k_0}^2 & x_{k_1}^3 & \sum_{i=0}^n x_i^4 & \cdots & \sum_{i=0}^n x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & x_{k_1}^{s+1} & \sum_{i=0}^n x_i^{s+2} & \cdots & \sum_{i=0}^n x_i^{2s} \end{vmatrix} = \begin{vmatrix} x_{k_0}^0 & \sum_{i=0}^n x_i^1 & \sum_{i=0}^n x_i^2 & \cdots & \sum_{i=0}^n x_i^s \\ x_{k_0}^1 & \sum_{i=0}^n x_i^2 & \sum_{i=0}^n x_i^3 & \cdots & \sum_{i=0}^n x_i^{s+1} \\ x_{k_0}^2 & \sum_{i=0}^n x_i^3 & \sum_{i=0}^n x_i^4 & \cdots & \sum_{i=0}^n x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & \sum_{i=0}^n x_i^{s+1} & \sum_{i=0}^n x_i^{s+2} & \cdots & \sum_{i=0}^n x_i^{2s} \end{vmatrix} = \sum_{k_1=0}^n \chi_{k_1} \, [k_0]$$

where

$$\chi_{k_1}[k_0] = \begin{vmatrix} x_{k_0}^0 & x_{k_1}^1 & \sum_{i=0}^n x_i^2 & \cdots & \sum_{i=0}^n x_i^s \\ x_{k_0}^1 & x_{k_1}^2 & \sum_{i=0}^n x_i^3 & \cdots & \sum_{i=0}^n x_i^{s+1} \\ x_{k_0}^2 & x_{k_1}^3 & \sum_{i=0}^n x_i^4 & \cdots & \sum_{i=0}^n x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & x_{k_1}^{s+1} & \sum_{i=0}^n x_i^{s+2} & \cdots & \sum_{i=0}^n x_i^{2s} \end{vmatrix}$$

and therefore

$$|\alpha| = \sum_{k_0=0}^n \sum_{k_1=0}^n \chi_{k_1}[k_0].$$

Further condensation of χ_{k_1} gives

$$\chi_{k_1} = \sum_{k_2=0}^n \chi_{k_2}[k_0, k_1] \qquad \text{where} \quad \chi_{k_2}[k_0, k_1] = \begin{vmatrix} x_{k_0}^0 & x_{k_1}^1 & x_{k_2}^2 & \cdots & \sum_{i=0}^n x_i^s \\ x_{k_0}^1 & x_{k_1}^2 & x_{k_2}^2 & \cdots & \sum_{i=0}^n x_i^{s+1} \\ x_{k_0}^2 & x_{k_1}^3 & x_{k_2}^3 & \cdots & \sum_{i=0}^n x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & x_{k_1}^{s+1} & x_{k_2}^{s+1} & \cdots & \sum_{i=0}^n x_i^{2s} \end{vmatrix},$$

and

$$|\alpha| = \sum_{k_0=0}^n \sum_{k_1=0}^n \sum_{k_2=0}^n \chi_{k_2}[k_0, k_1].$$

Continuing in this manner and by setting

$$\chi_{k_s}[k_0, k_1, \cdots, k_{s-1}] = \begin{vmatrix} x_{k_0}^0 & x_{k_1}^1 & x_{k_2}^2 & \cdots & x_{k_s}^s \\ x_{k_0}^1 & x_{k_1}^2 & x_{k_2}^2 & \cdots & x_{k_s}^s \\ x_{k_0}^2 & x_{k_1}^3 & x_{k_2}^3 & \cdots & x_{k_s}^s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & x_{k_1}^{s+1} & x_{k_2}^{s+1} & \cdots & x_{k_s}^s \end{vmatrix},$$

we have

$$|\alpha| = \sum_{k_0=0}^n \sum_{k_1=0}^n \sum_{k_2=0}^n \cdots \sum_{k_s=0}^n \chi_{k_s}[k_0, k_1, k_2, \cdots, k_{s-1}]$$

This a sum of an $(n+1)^{(s+1)}$ determinants and the proof is complete. □

Theorem 2.3. *At least*

$$(n+1)^{s+1} - \frac{(n+1)!}{(n-s)!}$$

determinants are zero.

Proof. From theorem 2.2, χ_{k_s} represents the sum of determinant of an $(n+1)^{(s+1)}$ matrices that will be equal to the determinant of α and

$$\chi_{k_s} = \begin{vmatrix} x_{k_0}^0 & x_{k_1}^1 & x_{k_2}^2 & \cdots & x_{k_s}^s \\ x_{k_0}^1 & x_{k_1}^2 & x_{k_2}^2 & \cdots & x_{k_s}^s \\ x_{k_0}^2 & x_{k_1}^3 & x_{k_2}^3 & \cdots & x_{k_s}^s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & x_{k_1}^{s+1} & x_{k_2}^{s+1} & \cdots & x_{k_s}^s \end{vmatrix} = 0.$$

The determinant of any $\chi_{k_i} (i = 0, 1, \cdots, s)$ will only be zero if any of the k_i's is repeated. Since there are $(s+1)$ k_i's $(i = 0, 1, \cdots, s)$ and each of the k_i's can take exactly one of $(n+1)$ values at a time (i.e. $k_i = 0, 1, \cdots, n$) and in order not to have a zero determinant, we avoid repeating the k_i's and the total number of ways to arrange $s+1$ k_i's from the $n+1$ choices without repeating any k_i is exactly ${}^{n+1}P_{s+1}$. Thus, there are exactly

$${}^{n+1}P_{s+1} = \frac{(n+1)!}{(n-s)!}$$

without repeated columns. Hence at least

$$(n+1)^{s+1} - \frac{(n+1)!}{(n-s)!} = (n+1)^{s+1} - {}^{n+1}p_{s+1} \text{ determinants are zero.}$$

□

Theorem 2.4 (Existence and Uniqueness Theorem)**.** *α is invertible.*

Proof. By hypothesis, the data (x_i, y_i) are not repeated and therefore there is no repeated column in the coefficient matrix α. Consider any two distinct columns p and q of A (say, $A_p$and A_q) such that

$$A_p = \begin{pmatrix} \alpha_{p-1} & \alpha_p & \cdots & \alpha_{p+s-1} \end{pmatrix}^T \quad \text{and } A_q = \begin{pmatrix} \alpha_{q-1} & \alpha_q & \cdots & a_{q+s-1} \end{pmatrix}^T$$

A linear combination of A_p and $A_q$gives

$$aA_p + bA_q = 0 \tag{2.4}$$

substituting valuesA_p and $A_q$into (2.4)we get

$$\begin{aligned} a\begin{pmatrix} \alpha_{p-1} & \alpha_p & \cdots & \alpha_{p+s-1} \end{pmatrix}^T + b\begin{pmatrix} \alpha_{q-1} & \alpha_q & \cdots & \alpha_{q+s-1} \end{pmatrix}^T &= \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}^T \\ \begin{pmatrix} a\alpha_{p-1} + b\alpha_{q-1} & a\alpha_p + b\alpha_q & \cdots & a\alpha_{p+s-1} + b\alpha_{q+s-1} \end{pmatrix}^T &= \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}^T \\ \begin{pmatrix} \sum_{j=1}^n ax_j^{p-1} + bx_j^{q-1} & \sum_{j=1}^n ax_j^p + bx_j^q & \cdots & \sum_{j=1}^n ax_j^{p+s-1} + bx_j^{q+s-1} \end{pmatrix}^T &= \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}^T \end{aligned}$$

and generally, we have

$$\sum_{j=1}^n ax_j^{p-1+k} + bx_j^{q-1+k} = 0 \Rightarrow ax_j^{p-1+k} + bx_j^{q-1+k} = 0. \quad \text{where } k = 0, 1, 2, \cdots, s$$

and hence $a = b = 0$ (since$p \neq q$). Thus, any two columns are linearly independent and the matrix α contain $(s+1)$ linearly independent columns (i.e. $rank\,(\alpha) = n$) therefore α is invertible. This invertibility of α verifies that there exists a unique solution of equation 2.3. □

3 Numerical Illustration of ELS Scheme

It is worth mentioning that the Maclaurin series require the evaluation of higher derivatives up to the order of interest which ELS scheme does not require. In this section, Maclaurin series and ELS polynomials are compared for exponential function, sine function and cosine function. ELS scheme is further used to forecast power consumption based on the data obtained from Energy Information Administration of the United States of America.

Example 3.1. The first five terms of the Maclaurin series for an exponential function is

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

while the ELS Scheme for exponential function of order 4 is given as

$$P_4 = 1 + 0.9985x + 0.5101x^2 + 0.1403x^3 + 0.0691x^4$$

(where the a_i's are evenly spaced in the $[0, 1]$). The mean square error for Maclaurin's series and ELS scheme for data set between 0 and 2 are 2.4965 and 0.1774 respectively. It becomes obvious from figure 1a below that ELS4 agrees more with the exact graph than the Maclaurin series of order 4.

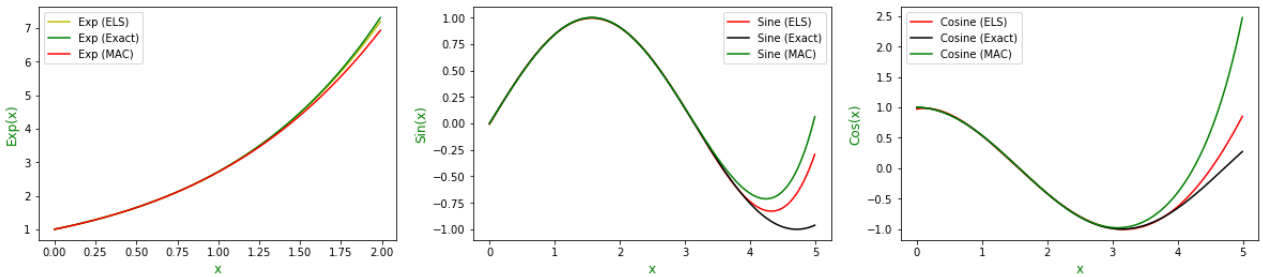
Example 3.2. The first five terms of the Maclaurin series for the sine function is

$$\exp(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

while the ELS Scheme for sine function of order 4 is given as

$$P_4 = -0.0066 + 1.0349x - 0.0205x^2 - 0.1973x^3 + 0.0316x^4$$

(where the a_i's are evenly spaced in the $[0, 1]$). The mean square error for Maclaurin's series and ELS scheme for data set between 0 and 2 are 23.938 and 9.0044 respectively. It becomes obvious from figure 1b below that ELS4 agrees more with the exact graph than the Maclaurin series of order 8.



(a) Graph of an Exponential Function (b) Graph of Sine Function (c) Graph of Cosine Function

Figure 1: Comparison between Maclaurin scheme and ELS scheme

Example 3.3. The first five terms of the Maclaurin series for the cosine function is

$$\exp(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

while the ELS Scheme for sine function of order 4 is given as

$$P_4 \quad = \quad -0.0066 + 1.0349x - 0.0205x^2 - 0.1973x^3 + 0.0316x^4$$

(where the a_i's are evenly spaced in the $[0, 1]$). The mean square error for Maclaurin's series and ELS scheme for data set between 0 and 2 are 116.3717 and 7.4402 respectively. It becomes obvious from figure 1c below that ELS4 agrees more with the exact graph than the Maclaurin series of order 8.

Example 3.4. Table 1 gives the recorded energy usage from the year 2006 to 2016 in the United States of America. Using the data from 2006 to 2015, ELS scheme of order 4 is obtained as

$$P_4(x) \quad = \quad 333.1343 + 24.3969x - 5.5276x^2 + 0.8039x^3 - 0.0391x^4.$$

This polynomial forecast for the year 2016 as 437.2333 which is pretty close to the exact value 440.1.

Africa Energy Consumption						
YEAR	2006	2007	2008	2009	2010	2011
Energy Consumption (thousand barrels) per day	334.8	347.9	369.5	373.4	388.9	388

YEAR	2012	2013	2014	2015	2016	
Energy Consumption (thousand barrels) per day	402.9	415.4	427.9	433.5	440.1	

Table 1: Africa energy consumption between 2010 and 2016, data was extracted from Energy Information Administrative EIA `https://www.bp.com/content/dam/bp/en/corporate/pdf/energy-economics/statistical-review-2017/bp-statistical-review-of-world-energy-2017-full-report.pdf`

4 Conclusion

A general formula for obtaining the Euclidean Least Square (ELS) scheme of arbitrary order is developed. The ELS polynomial $P_s(x)$ of order s is derived as

$$P_s(x) = \sum_{m=0}^{s} a_m x^m$$

where

$$A = \alpha^{-1}\beta,$$

$$\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{s+1} \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_s & \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{2s} \end{pmatrix}, \ A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_s \end{pmatrix}, \ \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_s \end{pmatrix}$$

The existence and uniqueness of the inverse of α suffices as a sufficient condition for the existence and uniqueness of ELS schemes. The schemes are used to approximate some selected transcendental functions. On comparing the resulting approximations with the classical Maclaurin's series, it is observed that the ELS schemes provide better approximations to this functions. It is worth noting that although the inversion of matrices may be deterrent to development of the ELS schemes, once the scheme is obtained, it is more accurate than the classical Maclaurin series.

Conflict of interest

Authors declare that there is no conflict of interest

References

[1] Adam, J. 1982. "A Detailed Study of the Duality Relation for the Least Squares Adjustment In Euclidean Spaces." *Satellite Geodetic Observatory* 56:180–195.

[2] Annaby, M. A. and H. A. Hassan. 2018. "Cover image Trigonometric sums by Hermite interpolations." *Applied Mathematics and Computation* 330:213–224.

[3] Chatzinakos, C. and G. Zioutas. 2014. "Least trimmed euclidean deviations for robust leverage in regression estimates." *Simulation Modelling Practice and Theory* 47:110–120.

[4] Chongbin, Z., B. E. Hobbs, H. B. Mug Hlhaus and A. Ord. 1999. "A Consistent Point-searching Algorithm for Solution Interpolation in Unstructured Meshes Consisting of 4-node Bilinear Quadrilateral Elements." *International Journal for Numerical Methods in Engineering* 45:1509–1526.

[5] Desmet, P. J. J. 1996. "Effects of Interpolation Errors on the Analysis of Dems." *Earth Surface Processes and Landforms* 22:563–580.

[6] Dorst, L. 2014. "Total Least Squares Fitting of k-Spheres in n-D Euclidean Space Using an (n+2)-D Isometric Representation." *J Math Imaging Vis* 14:1–20.

[7] Froncioni, A. M., P. Labbe, A. Garon and R. Camarero. 1997. "Interpolation Free Space Time Remeshing for the Burgers Equation." *Communications in Numerical Methods in Engineering,* 13:875–884.

[8] Gabrielle, M. A., K. M. David and C. G. Simon. 1997. "Twitch Interpolation of the Elbow Flexor Muscles at High Forces." *Voluntary Drive to the Elbow Flexors* 21:318–328.

[9] Hill, D. L. and E. A. Baskharone. 1997. "Development of a Multiblock Pressure-based Algorithm Using Mixed Interpolation for Turbulent Flows." *International Journal for Numerical Methods in Fluids* 25:615–631.

[10] Hon, Y. C. and W. Zongmin. 2000. "A Quasi-interpolation Method for Solving Stiff Ordinary Differential Equations." *International Journal for Numerical Methods In Engineering* 48:1187–1197.

[11] Kang, I. G. and F. C. Park. 1999. "Cubic Spline Algorithms for Orientation Interpolation." *International Journal for Numerical Methods in Engineering* 46:45–64.

[12] Karl, T. 1996. "On the Choice of a Derivative Boundary Element Formulation Using Hermite Interpolation." *International Journal for Numerical Methods in Engineering* 39:451–468.

[13] Moez, S. and C. Abdelkader. 2013. "A novel weighted recursive least squares based on Euclidean particle swarm optimization." *Electrical engineering, High School of Sciences and Engineering of Tunis* 42:268–281.

[14] Papageorgakopoulos, J., G. Arampatzis, D. Assimacopoulos and N. C. Markatos. 2000. "Enhancement of the momentum interpolation method on non-staggered grids." *International Journal for Numerical Methods in Fluids* 33:1–22.

[15] Wei, H., Z. Jian, C. Hao, W. Liyuan, W. Kang, W. Weifeng and L Xinghui. 2012. "Validation of origins of tea samples using partial least squares analysis and Euclidean distance method with near-infrared spectroscopy data." *Spectrochimica Acta Part A: Molecular and Biomolecular Spectroscopy* 86:399–404.

[16] Xiao-Wen, C. and C.P. Christopher. 2007. "Euclidean distances and least squares problems for a given set of vectors." *Applied Numerical Mathematics* 57:1240–1244.

[17] Youngmo, H. and F. C. Park. 2000. "Least Squares Tracking on the Euclidean Group." *Ieee Transactions on Automatic Control* 46:1127–1132.

[18] Zhiye, Z. 1998. "Error Estimation in Adaptive Bem by Postprocessing Interpolation." *Communications in Numerical Methods in Engineering* 14:633–645.