

SOME CHARACTERIZATIONS OF TRANSMUTED MODIFIED BURR III DISTRIBUTION

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ABSTRACT

In this paper, Transmuted Modified Burr III (TMBIII) distribution (Ali and Ahmad; 2015) is characterized through (i) ratio of truncated moments; (ii) doubly truncated moments; (iii) hazard rate function; (iv) reverse hazard rate function and (v) elasticity function. The applications of characterizations of TMBIII distribution will be beneficial for scientists in different areas of science.

KEY WORDS: Characterizations; Truncated moments; Hazard rate; Reverse hazard rate; Elasticity;

1. INTRODUCTION

1.1 The Burr Family of Distributions

Burr (1942) has suggested 12 distributions as Burr family to fit cumulative frequency functions on frequency data. Burr distributions XII, III and X are frequently used. Burr III (BIII) distribution has wide applications in business failure data, modeling finance, insurance data failure time modeling, reliability and acceptance sampling plans. The BIII distribution for the random variable Y has the following cumulative distribution function (cdf) and probability density function (pdf)

$$G(y) = (1 + y^{-\beta})^{-\alpha} \quad \alpha > 0, \beta > 0, y \geq 0, \quad (1)$$

$$g(y) = \alpha\beta y^{-(\beta+1)} (1 + y^{-\beta})^{-(\alpha+1)}, \alpha > 0, \beta > 0, y > 0. \quad (2)$$

Ali et.al, (2015) studied modified Burr III (MBIII) distribution along with some of its properties. The MBIII distribution with parameters α, β and γ , has the following pdf and cdf

$$f_{\text{MBIII}}(x) = \alpha\beta x^{-\beta-1} (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1}, \quad x > 0, \quad (3)$$

and
$$F_{\text{MBIII}}(x) = (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}}, \quad x \geq 0. \quad (4)$$

1.2 Transmuted Modified Burr III Distribution

Ali and Ahmad (2015) studied the transmuted modified Burr III (TMBIII) distribution along with its properties. The TMBIII distribution with parameters α, β, γ and λ , has the following pdf and cdf

$$f(x) = \alpha\beta x^{-\beta-1} (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1} \left[1 + 2\lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right] \quad x > 0, \quad |\lambda| \leq 1, \quad (5)$$

$$F(x) = (1 + \lambda) (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} - \lambda (1 + \gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}}, \quad x \geq 0, \quad |\lambda| \leq 1. \quad (6)$$

The hazard, reverse hazard functions and Mills ratio of a random variable X with TMBIII distribution are given, respectively, by

$$h_F(x) = \frac{\alpha \beta x^{-\beta-1} (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1} \left(1 + 2\lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right)}{1 - \left((1 + \lambda) (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} - \lambda (1 + \gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}} \right)}, \quad (7)$$

$$r_F(x) = \frac{\alpha \beta x^{-\beta-1} (1 + \gamma x^{-\beta})^{-1} \left[1 + 2\lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right]}{(1 + \lambda) - \lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}}}, \quad (8)$$

and

$$m(x) = \frac{1 - \left((1 + \lambda) (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} - \lambda (1 + \gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}} \right)}{\alpha \beta x^{-\beta-1} (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1} \left(1 + 2\lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right)}. \quad (9)$$

The elasticity $e(x) = \frac{d \ln F(x)}{d \ln x} = x r_F(x)$ of TMBIII distribution is

$$e_F(x) = \frac{\alpha \beta x^{-\beta} (1 + \gamma x^{-\beta})^{-1} \left[1 + 2\lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right]}{(1 + \lambda) - \lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}}}. \quad (10)$$

2. CHARACTERIZATIONS

In order to develop a stochastic function in a certain problem, it is necessary to know whether the selected function fulfills the requirements of the specific underlying probability distribution. To this end, it is required to study characterizations of the specific probability distribution. Different characterization techniques have developed. Glänzel (1987, 1988 and 1990), Hamedani (1993, 2002, 2011 and 2015), Ahsanullah and Hamedani (2007, 2012), Ahsanullah et. al (2013), Shakil et al. (2014), Ali et.al, (2015) and Merovci et.al (2016) have worked on characterizations.

In this paper, TMBIII distribution is characterized through (i) ratio of truncated moments; (ii) doubly truncated moments; (iii) hazard rate function; (iv) reverse hazard rate function and (v) elasticity function.

2.1 Characterizations of TMBIII Distribution through Ratio of Truncated Moments

TMBIII distribution is characterized using Theorem 1 (Glänzel; 1987) on the basis of a simple relationship between two truncated moments of functions of X. Theorem 1 is given in Appendix A.

Proposition 2.1.1: Let $X : \Omega \rightarrow (0, +\infty)$ be a continuous random variable and let the two functions of theorem 1

in Appendix A to be $h_1(x) = \frac{(1 + \gamma x^{-\beta})^{\frac{\alpha}{\gamma} + 1}}{\alpha \left[1 + 2\lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right]}$ and $h_2(x) = \frac{(\beta + 1)(1 + \gamma x^{-\beta})^{\frac{\alpha}{\gamma} + 1}}{\alpha \beta x \left[1 + 2\lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right]}$ $x > 0$. The

random variable X has pdf (5), if and only if the function $p(x)$ in Theorem 1 has the form $p(x) = x$.

Proof:

Let the random variable X have pdf (5), then

$$(1 - F(x))E[h_1(X)|X \geq x] = x^{-\beta} \text{ and } (1 - F(x))E[h_2(X)|X \geq x] = x^{-(\beta+1)},$$

$$\frac{E[h_1(X)|X \geq x]}{E[h_2(X)|X \geq x]} = p(x) = x, \quad x > 0.$$

Conversely, if $p(x)$ has the given form, then $p'(x) = 1$. The differential equation

$$s'(x) = \frac{p'(x)h_2(x)}{p(x)h_2(x) - h_1(x)} = (\beta + 1)x^{-1}$$

has solution $s(x) = (\beta + 1)\ln x$.

Therefore, in light of theorem 1, the random variable X has pdf (5).

Corollary 2.1.1: Let $X : \Omega \rightarrow (0, +\infty)$ be a continuous random variable and let

$$h_2(x) = \frac{(\beta + 1)(1 + \gamma x^{-\beta})^{\frac{\alpha}{\gamma} + 1}}{\alpha \beta x \left[1 + 2\lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right]}$$
 $x > 0$. The pdf of X is (5), if and only if there exist functions

$p(x)$ and $h_1(x)$ (defined in Theorem 1), satisfying the differential equation

$$\frac{p'(x)}{p(x)h_2(x) - h_1(x)} = \frac{\alpha \beta \left(1 + 2\lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right)}{(1 + \gamma x^{-\beta})^{\frac{\alpha}{\gamma} + 1}}, \quad x > 0. \quad (11)$$

Remark 2.1.1: The general solution of (11) is

$$p(x) = x^{\beta+1} \int \left(-\alpha \beta x^{-\beta-1} (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1} \left[1 + 2\lambda (1 + \gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right] \right) h_1(x) dx + C,$$

where C is a constant.

2.2 Characterization via Doubly Truncated Moments

Here TMBIII distribution is characterized via doubly truncated moments.

Proposition 2.2.1: Let $X: \Omega \rightarrow (0, +\infty)$ be a continuous random variable. Then X has pdf (5) if and only if

$$E \left[\left[1 + 2\lambda \left(1 + \gamma X^{-\beta} \right)^{-\frac{\alpha}{\gamma}} \right]^{-1} \middle| x < X < y \right] = \frac{\left(1 + \gamma y^{-\beta} \right)^{-\frac{\alpha}{\gamma}} - \left(1 + \gamma x^{-\beta} \right)^{-\frac{\alpha}{\gamma}}}{\left[\bar{F}(x) - \bar{F}(y) \right]}, \quad x > 0, y > 0. \quad (12)$$

Proof:

For random variable X with pdf (5), we have

$$\begin{aligned} E \left[\left[1 + 2\lambda \left(1 + \gamma X^{-\beta} \right)^{-\frac{\alpha}{\gamma}} \right]^{-1} \middle| x < X < y \right] &= \frac{\int_y^x \left[1 + 2\lambda \left(1 + \gamma u^{-\beta} \right)^{-\frac{\alpha}{\gamma}} \right]^{-1} f(u) du}{\left[\bar{F}(x) - \bar{F}(y) \right]}, \\ &= \frac{\int_y^x \left[1 + 2\lambda \left(1 + \gamma u^{-\beta} \right)^{-\frac{\alpha}{\gamma}} \right]^{-1} \alpha \beta u^{-\beta-1} \left(1 + \gamma u^{-\beta} \right)^{-\frac{\alpha}{\gamma}-1} \left[1 + 2\lambda \left(1 + \gamma u^{-\beta} \right)^{-\frac{\alpha}{\gamma}} \right] du}{\left[\bar{F}(x) - \bar{F}(y) \right]}, \\ &= \frac{\int_y^x \alpha \beta u^{-\beta-1} \left(1 + \gamma u^{-\beta} \right)^{-\frac{\alpha}{\gamma}-1} du}{\left[F(y) - F(x) \right]} = \frac{\left(1 + \gamma y^{-\beta} \right)^{-\frac{\alpha}{\gamma}} - \left(1 + \gamma x^{-\beta} \right)^{-\frac{\alpha}{\gamma}}}{\left[\bar{F}(x) - \bar{F}(y) \right]}. \end{aligned}$$

Conversely, if (12) holds, then

$$\frac{\int_x^y \left[1 + 2\lambda \left(1 + \gamma u^{-\beta} \right)^{-\frac{\alpha}{\gamma}} \right]^{-1} f(u) du}{\left[\bar{F}(x) - \bar{F}(y) \right]} = \frac{\left(1 + \gamma y^{-\beta} \right)^{-\frac{\alpha}{\gamma}} - \left(1 + \gamma x^{-\beta} \right)^{-\frac{\alpha}{\gamma}}}{\left[\bar{F}(x) - \bar{F}(y) \right]},$$

$$\int_x^y \left[1 + 2\lambda \left(1 + \gamma u^{-\beta} \right)^{-\frac{\alpha}{\gamma}} \right]^{-1} f(u) du = \left(1 + \gamma y^{-\beta} \right)^{-\frac{\alpha}{\gamma}} - \left(1 + \gamma x^{-\beta} \right)^{-\frac{\alpha}{\gamma}}.$$

Differentiating with respect to y , we have

$$\left[1 + 2\lambda \left(1 + \gamma y^{-\beta} \right)^{-\frac{\alpha}{\gamma}} \right]^{-1} f(y) = \alpha \beta y^{-\beta-1} \left(1 + \gamma y^{-\beta} \right)^{-\frac{\alpha}{\gamma}-1},$$

or

$$f(y) = \alpha \beta y^{-\beta-1} (1 + \gamma y^{-\beta})^{-\frac{\alpha}{\gamma}-1} \left[1 + 2\lambda (1 + \gamma y^{-\beta})^{-\frac{\alpha}{\gamma}} \right] \quad y > 0, \quad |\lambda| \leq 1,$$

which is pdf of TMBIII distribution.

2.3 Characterization Based On Hazard Function

Here TMBIII distribution is characterized through hazard rate function.

Definition 2.3.1: Let $X: \Omega \rightarrow (0, +\infty)$ be a continuous random variable with cdf $F(x)$ and pdf $f(x)$. The hazard function, h_F , of a twice differentiable distribution function F , can be defined as the one satisfying the differential equation

$$\frac{d}{dx} [\ln f(x)] = \frac{h'_F}{h_F} - h_F. \quad (13)$$

Proposition 2.3.1: Let $X: \Omega \rightarrow (0, +\infty)$ be a continuous random variable. Then, X has pdf (5), if and if its hazard function satisfies the differential equation

$$\left[\begin{aligned} &h'(x) \left(1 - \left((1+\lambda)(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} - \lambda(1+\gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}} \right) \right) + \\ &h(x) \left(-\alpha \beta x^{-\beta-1} (1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1} \left((1+\lambda) - 2\lambda(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right) \right) \end{aligned} \right] = \alpha \beta x^{-\beta-1} (1+\gamma x^{-\beta})^{-1} \left[\begin{aligned} &2\alpha \beta \lambda x^{-\beta-1} (1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1} \\ &- \left((\beta+1)x^{-1} - \frac{\gamma}{\alpha} (1+\gamma x^{-\beta})^{-1} \right) \left(1 + 2\lambda (1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right) \end{aligned} \right], x > 0.$$

Proof:

If X has pdf (5), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\begin{aligned} \frac{d}{dx} \left[h(x) \left(1 - \left((1+\lambda)(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} - \lambda(1+\gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}} \right) \right) \right] &= \frac{d}{dx} \left[\alpha \beta x^{-\beta-1} (1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1} \left(1 + 2\lambda (1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right) \right] \\ \frac{f(x)}{1-F(x)} &= \frac{\alpha \beta x^{-\beta-1} (1+\gamma x^{-\beta})^{-1} \left[1 + 2\lambda (1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right]}{\left(1 - \left((1+\lambda)(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} - \lambda(1+\gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}} \right) \right)}. \end{aligned}$$

After integrating the above equation and simplifying, we arrive at

$$F(x) = (1+\lambda)(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} - \lambda(1+\gamma x^{-\beta})^{-\frac{2\alpha}{\gamma}}, \quad x > 0, \quad |\lambda| \leq 1,$$

which is cdf of TMBIII distribution.

2.4 Characterization Based On Reverse Hazard Function

Here TMBIII distribution is characterized via reverse hazard rate function.

Definition 2.4.1: Let $X:\Omega \rightarrow (0, +\infty)$ be a continuous random variable with cdf $F(x)$ and pdf $f(x)$. The reverse hazard function, r_F , of a twice differentiable distribution function F , can be defined as the one satisfying the differential equation

$$\frac{d}{dx} [\ln f(x)] = \frac{r'_F}{r_F} + r_F. \quad (14)$$

Proposition 2.4.1: Let $X:\Omega \rightarrow (0, \infty)$ be a continuous random variable. Then, X has pdf (5), if and only if its reverse hazard function $r_F(x)$, satisfies the differential equation

$$r'_F(x) \left[(1+\lambda) - \lambda(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right] - r_F(x) \left(\alpha \beta \lambda x^{-\beta-1} (1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}-1} \right) = \frac{\beta x^{-\beta-1}}{(1+\gamma x^{-\beta})} \left(-\alpha(\beta+1)x^{-1} + \gamma(1+\gamma x^{-\beta})^{-1} \right) \left(1+2\lambda(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right). \quad (15)$$

Proof If X has pdf as in (5), then (15) holds. Now if (15) holds, then

$$\begin{aligned} \frac{d}{dx} \left(r_F(x) \left((1+\lambda) - \lambda(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right) \right) &= \frac{d}{dx} \left[\alpha \beta x^{-\beta-1} (1+\gamma x^{-\beta})^{-1} \left[1+2\lambda(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right] \right], \\ \frac{f(x)}{F(x)} &= \frac{\alpha \beta x^{-\beta-1} (1+\gamma x^{-\beta})^{-1} \left[1+2\lambda(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}} \right]}{(1+\lambda) - \lambda(1+\gamma x^{-\beta})^{-\frac{\alpha}{\gamma}}}. \end{aligned}$$

After integrating the above equation and simplifying, we obtain

$$F(x) = (1+\lambda) \left(1+\gamma x^{-\beta} \right)^{\frac{\alpha}{\gamma}} - \lambda \left(1+\gamma x^{-\beta} \right)^{\frac{2\alpha}{\gamma}}, \quad x > 0, \quad |\lambda| \leq 1,$$

which is cdf of TMBIII distribution.

2.5 Characterization Based On Elasticity Function

Here TMBIII distribution is characterized through elasticity function.

Definition 2.5.1: Let $X:\Omega \rightarrow (0, +\infty)$ be a continuous random variable with cdf $F(x)$ and pdf $f(x)$. The elasticity function, e_F , of a twice differentiable distribution function F , can be defined as the one satisfying the differential equation

$$\frac{d}{dx} [\ln f(x)] = \frac{e'_F}{e_F} + \frac{e_F}{x} - \frac{1}{x}. \quad (16)$$

Proposition 2.5.1: Let $X:\Omega \rightarrow (0, +\infty)$ be continuous random variable. The pdf of X is (5), if and only if, its elasticity function, $e_F(x)$, satisfies the first order differential equation

$$e'_F(x) \left[(1+\lambda) - \lambda(1+\gamma x^{-\beta})^{\frac{\alpha}{\gamma}} \right] - \alpha\beta\lambda x^{-\beta-1} (1+\gamma x^{-\beta})^{\frac{\alpha}{\gamma}-1} e_F(x) = \alpha\beta^2 x^{-\beta-1} (1+\gamma x^{-\beta})^{-2} \left[2\alpha\lambda x^{-\beta} (1+\gamma x^{-\beta})^{\frac{\alpha}{\gamma}} - \left(1+2\lambda(1+\gamma x^{-\beta})^{\frac{\alpha}{\gamma}} \right) \right]. \quad (17)$$

Proof If X has pdf (5), then (17) holds. Now if (17) holds, then

$$\frac{d}{dx} \left[\left((1+\lambda) - \lambda(1+\gamma x^{-\beta})^{\frac{\alpha}{\gamma}} \right) e_F(x) \right] = \alpha\beta \frac{d}{dx} \left[\frac{1+2\lambda(1+\gamma x^{-\beta})^{\frac{\alpha}{\gamma}}}{x^\beta (1+\gamma x^{-\beta})} \right],$$

or

$$e_F(x) = \frac{\alpha\beta x^{-\beta} (1+\gamma x^{-\beta})^{\frac{\alpha}{\gamma}-1} \left[1+2\lambda(1+\gamma x^{-\beta})^{\frac{\alpha}{\gamma}} \right]}{\left[(1+\lambda)(1+\gamma x^{-\beta})^{\frac{\alpha}{\gamma}} - \lambda(1+\gamma x^{-\beta})^{\frac{2\alpha}{\gamma}} \right]},$$

which is the elasticity function of TMBIII distribution.

3. CONCLUDING REMARKS

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with the characterizations of the TMBIII distribution. These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) doubly truncated moments; (iii) hazard function; (iv) reverse hazard function and (v) elasticity function.

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Appendix A

Theorem 1: Let (Ω, F, P) be a probability space for given interval $[d_1, d_2]$ with $-\infty < d_1 < d_2 < d_2 + \infty$. Let $X : \Omega \rightarrow [d_1, d_2]$ be a continuous random variable with distribution function F. Let real functions h_1 and h_2 be

continuous on $[d_1, d_2]$ such that $\frac{E[h_1(X)|X \geq x]}{E[h_2(X)|X \geq x]} = p(x)$ is real function in simple form. Assume that

$h_1, h_2 \in C([d_1, d_2])$, $p(x) \in C^2([d_1, d_2])$ and F is two times continuously differentiable and strictly monotone function on $[d_1, d_2]$ ($[d_1, d_2]$). Finally, assume that the equation $h_2 p(x) = h_1$ has no real solution

in $[d_1, d_2]$. Then $F(x) = \int_{\ln k}^x K \left| \frac{p'(t)}{p(t)h_2(t) - h_1(t)} \right| \exp(-s(t)) dt$ is obtained from the functions $h_1, h_2, p(t)$

and $s(t)$, where $s(t)$ is obtained from equation $s'(t) = \frac{p'(t)h_2(t)}{p(t)h_2(t) - h_1(t)}$ and K is a constant, adopted such that

$$\int_{d_1}^{d_2} dF = 1.$$

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