

A COMPARISON BETWEEN MAXIMUM LIKELIHOOD AND BAYESIAN ESTIMATION METHODS FOR THE SCALE PARAMETER OF THE WEIBULL-EXPONENTIAL DISTRIBUTION

ABSTRACT

We considered the Bayesian analysis of the scale parameter of Weibull-Exponential distribution in this paper. We assumed a class of non-informative priors in deriving the corresponding posterior distributions. In particular, the Bayes estimators and associated risks were calculated under three different loss functions. The performance of the Bayes estimators has been evaluated and compared to the method of maximum likelihood under a comprehensive simulation study. It was discovered that for the said parameters to be estimated, the quadratic loss function under both uniform and Jeffrey's priors should be used for decreasing parameter values while the use of precautionary loss function can be preferred for increasing parameter values irrespective of the variations in sample size.

Keywords: Weibull Exponential, Bayesian Estimation, Mathematical Statistics, Maximum Likelihood Estimation, Simulation

1. Introduction

The Exponential distribution is regarded as *memoryless* and has a constant failure rate; this latter property makes the distribution unsuitable for real-life problems and hence there is need to generalize the Exponential distribution in order to increase its flexibility and capability to model some other real-life problems, Singh *et al.* (2013). Some of recent generalizations of the exponential distribution include the transmuted exponential distribution by Owoloko *et al.* (2015), transmuted inverse exponential distribution by Oguntunde and Adejumo (2015) and the Weibull-Exponential distribution (WED) by Oguntunde *et al.* (2015).

According to Oguntunde *et al.*, (2015), if X is a Weibull-exponential random variable, then the probability density function (pdf) and the cumulative distribution function (CDF) of X are respectively given by;

$$f(x) = \alpha\beta\lambda e^{\lambda\beta x} \left(1 - e^{-\lambda x}\right)^{\beta-1} e^{-\alpha(e^{\lambda x} - 1)^{\beta}} \quad (1)$$

and

$$F(x) = 1 - e^{-\alpha(e^{\lambda x} - 1)^{\beta}} \quad (2)$$

where; α and β are scale and shape parameters respectively while λ is a exponential parameter.

The classical and non-classical or Bayesian methods of estimation have gained wider applications in statistical theory and analysis. In classical scenario, the parameters are considered to be fixed while in the Bayesian concept, the parameters are viewed as unknown random variables. It is true that in many real-life phenomena which are represented by lifetime models, the parameters cannot be treated as

constant throughout the life testing period (Ibrahim *et al.*, 2001; Martz and Waller, 1982; Singpurwalla, 2006) hence the need for Bayesian estimation for lifetime models. Al-Aboud (2009) has considered Bayesian estimation for the extreme value distribution using progressive censored data and asymmetric loss, Himanshu and Arun (2009) estimated the shape parameter of the Generalized Pareto Distribution (GPD) using quasi, inverted gamma and uniform prior distributions under the LINEX, precautionary and entropy loss functions, Sanku (2010) estimated the shape parameter of Generalized Exponential distribution using extended Jeffrey's prior under the quadratic loss function, squared error loss function and general entropy loss function, Sanku and Tanujit (2011) also estimated the parameters of Rayleigh distribution using Bayesian approach, Sajid *et al.*, (2012) estimated the scale parameter of Laplace model using different asymmetric loss functions comprising precautionary, weighted squared, modified (quadratic) squared loss functions and Ahmed *et al.* (2011) considered Bayesian Survival Estimator for Weibull distribution with censored data. In addition, Guure *et al.*, (2012) has also considered the Bayesian estimation of Weibull distribution under three loss functions.

One of the important things that cannot be ignored in Bayesian approach is choosing the appropriate prior(s) for the parameters, so also is the choice of loss function. Pandey *et al.* (2011), Al-Athari (2011), Ahmad and Ahmad (2013), Ahmad *et al.* (2015), Ahmad *et al.* (2016), Gupta and Singh (2017) and Gupta (2017) and many others have however shown a number of symmetric and asymmetric loss functions to be functional in several applications.

The objective of this study is to introduce a statistical comparison between the Bayesian and Maximum Likelihood estimation (MLE) procedures for estimating the scale parameter of the Weibull-Exponential distribution. The resulting estimators are obtained by using squared error, Quadratic and precautionary loss functions.

The layout of the paper is as follows; in Section 2, Maximum likelihood estimate of the scale parameter are obtained, in Section 3, the posterior distributions are obtained under the two different prior distributions while the Bayes estimates and corresponding risks are obtained in section 4. Finally, comparison between MLE and Bayes estimates under the two priors and loss functions are made using simulation study in Section 5. Some concluding remarks are given in Section 6.

2. Maximum Likelihood Estimation

This section presents the estimation of the scale parameter of the Weibull-Exponential distribution using the method of maximum likelihood estimation. Let $X_1, X_2, \dots, X_n$ be a random sample from the WED with unknown parameter vector $\theta = (\alpha, \beta, \lambda)^T$. The total log-likelihood function for θ is obtained from $f(x)$ as follows:

$$L(X_1, X_2, \dots, X_n | \alpha, \beta, \lambda) = (\alpha\beta\lambda)^n e^{\lambda\beta \sum_{i=1}^n x_i} \sum_{i=1}^n \left(1 - e^{-\lambda x_i}\right)^{\beta-1} \exp\left\{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta\right\} \quad (3)$$

The likelihood function for the scale parameter, α , is given by;

$$L(\alpha | \underline{X}) \propto (\alpha)^n e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} \quad (4)$$

Let the log-likelihood function, $l = \log L(\alpha | \underline{X})$, therefore

$$l = n \log \alpha - \alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \quad (5)$$

Differentiating l partially with respect to α , the scale parameter, equating to zero and solving for $\hat{\alpha}$ gives;

$$\begin{aligned} \frac{\partial l}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \\ \Rightarrow \hat{\alpha} &= \frac{n}{\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} \end{aligned} \quad (6)$$

Hence, equation (6) is the estimator for the scale parameter of the Weibull-Exponential distribution obtained by the method of Maximum Likelihood estimation.

3. Posterior Distributions

To obtain the posterior distribution of a parameter once the data has been observed, we apply Bayes' Theorem which is given as:

$$p(\alpha | \underline{X}) = \frac{p(\alpha) L(\alpha | \underline{X})}{\int_0^\infty p(\alpha) L(\alpha | \underline{X}) d\alpha} \quad (7)$$

where $p(\alpha)$ and $L(\alpha | \underline{X})$ are the prior distribution and the Likelihood function respectively.

Here, Posterior distributions are derived by using uniform and Jeffrey's prior.

3.1 Posterior Distributions under the assumption of Uniform Prior

The uniform prior as a non-informative prior relating to the scale parameter α is defined as:

$$p(\alpha) \propto 1 \quad ; \quad 0 < \alpha < \infty \quad (8)$$

The posterior distribution of the scale parameter α for a given data under uniform prior is obtained from equation (5) using integration by substitution method as

$$p(\alpha | \underline{X}) = \frac{\alpha^n e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta}}{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^{-(n+1)} \Gamma(n+1)} \quad (9)$$

3.2 Posterior Distributions under the assumption of Uniform Prior

Also, the Jeffrey's prior as a non-informative prior relating to the scale parameter α of the WED distribution is defined as:

$$p(\alpha) \propto \frac{1}{\alpha} \quad ; \quad \alpha \geq 1 \quad (10)$$

The posterior distribution of the scale parameter α for a given data under Jeffrey prior is obtained from equation (5) using integration by substitution method as

$$p(\alpha | \underline{X}) = \frac{\alpha^{n-1} \left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^n e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta}}{\Gamma(n)} \quad (11)$$

105

106 4 Bayes Estimators and their respective corresponding Risks

107 Here, we estimate the scale parameter of the *WED* using three loss functions under the posterior
108 distributions obtained from both the uniform and Jeffrey's priors.

109 The Bayes estimators and their corresponding Bayes posterior risks using uniform prior are as follows:

110

111 Using Squared Error Loss Function (*SELF*) under uniform prior

112 From Legendre (1805) and Gauss (1810), the squared error loss function relating to the scale
113 parameter α is defined as

$$L(\alpha, \alpha_{SELF}) = (\alpha - \alpha_{SELF})^2 \quad (12)$$

115 where α_{SELF} is the estimator of the parameter α under *SELF*?

116 The derivation of Bayes estimator using *SELF* under uniform prior is given below:

$$\alpha_{SELF} = E(\alpha) = E(\alpha | \underline{X})$$

$$E(\alpha | \underline{X}) = \int_0^\infty \alpha p(\alpha | \underline{X}) d\alpha \quad (13)$$

119 Substituting for $p(\alpha | \underline{X})$ in equation (11); we have:

$$E(\alpha | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^{n+1}}{\Gamma(n+1)} \int_0^\infty \alpha^{n+1} e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} d\alpha \quad (14)$$

121 Now, using integration by substitution method in equation (14) and simplification, we obtained the
122 Bayes estimator using *SELF* under the uniform prior as:

$$\alpha_{SELF} = \frac{\Gamma(n+2)}{\Gamma(n+1) \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} \quad (15)$$

124 Using the Squared error loss function (*SELF*), the following risk $p(\alpha_{SELF})$ is defined as:

$$P(\alpha_{SELF}) = E(\alpha^2 | \underline{X}) - \{E(\alpha | \underline{X})\}^2 \quad (16)$$

126 And it is obtained as

$$P(\alpha_{SELF}) = \frac{\Gamma(n+3)\Gamma(n+1) - (\Gamma(n+2))^2}{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^2 (\Gamma(n+1))^2} \quad (17)$$

128 Using Quadratic Loss Function (*QLF*) under uniform prior

129 The Quadratic loss function (*QLF*) is defined as

$$L(\alpha, \alpha_{QLF}) = \left(\frac{\alpha - \alpha_{QLF}}{\alpha} \right)^2 \quad (18)$$

where α_{QLF} is the estimator of the parameter α under *QLF*?

The derivation of Bayes estimator using *QLF* under uniform prior is as follows:

$$\alpha_{QLF} = \frac{E(\alpha^{-1})}{E(\alpha^{-2})} = \frac{E(\alpha^{-1} | \underline{X})}{E(\alpha^{-2} | \underline{X})}$$

$$E(\alpha^{-1} | \underline{X}) = \int_0^{\infty} \alpha^{-1} p(\alpha | \underline{X}) d\alpha \quad (19)$$

Substituting for $p(\alpha | \underline{X})$ in equation (19), we have:

$$E(\alpha^{-1} | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^{\beta} \right)^{n+1}}{\Gamma(n+1)} \int_0^{\infty} \alpha^{n-1} e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^{\beta}} d\alpha \quad (20)$$

Using integration by substitution method in comparison (20) and simplifying, we obtained the Bayes estimator using *QLF* under the uniform prior as:

$$\alpha_{QLF} = \frac{\Gamma(n)}{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^{\beta} \right) \Gamma(n-1)} \quad (21)$$

Using the Quadratic loss function (*QLF*), the following risk $p(\alpha_{QLF})$ is defined as:

$$P(\alpha_{QLF}) = 1 - \frac{\left\{ E(\alpha^{-1} | \underline{X}) \right\}^2}{E(\alpha^{-2} | \underline{X})} \quad (22)$$

Therefore, the following risk under the uniform prior using the Quadratic loss function is given as:

$$P(\alpha_{QLF}) = \frac{\Gamma(n+1)\Gamma(n-1) - [\Gamma(n)]^2}{\Gamma(n+1)\Gamma(n-1)} \quad (23)$$

144

145 Using Precautionary Loss Function (*PLF*) under the uniform prior

146 The precautionary loss function (*PLF*) introduced by Norstrom (1996) is an asymmetric loss function
147 and is defined as

$$L(\alpha_{PLF}, \alpha) = \frac{(\alpha_{PLF} - \alpha)^2}{\alpha} \quad (24)$$

149 where α_{PLF} is the estimator of the parameter α under *PLF*?

150 Similarly, the derivation of Bayes estimator under *PLF* using uniform prior is given below:

$$\alpha_{PLF} = \left\{ E(\alpha^2) \right\}^{\frac{1}{2}} = \left\{ E(\alpha^2 | \underline{X}) \right\}^{\frac{1}{2}} = \sqrt{E(\alpha^2 | \underline{X})}$$

$$E(\alpha^2 | \underline{X}) = \int_0^{\infty} \alpha^2 p(\alpha | \underline{X}) d\alpha \quad (25)$$

153 Substituting for $p(\alpha | \underline{X})$ in equation (19); we have:

$$154 \quad E(\alpha^2 | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^{n+1}}{\Gamma(n+1)} \int_0^\infty \alpha^{n+2} e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} d\alpha \quad (26)$$

155 Again, using integration by substitution method in comparison (26) and simplifying, we obtained the
156 Bayes estimator using *PLF* under the uniform prior as:

$$157 \quad \alpha_{PLF} = \sqrt{\frac{\Gamma(n+3)}{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^2 \Gamma(n+1)}} \quad (27)$$

158 Applying the Precautionary loss function (*PLF*), the following risk $p(\alpha_{PLF})$ is defined as:

$$159 \quad P(\alpha_{PLF}) = 2 \{ \alpha_{PLF} - E(\alpha | \underline{X}) \} \quad (28)$$

160 and derived from being:

$$161 \quad P(\alpha_{PLF}) = 2 \left\{ \frac{\left\{ \Gamma(n+3) \Gamma(n+1) \right\}^{\frac{1}{2}} - \Gamma(n+2)}{\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \Gamma(n+1)} \right\} \quad (29)$$

162 Similarly, the Bayes estimators and posterior risks of the scale parameter of the *WED* using three loss
163 functions under the posterior distribution obtained from Jeffrey's prior are as follows:
164

166 **Using Squared Error Loss Function (*SELF*) under Jeffrey's prior**

167 The derivation of Bayes estimator under *SELF*-using Jeffrey's prior is given below:

$$168 \quad \alpha_{SELF} = E(\alpha) = E(\alpha | \underline{X})$$

$$169 \quad E(\alpha | \underline{X}) = \int_0^\infty \alpha p(\alpha | \underline{X}) d\alpha \quad (30)$$

170 Substituting for $p(\alpha | \underline{X})$ in equation (30); we have:

$$171 \quad E(\alpha | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^n}{\Gamma(n)} \int_0^\infty \alpha^n e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} d\alpha \quad (31)$$

172 Using integration by substitution method in comparison (31) and simplifying, we obtained the Bayes
173 estimator using *SELF* under Jeffrey prior as:

$$174 \quad \alpha_{SELF} = \frac{\Gamma(n+1)}{\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \Gamma(n)} \quad (32)$$

175 Using the Squared error loss function (*SELF*), the following risk $p(\alpha_{SELF})$ is defined as:

$$176 \quad P(\alpha_{SELF}) = E(\alpha^2 | \underline{X}) - \{E(\alpha | \underline{X})\}^2 \quad (33)$$

177 Therefore, the following risk under Jeffrey's prior using the squared error loss function is:

$$P(\alpha_{SELF}) = \frac{\Gamma(n+2)\Gamma(n) - (\Gamma(n+1))^2}{\left(\sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta\right)^2 (\Gamma(n))^2} \quad (34)$$

179

180 Using Quadratic Loss Function (QLF) under Jeffrey's prior

181 The derivation of Bayes estimator under QLF using Jeffrey's prior is given below:

$$\alpha_{QLF} = \frac{E(\alpha^{-1})}{E(\alpha^{-2})} = \frac{E(\alpha^{-1} | \underline{X})}{E(\alpha^{-2} | \underline{X})}$$

$$E(\alpha^{-1} | \underline{X}) = \int_0^\infty \alpha^{-1} p(\alpha | \underline{X}) d\alpha \quad (35)$$

184 Substituting for $p(\alpha | \underline{X})$ in equation (35); we have:

$$E(\alpha^{-1} | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta\right)^n}{\Gamma(n)} \int_0^\infty \alpha^{n-2} e^{-\alpha \sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta} d\alpha \quad (36)$$

186 Using integration by substitution method in comparison (36) and simplifying, we obtained the Bayes
187 estimator using QLF under Jeffrey prior as:

$$\alpha_{QLF} = \frac{\Gamma(n-1)}{\left(\sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta\right) \Gamma(n-2)} \quad (37)$$

189 Using the Quadratic loss function (QLF), the following risk $p(\alpha_{QLF})$ is defined as:

$$P(\alpha_{QLF}) = 1 - \frac{\{E(\alpha^{-1} | \underline{X})\}^2}{E(\alpha^{-2} | \underline{X})} \quad (38)$$

191 Hence, it is obtained as:

$$P(\alpha_{QLF}) = \frac{\Gamma(n)\Gamma(n-2) - [\Gamma(n-1)]^2}{\Gamma(n)\Gamma(n-2)} \quad (39)$$

193

194 Using Precautionary Loss Function (PLF) under Jeffrey's prior

195 Similarly, the derivation of Bayes estimator under PLF using Jeffrey's prior is given below:

$$\alpha_{PLF} = \{E(\alpha^2)\}^{\frac{1}{2}} = \{E(\alpha^2 | \underline{X})\}^{\frac{1}{2}} = \sqrt{E(\alpha^2 | \underline{X})}$$

$$E(\alpha^2 | \underline{X}) = \int_0^\infty \alpha^2 p(\alpha | \underline{X}) d\alpha \quad (40)$$

198 Substituting for $p(\alpha | \underline{X})$ in equation (40); we have:

$$E(\alpha^2 | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta\right)^n}{\Gamma(n)} \int_0^\infty \alpha^{n+1} e^{-\alpha \sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta} d\alpha \quad (41)$$

Using integration by substitution method in comparison (41) and simplifying, we obtained the Bayes estimator using *PLF* under Jeffrey prior as:

$$\alpha_{PLF} = \sqrt{\frac{\Gamma(n+2)}{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta\right)^2 \Gamma(n)}} \quad (42)$$

Applying the Precautionary loss function (PLF), the following risk $p(\alpha_{PLF})$ is defined as:

$$P(\alpha_{PLF}) = 2 \{ \alpha_{PLF} - E(\alpha | \underline{X}) \} \quad (43)$$

Hence, obtained as:

$$P(\alpha_{PLF}) = 2 \left\{ \frac{\left\{ \Gamma(n+2) \Gamma(n) \right\}^{\frac{1}{2}} - \Gamma(n+1)}{\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \Gamma(n)} \right\} \quad (44)$$

5 Simulation Study

We used a package in R software to generate random samples of size $n = (25, 35, 75, 125)$ from *WED* by using different values of α and assuming that the exponential parameter, λ and shape parameter, β are known, that is $(\lambda = \beta = 1)$. Tables 1 to 3 present the results of our simulation study by listing the estimates of the scale parameter under the appropriate estimation methods such as the Maximum Likelihood Estimation (*MLE*), Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*), and Precautionary Loss Function (*PLF*) under both Uniform and Jeffrey priors. The Maximum Likelihood estimates and Bayesian estimates were obtained under a reasonably high number of replications; different sample sizes were used to investigate the performance of the estimators about their biases and mean squared errors as well as the sample sizes.

The performance of the two methods was evaluated using the following performance measures:

$$\text{Bias} = \frac{1}{N} \sum_{i=1}^N (\hat{\alpha}_i - \alpha) \text{ and Mean Square Error, } MSE = \frac{1}{N} \sum_{i=1}^N (\hat{\alpha}_i - \alpha)^2$$

Where $\hat{\alpha}$ is the estimate of α (the scale parameter) from the i th simulated data and N is the number of Monte Carlo samples.

Note: the estimate with the minimum bias and MSE will be considered as the best.

Table 1: Simulation of Bayes Estimates, their Biases and Mean Squared Errors based on the replications and sample sizes where $\alpha = 0.5$ assuming that λ and β are known (i.e. $\lambda = \beta = 1$).

Sample sizes	Measures	MLE	Uniform Prior			Jeffrey's Prior		
			SELF	QLF	PLF	SELF	QLF	PLF

25	Estimate	0.4437	0.4615	0.4260	0.4703	0.4437	0.4082	0.4525
	BIAS	0.0840	0.0899	0.0813	0.0940	0.0840	0.0818	0.0866
	MSE	0.0123	0.0145	0.0109	0.0160	0.0123	0.0105	0.0133
35	Estimate	0.4611	0.4743	0.4479	0.4808	0.4611	0.4348	0.4676
	BIAS	0.0699	0.0732	0.0686	0.0756	0.0699	0.0690	0.0713
	MSE	0.0083	0.0094	0.0076	0.0101	0.0083	0.0074	0.0087
75	Estimate	0.4645	0.4707	0.4583	0.4738	0.4645	0.4521	0.4676
	BIAS	0.0464	0.0475	0.0460	0.0482	0.0464	0.0462	0.0469
	MSE	0.0035	0.0037	0.0034	0.0039	0.0035	0.0034	0.0036
125	Estimate	0.4644	0.4681	0.4607	0.4699	0.4644	0.4570	0.4663
	BIAS	0.0360	0.0364	0.0358	0.0368	0.0360	0.0358	0.0362
	MSE	0.0021	0.0021	0.0020	0.0022	0.0021	0.0020	0.0021

Table 2: Simulation of Bayes Estimates, their Biases and Mean Squared Errors based on the replications and sample sizes where $\alpha = 1.5$ assuming that λ and β are known (i.e. $\lambda = \beta = 1$).

Sample sizes	Measures	MLE	Uniform Prior			Jeffrey's Prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	1.3311	1.3844	1.2779	1.4108	1.3311	1.2247	1.3575
	BIAS	0.2521	0.2699	0.2438	0.2820	0.2521	0.2454	0.2598
	MSE	0.1106	0.1309	0.0985	0.1440	0.1106	0.0945	0.1197
35	Estimate	1.3833	1.4228	1.3438	1.4425	1.3833	1.3043	1.4029
	BIAS	0.2099	0.2197	0.2057	0.2267	0.2098	0.2071	0.2140
	MSE	0.0743	0.0842	0.0685	0.0905	0.0743	0.0667	0.0787
75	Estimate	1.3935	1.4121	1.3749	1.4213	1.3935	1.3563	1.4027
	BIAS	0.1393	0.1424	0.1380	0.1446	0.1393	0.1386	0.1406
	MSE	0.0317	0.0337	0.0305	0.0350	0.0317	0.0302	0.0326
125	Estimate	1.3932	1.4044	1.3821	1.4099	1.3932	1.3709	1.3988
	BIAS	0.1079	0.1093	0.1073	0.1104	0.1079	0.1075	0.1085
	MSE	0.0186	0.0193	0.0182	0.0198	0.0186	0.0180	0.0189

Table 3: Simulation of Bayes Estimates, their Biases and Mean Squared Errors based on the replications and sample sizes where $\alpha = 3.5$ assuming that λ and β are known (i.e. $\lambda = \beta = 1$).

Sample sizes	Measures	MLE	Uniform Prior			Jeffrey's Prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	3.1060	3.2302	2.9818	3.2918	3.1060	2.8575	3.1675
	BIAS	0.5882	0.6298	0.5688	0.6579	0.5882	0.5726	0.6062
	MSE	0.6024	0.7127	0.5363	0.7838	0.6024	0.5146	0.6515
35	Estimate	3.2277	3.3199	3.1355	3.3657	3.2277	3.0433	3.2735
	BIAS	0.4897	0.5126	0.4799	0.5291	0.4897	0.4833	0.4994
	MSE	0.4048	0.4582	0.3731	0.4928	0.4048	0.3632	0.4286
75	Estimate	3.2515	3.2948	3.2081	3.3164	3.2515	3.1648	3.2731
	BIAS	0.3249	0.3323	0.3219	0.3375	0.3249	0.3233	0.3281
	MSE	0.1724	0.1833	0.1660	0.1904	0.1724	0.1642	0.1773
125	Estimate	3.2509	3.2769	3.2249	3.2898	3.2509	3.1989	3.2638
	BIAS	0.2518	0.2551	0.2503	0.2575	0.2518	0.2509	0.2532
	MSE	0.1012	0.1051	0.0989	0.1076	0.1012	0.0983	0.1030

6 Conclusions

We have obtained the Bayesian estimators of the scale parameter of Weibull-Exponential distribution in this study, the Posterior distributions of this parameter are derived by using

Uniform and Jeffrey's priors. Bayes estimators and their associated risks have been obtained by using three loss functions under the two prior distributions. The three loss functions used in this study are the Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*) and Precautionary Loss Function (*PLF*). The biases and mean square errors based on all the priors and for all the loss functions relating to the scale parameter of the Weibull-Exponential distribution expectedly decrease with increase in sample size.

For $\alpha = 0.5, \lambda = \beta = 1$, the result indicates that using *QLF* under Jeffrey prior produces the best estimator with minimum bias and *MSE* followed by *QLF* under Uniform prior and these performances are found to be consistent irrespective of the different sample sizes used.

For $\alpha = 1.5, \lambda = \beta = 1$, the result indicates that using *QLF* under Jeffrey's prior produces the best estimator with minimum bias and *MSE* followed by *QLF* under Uniform prior and these performances are found to be consistent irrespective of the different sample sizes used and this generally means that increasing the exponential or shape parameter has a huge impact on the estimate of the scale parameter of the Weibull-Exponential distribution.

Also, for $\alpha = 3.5, \lambda = \beta = 1$, our result shows that using *QLF* under Jeffrey's prior produces the best estimator with minimum bias and *MSE* followed by *QLF* under Uniform prior and again these performances are found to be consistent irrespective of the different sample sizes considered, and this proves that increasing the scale parameter alone with a constant shape and exponential parameter has a very little or no effect on the estimate of the scale parameter of the Weibull-Exponential distribution. Most importantly, we found that using the *QLF* under both priors is a better approach for estimating the scale parameter of the *WED* irrespective of the values of the exponential and shape parameters and the sample sizes.

REFERENCES

- Ahmad, K., Ahmad, S. P. and Ahmed, A. (2015). Bayesian analysis of generalised gamma distribution using R software. *Journal of Statistics Applications & Probability*, 4(2): 323–335.
- Ahmad, K., Ahmad, S. P. and Ahmed, A. (2016). Classical and Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. *Journal of Probability and Statistics*, Volume 2016, Article ID 7581918, eight pages <http://dx.doi.org/10.1155/2016/7581918>
- Ahmad, S. P. and Ahmad, K. (2013) "Bayesian analysis of Weibull distribution using R software," *Australian Journal of Basic and Applied Sciences*, vol. 7, no. 9, pp. 156–164.
- Ahmed, A. O. M., Ibrahim, N. A., Arasan, J. and Adam, M. B. (2011) "Extension of Jeffreys' prior estimate for Weibull censored data using Lindley's approximation," *Australian Journal of Basic and Applied Sciences*, vol. 5, no. 12, pp. 884–889.
- Al-Aboud, F. M. (2009) "Bayesian estimations for the extreme value distribution using progressive censored data and asymmetric loss," *International Mathematical Forum*, vol. 4, no. 33, pp. 1603–1622.

- 280 Al-Athari, F. M. (2011) "Parameter estimation for the double-Pareto distribution," *Journal of*
281 *Mathematics and Statistics*, vol. 7, no. 4, pp. 289–294.
- 282 Gupta, I. (2017). Estimation of Parameter And Reliability Function of Exponentiated Inverted
283 Weibull Distribution using Classical and Bayesian Approach. *International Journal of*
284 *Recent Scientific Research*, 8(7): 18117-18819.
- 285 Gupta, P.K. and Singh, A. K. (2017). Classical and Bayesian estimation of Weibull
286 distribution in presence of outliers. *Cogent Mathematics*, 4: 1300975.
- 287 Guure C. B., Ibrahim N. A., Ahmed O. M. (2012). Bayesian Estimation of two-parameter
288 Weibull Distribution using Extension of Jeffrey's Prior Information with Three Loss
289 Functions, *Mathematical Problems in Engineering*, Article ID 589640, 13 pages.
- 290 Himanshu, P. and Arun, K. R. (2009). Bayesian Estimation of the Shape Parameter of
291 Generalized Pareto Distribution Under Asymmetric Loss Functions. *Hacettepe*
292 *Journal of Mathematics and Statistics*, 38 (1), 69-83.
- 293 Ibrahim, J. G., Chen, M.-H., & Sinha, D. (2001). *Bayesian survival analysis*. New York, NY:
294 Springer-Verlag.
- 295 Martz, H. F., & Waller, R. A. (1982). *Bayesian reliability analysis*. New York, NY: John
296 Wiley.
- 297 Norstrom, J. G. (1996). The use of Precautionary Loss Functions in Risk Analysis. *IEEE*
298 *Transactions on reliability*, (3), pp.400-403
- 299 Oguntunde, P. E. And Adejumo, A. O. (2015). The transmuted inverse exponential
300 distribution. *International Journal Advanced Statistics and Probability*, 3(1):1–7
- 301 Oguntunde, P. E., Balogun, O. S., Okagbue, H. I. and Bishop, S. A. (2015). The Weibull-
302 Exponential Distribution: Its Properties and Applications. *Journal of Applied Science*.
303 15(11): 1305-1311.
- 304 Owoloko, E. A., oguntunde, P. E. and Adejumo, A. O. (2015). Performance rating of the
305 transmuted exponential distribution: an analytical approach. *Springerplus* 4:818.
- 306 Pandey, B. N., Dwividi, N. and Pulastya, B. (2011) "Comparison between Bayesian and
307 maximum likelihood estimation of the scale parameter in Weibull distribution with
308 known shape under linex loss function," *Journal of Scientific Research*, vol. 55, pp.
309 163–172.
- 310 Sajid, A., Muhammad, A., Nasir, A. and Syed, M. A. (2012). Scale Parameter Estimation of
311 the Laplace Model Using Differen Asymmetric Loss Functions. *International Journal*
312 *of Statistics and Probability* , I (1), 105-127.
- 313 Sanku, D. (2010). Bayesian Estimation of the Shape Parameter of the Generalised
314 Exponential Distribution under Different Loss Functions. *Pakistan Journal of*
315 *statistics and Operations Research* , VI (2), 163-174.
- 316 Sanku, D. and Tanujit, D. (2011). Rayleigh Distribution; Revisited via extension of Jeffrey's
317 prior information and a new loss function. *Statistical Journal* , IX (3), 213-226.
- 318 Singh, S. K., Singh, U. and Kumar, M. (2013). Estimation of Parameters of Generalized
319 Inverted Exponential Distribution for Progressive Type-II Censored Sample with

320 Binomial Removals. *Journal of Probability and Statistics*, Article ID :183-652.
321 <http://dx.doi.org/10.1155/2013/183652>.
322 Singpurwalla, N. D. (2006). *Reliability and risk: A Bayesian perspective*. Chichester: John
323 Wiley.