UNDER PEER REVIEW

Theoretical analysis of a new five parameter model: the Kumaraswamy exponentiated U-quadratic distribution

Original Research Article

Abstract

In this paper, a new lifetime model called Kumaraswamy exponentiated U-quadratic (KwEUq) distribution is proposed. Several mathematical and statistical properties are derived such as the quantile function, moments, moment generating function, order statistics, probability weighted moments, Shannon entropy and Renyi entropy. We also found that the usual maximum likelihood estimates (MLEs) fail to hold for the Kumaraswamy Exponentiated U-quadratic (KwEUq) distribution. Two alternative methods are suggested for the parameter estimation, the alternative maximum likelihood estimation (AMLE) and modified maximum likelihood estimation (MMLE). Finally, we compare the fit of the two methods by the use of real data as measured by Kolmogorov Smirnov test, showing that both methods are suitable for the estimation of the parameters of KwEUq.

Keywords: Kumaraswamy-G distribution, exponentiated Uqadrtic distribution, Moments, Maximum likelihood estimation.

2010 Mathematics Subject Classification: 62E05; 62F10

1 Introduction

In statistical modeling, problems that occur in practical applications encountered researchers to propose new models, so that lifetime data set can be investigated better. In another word, there is a high need to introduce useful models to analyze the real-life phenomenon which arises in the fields of studies such as physics, medical science, computer science, industry, communication, biology, engineering, public health among others. For example, the well-known classical distributions such as exponential, Weibull are not able to show wide flexibility in modeling data with U-shape or bimodal density.

The method of generalization of distributions by the Kumaraswamy-G generator proposed by (8) is one of the popular technique in distribution theory and were considered by many authors in recent years. For instance, the Kumaraswamy-Weibull distribution (9), Kumaraswamy binomial (22), Kumaraswamy generalized gamma (13), the Kumaraswamy-Gumbel (10), Kumaraswamy generalized half-normal (11), Kumaraswamy log-logistic (14), Kumaraswamy-Birnbaum-Saunders (27), Kumaraswamy

UNDER PEER REVIEW

inverse Weibull (28), Kumaraswamy double inverse exponential (4), Kumaraswamy power series (6), Kumaraswamy-Pareto (7), the Kumaraswamy generalized linear failure rate (15), Kumaraswamy exponentiated Pareto (16), Kumaraswamy quasi-Lindley (17), Kumaraswamy generalized Pareto (24), Kumaraswamy-Burr XII distribution (26), Kumaraswamy generalized exponentiated Pareto (29), Kumaraswamy generalized Lomax (30), Kumaraswamy geometric (2), Kumaraswamy modified Weibull (12), Kumaraswamy half-Cauchy distribution due to Ghosh (2014) by (18), the Kumaraswamy generalized Rayleigh (19), Kumaraswamy-Dagum distribution (20), Kumaraswamy inverse Rayleigh (21), Kumaraswamy inverse exponential (25), Kumaraswamy-Kumaraswamy (1), the Kumaraswamy transmuted exponentiated modified Weibull (3), Kumaraswamy odd log-logistic (5), Kumaraswamy skew-normal (23).

The Kumaraswamy-G generator of distributions is defined according to (8) as follows.

Definition 1.1. Let $G(x)$ be any base line cumulative distribution function and $g(x)$ be the density function of $G(x)$, let $\lambda,\ \delta > 0$, then, the cumulative distribution function (cdf) and density function of the Kumaraswamy-G generator of distributions satisfy the relationship

$$F^*(x) = 1 - [1 - G(x)^{\lambda}]^{\delta}. \tag{1.1}$$

and

$$f(x)^* = \delta\lambda g(x) G(x)^{\lambda-1}[1 - G(x)^{\lambda}]^{\delta-1}, \tag{1.2}$$

respectively.

In this case, let the baseline distribution $G(x)$ be the cumulative distribution function of the exponentiate U-quadratic distribution proposed by (33), given by

$$G(x) = \left(\frac{\alpha}{3}\right)^{\theta} ((x-\beta)^3 + (\beta - a)^3)^{\theta}, \quad x \in [a, b] \tag{1.3}$$

where $\theta > 0$ is the shape parameter, a the lower limit, b as the upper limit and $a \in (-\infty, \infty)$, $b \in (a, \infty)$ or $\alpha \in (0, \infty)$, $\beta \in (-\infty, \infty)$ such that $\alpha = \frac{12}{(b-a)^3}$ and $\beta = \frac{b+a}{2}$. The density function of EUq is given as

$$g(x) = \theta\alpha^{\theta} 3^{1-\theta}(x-\beta)^2((x-\beta)^3 + (\beta - a)^3)^{\theta-1}. \tag{1.4}$$

The main objective of this paper is to propose a new probability model, called, Kumaraswamy exponentiated U-quadratic (KwEUq) distribution and studied some of its important mathematical and statistical properties. In addition, the to investigate the parameters of the new model.

The rest of the paper is organized as follows. In section 2, the exponentiated generalized U-quadratic (EGUq) distribution and some important properties are discussed. In section 3, the maximum likelihood and alternative maximum likelihood estimation methods of the new model are analyzed. Conclusions in Section 4.

2 The Kumaraswamy Exponentiated U-quadratic Distribution

The cumulative distribution of the kumaraswamy exponentiated u-quadratic (KwEUq) distribution can be defined by substituting $G(x)$ into equation (1.1), thus

$$F(x) = 1 - \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x-\beta)^3 + (\beta - a)^3)^{\theta\lambda}\right]^{\delta}, \tag{2.1}$$

and the corresponding density function is defined by

$$f(x) = \delta\lambda\theta\alpha^{\theta\lambda} 3^{1-\theta\lambda}(x-\beta)^2((x-\beta)^3 + (\beta - a)^3)^{\theta\lambda-1}\left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x-\beta)^3 + (\beta - a)^3)^{\theta\lambda}\right]^{\delta-1}, \tag{2.2}$$

UNDER PEER REVIEW

where θ, $\lambda > 0$ and $\delta > 0$ are the shape parameters. The survival function $s(x)$ and the hazard rate function $h(x)$ of the KwEUq distribution are respectively given by

$$s(x) = 1 - \left[1 - \left(1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x-\beta)^3 + (\beta - a)^3)^{\theta\lambda}\right)^{\delta}\right], \tag{2.3}$$

and

$$h(x) = \frac{\delta\lambda\theta\alpha^{\theta\lambda}3^{1-\theta\lambda}(x-\beta)^2((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda-1}\left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda}\right]^{\delta-1}}{1 - \left[1 - \left(1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda}\right)^{\delta}\right]}. \tag{2.4}$$

Figure1, 2 and 3 display the plot of the cumulative distribution function, probability density function and hazard rate function of the Kumaraswamy exponentiated U-quadratic distribution for some parameter values.

Special Cases :

The special cases of the KwEUq distribution for some selected parameters values are:

1. When $\delta = 1$, we obtain Exponentiated u-quadratic distribution (EUq) with power parameter $\theta\lambda$.
2. If we take $\delta = 1$ and $\lambda = 1$, we also obtain Exponentiated u-quadratic distribution (EUq) but with power parameter θ.
3. Setting $\theta = 1$, we get Kumaraswamy u-quadratic distribution (KwUq).
4. If $\theta = 1, \lambda = 1$ and $\delta = 1$, we get U-quadratic distribution (Uq).

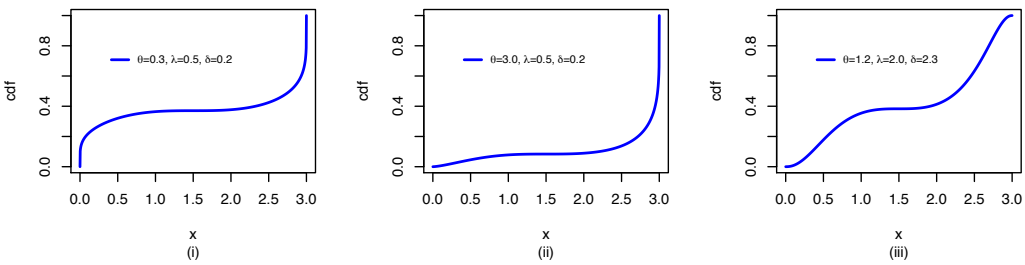

Figure 1: plot of the cumulative distribution function of the KwEUq distribution some different values of parameter

2.1 Quantile and Moments

The quantile function of the KwEUq distribution can be obtained by inverting (2.1) and can be used to determine some measures such as median and other percentiles. The quantile function can also be used to generate a random data which follow according to the kumaraswamy exponentiated u-quadratic distribution (KwEUq). The quantile function of KwEUq distribution is obtained as

$$\Omega(p) = \left[\frac{(1-(1-p)^{1/\delta})^{1/\theta\lambda}}{\left(\frac{\alpha}{3}\right)} - (\beta - a)^3\right]^{1/3} + \beta. \tag{2.5}$$

UNDER PEER REVIEW

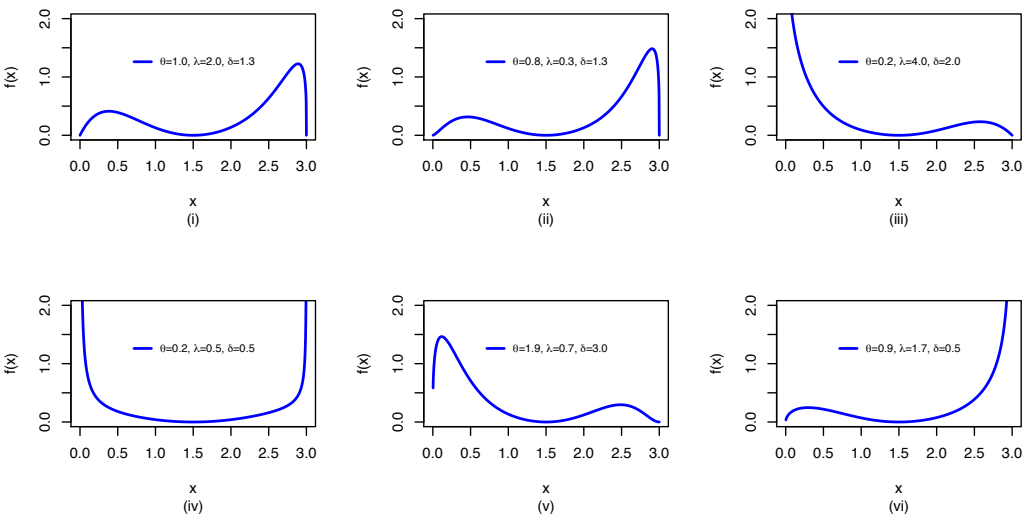

Figure 2: plot of the probability density function of the KwEUq distribution for some parameter values

Proposition 2.1. *Let* $Y \sim U(0,1)$*, then* $X = \sqrt[3]{\frac{(1-(1-y)^{1/\delta})^{1/\theta\lambda}}{\left(\frac{a}{3}\right)} - (\beta - a)^3} + \beta$ *is the random variable with* $KwEUq(\theta, \lambda, \delta, a, b)$*, where* $U(0,1)$ *is uniform distribution.*

The following lemma 2.1 is very useful in computations of several important properties of the Kumaraswamy exponentiated u-quadratic (KwEUq) distribution.

Lemma 2.1. *For* w_1, w_2, w_3 *and* $w_4 \in \mathbb{R}$*, let*

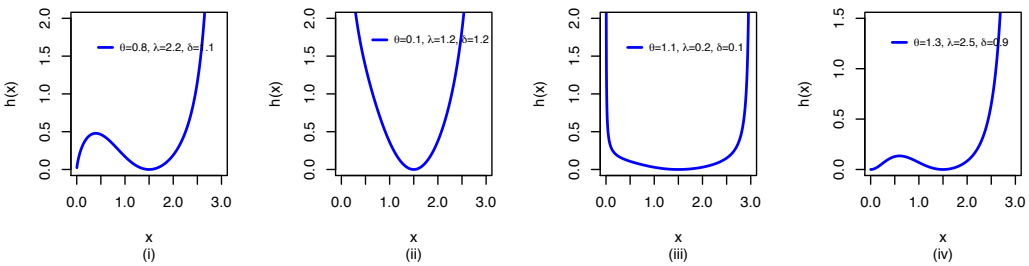

Figure 3: plot of the hazard function of the KwEUq distribution for some parameter values

UNDER PEER REVIEW

$$J(w_1,w_2,w_3,w_4)=\int_a^b x^{w_1}(x-\beta)^{w_2}((x-\beta)^3+(\beta-a)^3)^{w_3} \times\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{w_4}dx \tag{2.6}$$

then,

$$J(w_1,w_2,w_3,w_4)=\sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\Phi_{i,k}(t)\left[\frac{b^{w_1+k+1}-a^{w_1+k+1}}{w_1+k+1}\right]. \tag{2.7}$$

where $\Phi_{i,k}(t)=\binom{w_4}{i}\binom{w_3+\theta\lambda i}{t}\binom{w_2+3t}{k}(-1)^{i+w_2+3t-k}\alpha^{\theta\lambda i}3^{-\theta\lambda i}\beta^{w_2+3t-k}(\beta-a)^{3(w_3+\theta\lambda i-t)}$ and $x\in(a,2\beta-a)\cap(-\beta,\beta)$.

Proof. First, consider

$$J(w_1,w_2,w_3,w_4)=\int_a^b x^{w_1}(x-\beta)^{w_2}((x-\beta)^3+(\beta-a)^3)^{w_3} \times\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{w_4}dx,$$

applying the generalized binomial expansion of

$$\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{w_4}=\sum_{i=0}^{\infty}\binom{w_4}{i}(-1)^i\left(\frac{\alpha}{3}\right)^{\theta\lambda i}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda i}dx$$

we obtain

$$J(w_1,w_2,w_3,w_4)=\sum_{i=0}^{\infty}\binom{w_4}{i}(-1)^i\alpha^{\theta\lambda i}3^{-\alpha\lambda i}\int_a^b x^{w_1}(x-\beta)^{w_2}((x-\beta)^3+(\beta-a)^3)^{w_3-\theta\lambda i}dx,$$

similarly, by the expansion of

$$((\beta-a)^3+(x-\beta)^3)^{w_3-\theta\lambda i}=(\beta-a)^{3(w_3-\theta\lambda i)}\sum_{t=0}^{\infty}\binom{w_3+\theta\lambda i}{t}\left(\frac{x-\beta}{\beta-a}\right)^{3t}$$

for $|\frac{x-\beta}{\beta-a}|<1$, we get

$$J(w_1,w_2,w_3,w_4)=\sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\binom{w_4}{i}\binom{w_3+\theta\lambda i}{t}(-1)^i\alpha^{\theta\lambda i}3^{-\alpha\lambda i}(\beta-a)^{3(w_3-\theta\lambda i)} \times\int_a^b\frac{x^{w_1}}{(\beta-a)^{3t}}(x-\beta)^{w_2+3t}dx$$

again expanding

$$(x-\beta)^{w_2+3t}=(-\beta)^{w_2+3t}\left(1+\left(\frac{x}{-\beta}\right)\right)^{w_2+3t} =\sum_{k=0}^{\infty}\binom{w_2+3t}{k}(-\beta)^{w_2+3t-k}x^k$$

UNDER PEER REVIEW

for $|\frac{x}{-\beta}| < 1$ we obtained

$$J(w_1, w_2, w_3, w_4) = \sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\sum_{k=0}^{\infty} \binom{w_4}{i}\binom{w_3+\theta\lambda i}{t}\binom{w_2+3t}{k}(-1)^{i+w_2+3t-k}\alpha^{\theta\lambda i}3^{-\theta\lambda i}\beta^{w_2+3t-k}$$
$$\times (\beta-a)^{3(w_3+\theta\lambda i-t)}\int_a^b x^{w_1+k}dx$$

finally, by integration, we get

$$J(w_1, w_2, w_3, w_4) = \sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\Phi_{i,k}(t)\left[\frac{b^{w_1+k+1}-a^{w_1+k+1}}{w_1+k+1}\right].$$

where $\Phi_{i,k}(t) = \binom{w_4}{i}\binom{w_3+\theta\lambda i}{t}\binom{w_2+3t}{k}(-1)^{i+w_2+3t-k}\alpha^{\theta\lambda i}3^{-\theta\lambda i}\beta^{w_2+3t-k}(\beta-a)^{3(w_3+\theta\lambda i-t)}$. □

Table 1 provide some rumerical values of $J(w_1, w_2, w_3, w_4)$ with absolute error in paranthesis for some values of parameters and $w_1, w_2, w_3,\ w_4$. The values are obtained numerically using `R-software`.

Table 1: Numerical values of $J(w1, w2, w3, w4)$ for some values of parameter.

a	b	θ	λ	δ	$w1$	$w2$	$w3$	$w4$	$J(w1, w2, w3, w4)$
0.0	5.0	1.0	4.0	5.0	7.0	0.2	0.5	0.5	1.9735 (2.2e^{-5})
0.0	5.0	1.0	4.0	5.0	0.7	0.2	0.5	0.5	1.1965 (9.0e^{-5})
0.0	5.0	1.5	0.4	5.0	0.7	1.2	1.5	2.5	0.2157 (3.7e^{-8})
0.0	5.0	1.5	0.2	0.5	1.7	1.2	1.5	2.5	0.0623 (6.7e^{-7})
0.0	5.0	1.4	0.2	1.2	1.0	2.0	1.5	2.0	0.1114 (3.6e^{-7})
0.0	5.0	0.1	0.3	1.8	1.3	4.0	1.5	3.1	0.0815 (3.8e^{-7})
0.0	5.0	3.1	2.3	3.8	9.3	2.1	7.5	6.1	1.0993 (1.1e^{-6})
-2.0	3.0	3.0	0.3	0.8	0.3	0.1	0.5	0.1	1.6828 (1.9e^{-14})
-5.0	5.0	3.0	0.3	1.1	0.3	2.0	2.5	1.1	0.3105 (3.4e^{-15})
-10	15	1.9	5.4	2.0	0.07	0.2	1.5	0.5	1.9765 (2.2e^{-14})

By considering (2.1), we can obtain the rth moment and moment generating function of the KwEUq distribution which are very useful in the study of characteristics and features of a distribution. The rth moments of the KwEUq distribution can be obtained using

$$E[X^r] = \int_a^b x^r f(x)dx = \delta\lambda\theta\alpha^{\theta\lambda}3^{1-\theta\lambda}\int_a^b x^r(x-\beta)^2((x-\beta)^3+(\beta-a)^3)^{\theta\lambda-1}$$
$$\times\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\delta-1}dx,$$

hence,

$$E[X^r] = \frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}}J(r, 2, \theta\lambda-1, \delta-1). \tag{2.8}$$

The moment generating function (mgf) of the KwEUq distribution can be computed directly using $M_X(t) = E[e^{tX}] = \sum_{r=0}^{\infty}\frac{t^r}{r!}E[X^r]$, thus, by substituting (2.8) we have

$$M_X(t) = \sum_{r=0}^{\infty}\frac{\delta\lambda\theta\alpha^{\theta\lambda}t^r}{3^{\theta\lambda-1}r!}J(r, 2, \theta\lambda-1, \delta-1).$$

UNDER PEER REVIEW

2.2 Entropy

An entropy of a random variable X can be defined as a measure of variation of uncertainty. In this section, we consider the two most important and popular entropies known as the Shannon and Renyi entropies. The Shannon entropy measure of a random variable X with KwEUq distribution can be defined by $E[-\log f(x)]$, then, the Shannon entropy of X can be computed by considering the lemma 2.2 as follows.

Lemma 2.2. *Let X be a random variable distributed according to the KwEUq distribution with pdf given by (2.2), then,*

$$E[\log(x-\beta)] = \frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2+t,\theta\lambda-1,\delta-1)|_{t=0}, \tag{2.9}$$

$$E[\log((x-\beta)^3+(\beta-a)^3)] = \frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2,\theta\lambda+t-1,\delta-1)|_{t=0}, \tag{2.10}$$

$$E\left[\log\left(1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right)\right] = \frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2,\theta\lambda-1,\delta+t-1)|_{t=0}. \tag{2.11}$$

Proof. We consider the first part

$$\begin{aligned}
E[\log(x-\beta)] &= \int_a^b \log(x-\beta)f(x)dx \\
&= \frac{\partial}{\partial t}\int_a^b (x-\beta)^t f(x)dx|_{t=0} \\
&= \delta\lambda\theta\alpha^{\theta\lambda}3^{1-\theta\lambda}\frac{\partial}{\partial t}\int_a^b (x-\beta)^{2+t}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda-1} \\
&\times\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\delta-1}dx|_{t=0},
\end{aligned}$$

hence,

$$E[\log(x-\beta)] = \frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2+t,\theta\lambda-1,\delta-1)|_{t=0},$$

where $J(.,.,.,.,.,)$ is given by (2.7). Equation (2.10) and (2.11) follow similar. □

Proposition 2.2. *Let $X \sim KwEUq(\theta,\lambda,\delta,a,b)$, then, the Shannon entropy of X can be expressed as*

$$\begin{aligned}
E[-\log f(x)] = &-\log\left(\frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}}\right) - \frac{2\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2+t,\theta\lambda-1,\delta-1)|_{t=0} \\
&- \frac{\delta\lambda\theta\alpha^{\theta\lambda}(\theta\lambda-1)}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2,\theta\lambda+t-1,\delta-1)|_{t=0} \\
&- \frac{\delta\lambda\theta\alpha^{\theta\lambda}(\delta-1)}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2,\theta\lambda-1,\delta+t-1)|_{t=0}.
\end{aligned}$$

Proof.

$$\begin{aligned}
E[-\log f(x)] = &-E\left[\log(\delta\lambda\theta\alpha^{\theta\lambda}3^{1-\theta\lambda})\right] - 2E[\log(x-\beta)] \\
&- (\theta\lambda-1)E[\log((x-\beta)^3+(\beta-a)^3)] \\
&- (\delta-1)E\left[\log\left(1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right)\right],
\end{aligned}$$

we obtain the result by considering the lemme 2.2. □

UNDER PEER REVIEW

The Renyi entropy of a random variable X is defined by $I_{R(\rho)} = \frac{1}{1-\rho}\log\left[\int_a^b f(x)^\rho dx\right]$,where $\rho > 0$ and $\rho \neq 1$. The Renyi entropy of X having KwEUq distribution can be obtained as follows.

$$\int_a^b f^\rho(x)dx = \int_a^b \delta^\rho\lambda^\rho\theta^\rho\alpha^{\rho\theta\lambda}3^{\rho(1-\theta\lambda)}(x-\beta)^{2\rho}((x-\beta)^3+(\beta-a)^3)^{\rho(\theta\lambda-1)} \times\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\rho(\delta-1)}dx \tag{2.12}$$

by applying lemma 2.7 the integral become

$$\int_a^b f^\rho(x)dx = \frac{\delta^\rho\lambda^\rho\theta^\rho\alpha^{\rho\theta\lambda}}{3^{\rho(\theta\lambda-1)}}J(0,2\rho,\rho(\theta\lambda-1),\rho(\delta-1)). \tag{2.13}$$

Hence, the Renyi entropy of X is

$$I_{R(\rho)} = \frac{1}{1-\rho}\log\left[\frac{\delta^\rho\lambda^\rho\theta^\rho\alpha^{\rho\theta\lambda}}{3^{\rho(\theta\lambda-1)}}J(0,2\rho,\rho(\theta\lambda-1),\rho(\delta-1))\right].$$

2.3 Order Statistics

Order statistics play an essencial role in reliability and quantity control, it is also an inportant tool in non parametric statistic. Let $X_1, X_2, X_3, \ldots, X_n$ be a simple random sample obtained from KwEUq distribution, let $X_{1:n} \leq \cdots \leq X_{n:n}$, be the order statistics obtained from this random sample, then for $j = 1, 2, 3, \ldots, n$, the corresponding pdf of $X_{j:n}$ say $f_{X_{j:n}}(x)$ is obtained as follows. The pdf of the jth order statistics can be computed from

$$f_{X_{j:n}}(x;\alpha,\beta,\lambda,\theta,\delta) = \frac{n!}{(j-1)!(n-j)!}f(x)(F(x))^{j-1}(1-F(x))^{n-j},$$

where $f(x)$ and $F(x)$ are the pdf and cdf of KwEUq distribution.

$$f_{X_{j:n}}(x;\alpha,\beta,\lambda,\theta,\delta) = \sum_{m-0}^{n-j}\frac{n!(-1)^m}{(j-1)(n-j-m)!m!}f(x)(F(x))^{j+m-1}, \tag{2.14}$$

the binomial expansion of $(F(x))^{j+m-1}$ is obtained as

$$(F(x))^{j+m-1} = \sum_{l=0}^{j+m-1}\binom{j+m-1}{l}(-1)^l\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\delta l} \tag{2.15}$$

by substituting (2.15) into (2.14), we get

$$f_{X_{j:n}}(x;\alpha,\beta,\lambda,\theta,\delta) = \sum_{m-0}^{n-j}\sum_{l=0}^{j+m-1}\frac{\binom{j+m-1}{l}(-1)^{m+l}n!\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\delta l}}{(j-1)!(n-j-m)!m!}f(x)$$

after some algebraic manipulations, we obtained the pdf as

$$f_{X_{j:n}}(x;\alpha,\beta,\lambda,\theta,\delta) = \sum_{m-0}^{n-j}\sum_{l=0}^{j+m-1}\frac{\binom{j+m-1}{l}n!(-1)^{m+l}f(x;\alpha,\beta,\lambda,\theta,\delta(l+1))}{(l+1)(j-1)!(n-j-m)!m!},$$

where $f(x;\alpha,\beta,\lambda,\theta,\delta(l+1))$ is the pdf of the KwEUq distribution with power $\delta(l+1)$. The moments of order statistics of the KwEUq distribution can be obtained using (2.3).

UNDER PEER REVIEW

2.4 Probability Weighted Moments

The PWM of X following the KwEUq distribution, say $\rho_{r,s}$, is formally defined by

$$\rho_{r,s} = E[X^r F(X)^s] = \int_a^b x^r F^s(x) f(x) dx, \tag{2.16}$$

where $F(x)$ and $f(x)$ are the cdf and pdf of the KwEUq distribution, by expansion of $F^s(x)$ as

$$F^s(x) = \sum_{u=0}^{s} \binom{s}{u} (-1)^u \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x-\beta)^3 + (\beta - a)^3)^{\theta\lambda}\right]^{\delta u} \tag{2.17}$$

and inserting (2.17) into (2.16), we have

$$\rho_{r,s} = \sum_{u=0}^{s} \binom{s}{u} (-1)^u \int_a^b x^r \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x-\beta)^3 + (\beta - a)^3)^{\theta\lambda}\right]^{\delta u} f(x) dx \tag{2.18}$$

after some algebraic manipulations, we obtained the $\rho_{r,s}$ as

$$\rho_{r,s} = \sum_{u=0}^{s} \binom{s}{u} \frac{(-1)^u}{(u+1)} \int_a^b x^r f(x; \alpha, \beta, \lambda, \theta, \delta(u+1)) dx,$$

hence,

$$\rho_{r,s} = \sum_{u=0}^{s} \binom{s}{u} \frac{(-1)^u}{(u+1)} \frac{\delta(u+1)\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}} J(r, 2, \theta\lambda - 1, \delta(u+1) - 1).$$

3 Estimation

The maximum likelihood estimation (MLEs) method is one of the most popular and commonly used estimation methods for finding the unknown parameters of a distribution. However, we found that there is an irregularity in the log-likelihood function of the KwEUq at the minimum and maximum order statistics, thus, the usual maximum likelihood fail to exist. Due to these, the use of Alternative Maximum Likelihood and Modified Maximum Likelihood Method are employed.

3.1 Maximum Likelihood Estimation

Let $X_1, X_2, X_3, \ldots, X_n$ be an independant and identically distributied random sample obtained from KwEUq distribution, let $X_{1:n} \leq \cdots \leq X_{n:n}$, be the order statistics from this sample. Let $\Theta = (\delta, \theta, \lambda)^T$, let the approximate maximum likelihood estimates be $\hat{\Theta}_{MLE} = (\hat{\delta}, \hat{\theta}, \hat{\lambda})$, then, the the likelihood function $l(\Theta)$ of KwEUq distribution for complete data set is given by

$$\begin{aligned}\log l(\Theta) &= n\log\delta + n\log\lambda + n\log\theta + \theta\lambda n\log\alpha + (1-\theta\lambda)n\log 3 \\ &+ 2\sum_{i=1}^{n}\log(x_i - \beta) + (\theta\lambda - 1)\sum_{i=1}^{n}\log((x_i-\beta)^3 + (\beta - a)^3) \\ &+ (\delta - 1)\sum_{i=1}^{n}\log\left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3 + (\beta-a)^3)^{\theta\lambda}\right].\end{aligned} \tag{3.1}$$

The first partial derivative of $\log l(\Theta)$ with respect to δ, θ and λ are respectively given by

$$\frac{\partial(\log l(\Theta))}{\partial\delta} = \frac{n}{\delta} + \sum_{i=1}^{n}\log\left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3 + (\beta-a)^3)^{\theta\lambda}\right] \tag{3.2}$$

UNDER PEER REVIEW

$$\frac{\partial(\log l(\Theta))}{\partial\theta} = \frac{n}{\theta} + n\log\alpha - \lambda n\log 3 + \lambda\sum_{i=1}^{n}\log((x_i-\beta)^3+(\beta-a)^3)$$
$$+(\delta-1)\sum_{i=1}^{n}\frac{(-1)(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3))^{\theta\lambda}\ln\left(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3)\right)^{\lambda}}{\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]} \tag{3.3}$$

$$\frac{\partial(\log l(\Theta))}{\partial\lambda} = \frac{n}{\lambda} - \theta n\log 3 + (\theta-1)\sum_{i=1}^{n}\log((x_i-\beta)^3+(\beta-a)^3)$$
$$+(\delta-1)\sum_{i=1}^{n}\frac{(-1)(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3))^{\theta\lambda}\ln\left(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3)\right)^{\theta}}{\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]}. \tag{3.4}$$

We can determined $\hat{\delta}$ from (3.2) if the value of $\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}$ and $\hat{\theta}$ are known,

$$\delta = \frac{-n}{\sum_{i=1}^{n}\log\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]}. \tag{3.5}$$

The can be computed by solving (3.2), (3.3) and (3.4), but the following proposition descrived the irregularity of the MLE method.

Proposition 3.1. *Let $L(\theta,\lambda,\delta,a,b|\boldsymbol{x})$ denote the likelihood function of an independent and identically distributed (i.i.d) random sample of size $n \geq 1$ say $x_1, x_2, \cdots, x_n$, drawn from KwEUq$(\theta,\lambda,\delta,a,b)$ distribution, let $a = X_1 \leq X_2, \leq \cdots \leq X_n = b$ be the oder statistics from the sample, then, for $x_i = x_j$ or for $x_i \neq x_j$, the $\log L(\Theta)$ diverges at x_1 or x_n.*

Proof. Consider (3.1), at x_1, $\log((x_i-\beta)^3+(\beta-a)^3) = \log(0) = -\infty$, while at x_n, $\log\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right] = \log(0) = -\infty$. □

3.1.1 Alternative Maximum Likelihood Estimation

In the alternative maximum likelihood, firsly we set $a = x_1$ and $b = x_n$ the we delete all the data points corresponding to the minimum and maximum oder statistics, then applying the usual maximum likelihood technique to obtain the approximate alteranative maximum likelihood estimates, say $\hat{\Theta}_{AMLE} = (\hat{\theta}, \hat{\delta}, \hat{\lambda})$ are numerically by simultaneously solving (3.6),(3.7) and (3.8) using some mathematical package such as R-software.

$$\frac{\partial(\log l(\Theta))}{\partial\delta} = \frac{n}{\delta} + \sum_{x_i\neq x_1,x_n}^{n}\log\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right] = 0 \tag{3.6}$$

$$\frac{\partial(\log l(\Theta))}{\partial\theta} = \frac{n}{\theta} + n\log\alpha - \lambda n\log 3 + \lambda\sum_{x_i\neq x_1,x_n}^{n}\log((x_i-\beta)^3+(\beta-a)^3)$$
$$+(\delta-1)\sum_{x_i\neq x_1,x_n}^{n}\frac{(-1)(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3))^{\theta\lambda}\ln\left(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3)\right)^{\lambda}}{\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]} = 0 \tag{3.7}$$

UNDER PEER REVIEW

$$\frac{\partial(\log l(\Theta))}{\partial\lambda} = \frac{n}{\lambda} - \theta n \log 3 + (\theta-1)\sum_{x_i\neq x_1,x_n}^{n} \log((x_i-\beta)^3 + (\beta-a)^3)$$
$$+ (\delta-1)\sum_{x_i\neq x_1,x_n}^{n} \frac{(-1)(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3))^{\theta\lambda}\ln\left(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3)\right)^{\theta}}{\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]} = 0 \tag{3.8}$$

3.1.2 Modified Maximum Likelihood Estimation

For the modified Maximum Likelihood, set $a = x_1$ and $b = x_n$ then, shift the data points corresponding to the minimum and maximum order statistics by introducing appropriate τ, $\pi \in \mathbb{R}$ such that the new values correspond to x_1 and x_n say $x_1^* = x_1 + \tau$ and $x_n^* = x_n + \pi$. The approximate modified maximum likelihood estimates, say $\hat{\Theta}_{MMLE} = (\hat{\delta}, \hat{\theta}, \hat{\lambda})$ are obtained numerically by simultaneously solving (3.9), (3.10) and (3.11) when set equal to zero using some mathematical package such as R-software.

$$\frac{\partial(\log l(\Theta))}{\partial\delta} = \frac{n}{\delta} + \sum_{i=1}^{n}\log\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]|_{x_1=x_1^*,x_n=x_n^*}, \tag{3.9}$$

$$\frac{\partial(\log l(\Theta))}{\partial\theta} = \frac{n}{\theta} + n\log\alpha - \lambda n\log 3 + \lambda\sum_{i=1}^{n}\log((x_i-\beta)^3+(\beta-a)^3)$$
$$+ (\delta-1)\sum_{i=1}^{n}\frac{(-1)(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3))^{\theta\lambda}\ln\left(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3)\right)^{\lambda}}{\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]}|_{x_1=x_1^*,x_n=x_n^*}, \tag{3.10}$$

$$\frac{\partial(\log l(\Theta))}{\partial\lambda} = \frac{n}{\lambda} - \theta n\log 3 + (\theta-1)\sum_{i=1}^{n}\log((x_i-\beta)^3+(\beta-a)^3)$$
$$+ (\delta-1)\sum_{i=1}^{n}\frac{(-1)(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3))^{\theta\lambda}\ln\left(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3)\right)^{\theta}}{\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]}|_{x_1=x_1^*,x_n=x_n^*}. \tag{3.11}$$

3.2 Illustration

In this subsection, we consider one goodness of fit statistics, that is the Kolmogorov Smirnov (KS) statistics to compare the fit of the KwEUq distribution estimated by the alternative Maximum Likelihood and Modified Maximum Likelihood Estimation method using a real data set.

The data set is the lifetimes of fifty (50) devices provided by (31) and recently studied by (32),the data set are: .1 ,.2 ,1 ,1, 1 ,1 ,1, 2, 3, 6 ,7 ,11, 12, 18, 18, 18 ,18, 18, 21, 32 ,36 ,40, 45, 46, 47, 50, 55 ,60 ,63 ,63 ,67 ,67 ,67 ,67, 72, 75, 79, 82, 82 ,83 ,84 ,84, 84, 85, 85, 85, 85, 85, 86, 86.

The numerical value of the estimator and the KS are given in table 2. The result shows that both methods can be suitable for the estimation of the parameters of KwEUq distribution. Figure 4 display the plot of the histogram of the given data and the fitted KwEUq distribution estimated by the AMLE and MMLE methods.

UNDER PEER REVIEW

Table 2: Estimated values (standard error) and the numerical value of the KS and p-value

	θ	λ	δ	a	b	σ	KS	p-value
KwEUq$_{AMLE}$	0.4609	1.0797	0.3351	0.1	86	0.5	0.2921	0.000284
	(1.9431)	(4.5453)	(0.0518)					
KwEUq$_{MMLE}$	12.3257	0.0383	0.3258	0.1	86	0.5	0.2919	0.000287
	(7.2280)	(0.0245)	(0.0508)					

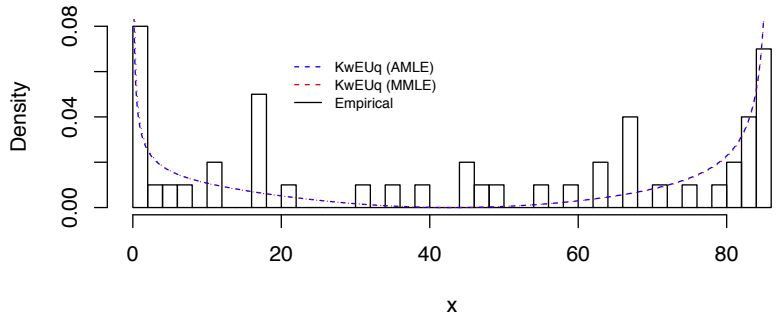

Figure 4: plot of the histogram and the fitted KwEUq distribution estimated by the AMLE and MMLE methods for the given data

4 CONCLUSIONS

We have proposed a new probability model called Kumaraswamy exponentiated U-quadratic (KwEUq). Several mathematical and statistical properties are investigated such as the quantile function, moments, moment generating function, order statistics, probability weighted moments, Shannon entropy and Renyi entropy. Parameter estimation was achieved by the alternative maximum likelihood estimation (AMLE) and modified maximum likelihood estimation (MMLE) due to the irregularity found in the usual maximum likelihood estimation method. We compare the fit of the two methods by the use of real data as measured by Kolmogorov Smirnov test which shows that both methods can be used for the parameter estimation of the KwEUq.

.

UNDER PEER REVIEW

References

[1] Ahmed MA, Mahmoud MR, ElSherbini EA (2015) The new Kumaraswamy Kumaraswamy family of generalized distributions with application. Pak J Stat Oper Res 11:159-180

[2] Akinsete A, Famoye F, Lee C (2014) The Kumaraswamy-geometric distribution. J Stat Distrib Appl. doi:10. 1186/s40488-014-0017-1

[3] Al-Babtain A, Fattah AA, Ahmedband AN, Merovci F (2015) The Kumaraswamy-transmuted exponentiated modified Weibull distribution. Commun Stat Simul Comput (to appear)

[4] Aleem M, Sufyan M, Khan NS, AliK(2013)Kumaraswamy double inverse exponential (Kw-DIE) distribution. In: Proceedings of the 11th international conference on statistical sciences, vol 25, pp 93-104

[5] Alizadeh M, Emadi M, Doostparast M, Cordeiro GM, Ortega EMM, Pescim RR (2015) A new family of distributions: the Kumaraswamy odd log-logistic, properties and applications. Hacet J Math Stat (to appear)

[6] Bidram H, Nekoukhou V (2013) Double bounded Kumaraswamy-power series class of distributions. SORT 37:211-230

[7] Bourguignon M, Silva RB, Zea LM, CordeiroGM(2013) TheKumaraswamy Pareto distribution. J Stat Theory Appl 12:129-144

[8] Cordeiro, G. M. and Castro M. (2010). A new family of generalized distributions. Journal of Statistical Computation and Simulation. DOI: 10.1080/0094965YY.

[9] Cordeiro GM, Ortega EMM, Nadarajah S (2010) The Kumaraswamy-Weibull distribution with application to failure data. J Frankl Inst 347:1399-1429

[10] Cordeiro GM, Nadarajah S, Ortega EMM (2012a) The Kumaraswamy Gumbel distribution. Stat Methods Appl 21:139-168

[11] Cordeiro GM, Pescim RR, Ortega EMM (2012b) The Kumaraswamy generalized half-normal distribution for skewed positive data. J Data Sci 10:195-224

[12] Cordeiro GM, Ortega EMM, Silva GO (2014) The Kumaraswamy modified Weibull distribution: theory and applications. J Stat Comput Simul 84:1387-1411

[13] de Pascoaa MAR, Ortega EMM, Cordeiro GM (2011) The Kumaraswamy generalized gamma distribution with application in survival analysis. Stat Methodol 8:411-433

[14] de Santana TVF, Ortega EMM, Cordeiro GM, Silva GO (2012) The Kumaraswamy-log-logistic distribution. J Stat Theory Appl 11:265-291

[15] Elbatal I (2013a) Kumaraswamy generalized linear failure rate distribution. Indian J Comput Appl Math 1:61-78

[16] Elbatal I (2013b) The Kumaraswamy exponentiated Pareto distribution. Econ Qual Control 28:1-8

[17] Elbatal I, Elgarhy M (2013) Statistical properties of Kumaraswamy quasi Lindley distribution. Int J Math Trends Technol 4:237-246

UNDER PEER REVIEW

[18] Ghosh I (2014) The Kumaraswamy-half-Cauchy distribution: properties and applications. J Stat Theory Appl 13:122-134

[19] Gomes AE, da Silva CQ, Cordeiro GM, Ortega EMM (2014) A new lifetime model: the Kumaraswamy generalized Rayleigh distribution. J Stat Comput Simul 84:290-309

[20] Huang S, Oluyede BO (2014) Exponentiated Kumaraswamy-Dagum distribution with applications to income and lifetime data. J Stat Distrib Appl. doi:10.1186/2195-5832-1-8

[21] Hussian MA, Amin EA (2014) Estimation and prediction for the Kumaraswamy-inverse Rayleigh distribution based on records. Int J Adv Stat Probab 2:21-27

[22] Li X, Huang Y, Zhao X (2011) The Kumaraswamy binomial distribution. Chin J Appl Probab Stat 27:511-521

[23] Mameli V (2015) The Kumaraswamy skew-normal distribution. Stat Probab Lett 104:75-81

[24] Nadarajah S, Eljabri S (2013) The Kumaraswamy GP distribution. J Data Sci 11:739-766

[25] Oguntunde PE, Babatunde O, Ogunmola A (2014) Theoretical analysis of the Kumaraswamy-inverse exponential distribution. Int J Stat Appl 4:113-116

[26] Paranaiba PF, Ortega EMM, Cordeiro GM, de Pascoa MAR (2013) The Kumaraswamy Burr XII distribution: theory and practice. J Stat Comput Simul 83:2117-2143

[27] Saulo H, Leao J, Bourguignon M (2012) The Kumaraswamy Birnbaum-Saunders distribution. J Stat Theory Pract 6:745-759

[28] Shahbaz MQ, Shahbaz S, Butt NS (2012) The Kumaraswamy-inverse Weibull distribution. Pak J Stat Oper Res 8:479-489

[29] Shams TM(2013a) The Kumaraswamy-generalized exponentiated Pareto distribution. Eur J Appl Sci 5:92-99

[30] Shams TM (2013b) The Kumaraswamy-generalized Lomax distribution. Middle-East J Sci Res 17:641-646

[31] M.V. Aarset. How to identify bathtub hazard rate. IEEE Trans. Reliab, 22(2) (1987), 106-108.

[32] Muhammad, M. ”Poisson-odd generalized exponential family of Distributions: Theory and Applications.” Hacettepe Journal of Mathematics and Statistics 47 (2016), 1-20.

[33] Muhammad. M, Ali. S. Rano, Ibrahim R. Sani, Maitama M. Ahmad and Sani A. Sulaiman. Parameter estimation of exponentiated U-quadratic distribution: alternative maximum likelihood and percentile methods, Asian Research Journal of Mathematics. (2018) to appear