

**Reviewer's Report on the Revision of
Ms AJPAS 41224**

General Comments:

The authors missed some of Reviewer's earlier corrections. Please see the attached pages.

Specific Comment:

The Reviewer asked the authors to provide a "Characterizations Section" for their proposed distribution. The Reviewer also mentioned that if the authors are not able to provide such a section, the Reviewer has already prepared a section characterizing the proposed distribution in different directions.

exponentiated Pareto [14], Kumaraswamy quasi-Lindley [15], Kumaraswamy generalized Pareto [16], Kumaraswamy-Burr XII distribution [17], Kumaraswamy generalized exponentiated Pareto [18], Kumaraswamy generalized Lomax [19], Kumaraswamy geometric [20], Kumaraswamy half-Cauchy distribution [21], Kumaraswamy generalized Rayleigh [22], Kumaraswamy-Dagum distribution [23], Kumaraswamy inverse Rayleigh [24], Kumaraswamy inverse exponential [25], Kumaraswamy -Kumaraswamy [26], Kumaraswamy transmuted exponentiated modified Weibull [27], Kumaraswamy odd log-logistic [28], Kumaraswamy skew-normal [29].

The Kumaraswamy-G generator of distributions is defined according to [1] as follows.

Definition 1.1. Let $G(x)$ be any base line cumulative distribution function and $g(x)$ be the density function corresponding to $G(x)$. Let $\lambda, \delta > 0$, then, the cumulative distribution function (cdf) and density function of the Kumaraswamy-G generator of distributions satisfy the relationship

$$F^*(x) = 1 - [1 - G(x)^\lambda]^\delta, \quad x \in \mathbb{R} \quad (1.1)$$

and

$$f^*(x) = \delta \lambda g(x) G(x)^{\lambda-1} [1 - G(x)^\lambda]^{\delta-1}, \quad x \in \mathbb{R} \quad (1.2)$$

respectively.

In this case, let the baseline distribution $G(x)$ be the cumulative distribution function of the exponentiate U-quadratic (EUq) distribution proposed by [30], given by

$$G(x) = \left(\frac{\alpha}{3}\right)^\theta ((x - \beta)^3 + (\beta - a)^3)^\theta, \quad x \in [a, b] \quad (1.3)$$

where $\theta > 0$ is the shape parameter, a the lower limit, b the upper limit and $a \in (-\infty, \infty)$, $b \in (a, \infty)$, $\alpha \in (0, \infty)$, and $\beta \in (-\infty, \infty)$ such that $\alpha = \frac{12}{(b-a)^3}$ and $\beta = \frac{b+a}{2}$. The density function of EUq is

$$g(x) = \theta \alpha^\theta 3^{1-\theta} (x - \beta)^2 ((x - \beta)^3 + (\beta - a)^3)^{\theta-1}, \quad x \in [a, b] \quad (1.4)$$

The main objective of this paper is to propose a new probability model called Kumaraswamy exponentiated U-quadratic (KwEUq) distribution and study some of its important mathematical and statistical properties. In addition, to investigate the parameters estimation and its application to real data.

The rest of the paper is organized as follows. In section 2, the KwEUq distribution and some important properties are discussed. In section 3, the parameters estimation via alternative maximum likelihood and modified maximum likelihood estimation methods are presented. Application of the new model to a real data set is provided in section 4. Conclusions are given in section 5.

2 The KwEUq model

The cumulative distribution of the kumaraswamy exponentiated U-quadratic (KwEUq) distribution can be defined by substituting (1.3) into equation (1.1), thus

$$F(x) = 1 - \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x - \beta)^3 + (\beta - a)^3)^{\theta\lambda}\right]^\delta, \quad x \in [a, b] \quad (2.1)$$

and the corresponding density function is

$$f(x) = \delta \lambda \theta \alpha^{\theta\lambda} 3^{1-\theta\lambda} (x - \beta)^2 ((x - \beta)^3 + (\beta - a)^3)^{\theta\lambda-1} \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x - \beta)^3 + (\beta - a)^3)^{\theta\lambda}\right]^{\delta-1}, \quad x \in [a, b] \quad (2.2)$$

$x \in [a, b]$

(a, b)

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where $\theta, \lambda > 0$ and $\delta > 0$ are the shape parameters. The survival function $s(x)$ and the hazard rate function $h(x)$ of the KwEUq distribution are respectively given by

$$s(x) = \left(1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x - \beta)^3 + (\beta - a)^3)^{\theta\lambda}\right)^{\delta}, \quad x \in [a, b] \quad (2.3)$$

and

$$h(x) = \frac{\delta\lambda\theta\alpha^{\theta\lambda}3^{1-\theta\lambda}(x - \beta)^2((x - \beta)^3 + (\beta - a)^3)^{\theta\lambda-1} \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x - \beta)^3 + (\beta - a)^3)^{\theta\lambda}\right]^{\delta-1}}{\left(1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x - \beta)^3 + (\beta - a)^3)^{\theta\lambda}\right)^{\delta}} \quad (2.4)$$

Figures 2, 2 and 2 display the plot of the cumulative distribution function, probability density function and hazard rate function of the KwEUq distribution for some parameter values.

Special Cases :

The special cases of the KwEUq distribution for some selected parameters values are:

1. When $\delta = 1$, we obtain Exponentiated U-quadratic (EUq) distribution with power parameter $\theta\lambda$.
2. If we take $\delta = 1$ and $\lambda = 1$, we also obtain Exponentiated U-quadratic (EUq) distribution but with power parameter θ [30].
3. Setting $\theta = 1$, we get Kumaraswamy U-quadratic (KwUq) distribution.
4. If $\theta = 1, \lambda = 1$ and $\delta = 1$, we get U-quadratic (Uq) distribution.

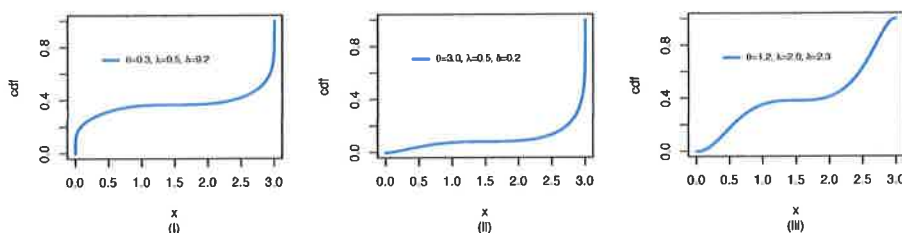


Fig. 1. Plots of the cumulative distribution function of the KwEUq distribution for some different values of parameter.

2.1 Quantile and Moments

The quantile function of the KwEUq distribution can be obtained by inverting (2.1) and can be used to determine some measures such as median and other percentiles. The quantile function can also be used to generate a random data which follow according to the KwEUq. The quantile function of KwEUq distribution is

$$\Omega(p) = \left[\frac{(1 - (1 - p)^{1/\delta})^{1/\theta\lambda}}{\left(\frac{\alpha}{3}\right)} - (\beta - a)^3 \right]^{1/3} + \beta. \quad (2.5)$$

) and kurtosis (γ^4) could be obtain from

$$\sigma^2 = \mu_2' - \mu_1'^2, \quad CV = \sqrt{\frac{\mu_2'}{\mu_1'^2}} - 1, \quad \gamma^3 = \frac{\mu_3' - 3\mu_2'\mu_1' + \mu_1'^3}{(\mu_2' - \mu_1'^2)^{3/2}},$$

$$\text{and } \gamma^4 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 + 3\mu_1'^4}{(\mu_2' - \mu_1'^2)^2}.$$

It is shown from the table 2 that for $a = 0$, $b = 5$, and θ, λ, δ are decreasing (i) the first six moments and kurtosis are decreasing –increasing–decreasing ... (ii) the variance σ^2 , CV, and skewness are increasing–decreasing–increasing ...

Table 2: First six moments, variance (σ^2), coefficient of variation (CV), skewness γ^3 , and kurtosis γ^4 of KwEUq for some parameter values and $a = 0$, $b = 5.0$

θ	λ	δ	μ_1'	μ_2'	μ_3'	μ_4'	μ_5'	μ_6'	σ^2	CV	γ^3	γ^4
0.1	0.1	0.1	3.7925	18.2964	89.6003	440.678	2172.340	10725.96	3.9139	0.5216	-1.2235	7.7291
0.15	0.16	0.14	3.6192	17.2358	83.7699	409.522	2008.19	9869.14	4.1373	0.5620	-1.0168	5.0123
0.2	0.2	0.2	3.2937	15.3845	73.9514	358.364	1743.962	8512.16	4.5359	0.6466	-0.6833	2.0012
0.4	0.35	0.5	2.3244	9.9411	45.5408	212.469	1000.096	4736.150	4.5384	0.9165	0.1382	0.7410
0.7	0.5	0.8	1.9182	7.6201	33.5861	152.071	697.054	3221.280	3.9408	1.0349	0.4923	1.4610
1.0	1.5	1.8	2.3154	8.5789	35.6902	153.764	673.034	2976.110	3.2180	0.7748	0.1603	2.7637
2.0	2.5	3.5	2.4048	8.4282	33.3092	137.093	574.643	2435.99	2.6467	0.6766	0.0734	5.5101
2.6	3.5	5.8	2.7853	10.0619	39.7472	162.328	673.681	2824.584	2.3039	0.5449	-0.3181	12.050
4.0	5.0	8.0	3.4457	13.4616	55.0097	229.155	964.900	4093.794	1.5882	0.3657	-1.1596	68.272
8.0	5.0	9.0	3.3475	12.8632	51.9585	214.342	894.438	3762.204	1.6573	0.3846	-1.0296	55.956

2.2 Entropy

An entropy of a random variable X can be defined as a measure of variation of uncertainty. In this section, we consider the two most important and popular entropies known as the Shannon and Renyi entropies. The Shannon entropy measure of a random variable X with KwEUq distribution can be defined by $E[-\log f(x)]$, then, the Shannon entropy of X can be computed by the following.

Lemma 2.3. Let X be a random variable with KwEUq distribution and pdf given by (2.2), then,

$$E[\log(x - \beta)] = \frac{\delta \lambda \theta \alpha^{\theta \lambda}}{3^{\theta \lambda - 1}} \frac{\partial}{\partial t} J(0, 2 + t, \theta \lambda - 1, \delta - 1)|_{t=0}, \quad (2.9)$$

$$E[\log((x - \beta)^3 + (\beta - a)^3)] = \frac{\delta \lambda \theta \alpha^{\theta \lambda}}{3^{\theta \lambda - 1}} \frac{\partial}{\partial t} J(0, 2, \theta \lambda + t - 1, \delta - 1)|_{t=0}, \quad (2.10)$$

$$E\left[\log\left(1 - \left(\frac{\alpha}{3}\right)^{\theta \lambda} ((x - \beta)^3 + (\beta - a)^3)^{\theta \lambda}\right)\right] = \frac{\delta \lambda \theta \alpha^{\theta \lambda}}{3^{\theta \lambda - 1}} \frac{\partial}{\partial t} J(0, 2, \theta \lambda - 1, \delta + t - 1)|_{t=0}. \quad (2.11)$$

Proof. We consider the first part

$$\begin{aligned} E[\log(x - \beta)] &= \int_a^b \log(x - \beta) f(x) dx \\ &= \frac{\partial}{\partial t} \int_a^b (x - \beta)^t f(x) dx|_{t=0} \\ &= \delta \lambda \theta \alpha^{\theta \lambda} 3^{1 - \theta \lambda} \frac{\partial}{\partial t} \int_a^b (x - \beta)^{2+t} ((x - \beta)^3 + (\beta - a)^3)^{\theta \lambda - 1} \\ &\quad \times \left[1 - \left(\frac{\alpha}{3}\right)^{\theta \lambda} ((x - \beta)^3 + (\beta - a)^3)^{\theta \lambda}\right]^{\delta - 1} dx|_{t=0}, \end{aligned}$$

hence,

$$E[\log(x - \beta)] = \frac{\delta \lambda \theta \alpha^{\theta \lambda}}{3^{\theta \lambda - 1}} \frac{\partial}{\partial t} J(0, 2 + t, \theta \lambda - 1, \delta - 1)|_{t=0},$$

where $J(\cdot, \cdot, \cdot, \cdot)$ is given by (2.7), (2.10) and (2.11) follow similar. $\square$

Proposition 2.2. Let $X \sim \text{KwEUq}(\theta, \lambda, \delta, a, b)$, then, the Shannon entropy of X can be expressed as

$$\begin{aligned} E[-\log f(x)] &= -\log \left(\frac{\delta \lambda \theta \alpha^{\theta \lambda}}{3^{\theta \lambda - 1}} \right) - \frac{2 \delta \lambda \theta \alpha^{\theta \lambda}}{3^{\theta \lambda - 1}} \frac{\partial}{\partial t} J(0, 2 + t, \theta \lambda - 1, \delta - 1)|_{t=0} \\ &\quad - \frac{\delta \lambda \theta \alpha^{\theta \lambda} (\theta \lambda - 1)}{3^{\theta \lambda - 1}} \frac{\partial}{\partial t} J(0, 2, \theta \lambda + t - 1, \delta - 1)|_{t=0} \\ &\quad - \frac{\delta \lambda \theta \alpha^{\theta \lambda} (\delta - 1)}{3^{\theta \lambda - 1}} \frac{\partial}{\partial t} J(0, 2, \theta \lambda - 1, \delta + t - 1)|_{t=0}. \end{aligned}$$

Proof.

$$\begin{aligned} E[-\log f(x)] &= -E \left[\log(\delta \lambda \theta \alpha^{\theta \lambda} 3^{1-\theta \lambda}) \right] - 2E[\log(x - \beta)] \\ &\quad - (\theta \lambda - 1)E[\log((x - \beta)^3 + (\beta - a)^3)] \\ &\quad - (\delta - 1)E \left[\log \left(1 - \left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \right) \right], \end{aligned}$$

and in view of lemma 2.3, we obtain the result. $\square$

The Renyi entropy of a random variable X is defined by $I_{R(\rho)} = \frac{1}{1-\rho} \log \left[\int_a^b f(x)^\rho dx \right]$, where $\rho > 0$ and $\rho \neq 1$. The Renyi entropy of X with KwEUq distribution can be obtained as follows.

$$\begin{aligned} \int_a^b f^\rho(x) dx &= \int_a^b \delta^\rho \lambda^\rho \theta^\rho \alpha^{\rho \theta \lambda} 3^{\rho(1-\theta \lambda)} (x - \beta)^{2\rho} ((x - \beta)^3 + (\beta - a)^3)^{\rho(\theta \lambda - 1)} \\ &\quad \times \left[1 - \left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \right]^{\rho(\delta - 1)} dx. \end{aligned}$$

Applying lemma 2.1, the integral becomes

$$\int_a^b f^\rho(x) dx = \frac{\delta^\rho \lambda^\rho \theta^\rho \alpha^{\rho \theta \lambda}}{3^{\rho(\theta \lambda - 1)}} J(0, 2\rho, \rho(\theta \lambda - 1), \rho(\delta - 1)).$$

Hence, the Renyi entropy of X is

$$I_{R(\rho)} = \frac{1}{1-\rho} \log \left[\frac{\delta^\rho \lambda^\rho \theta^\rho \alpha^{\rho \theta \lambda}}{3^{\rho(\theta \lambda - 1)}} J(0, 2\rho, \rho(\theta \lambda - 1), \rho(\delta - 1)) \right].$$

2.3 Order Statistics

Order statistics play an essential role in reliability and quality control. It is also an important tool in non parametric statistic. Let $X_1, X_2, X_3, \dots, X_n$ be a simple random sample obtained from KwEUq distribution and let $X_{1:n} \leq \dots \leq X_{n:n}$ be the corresponding order statistics. Then for $j = 1, 2, 3, \dots, n$, the corresponding pdf of $X_{j:n}$, say $f_{X_{j:n}}(x)$, is obtained as follows. The pdf of the j^{th} order statistics can be computed from

$$f_{X_{j:n}}(x; \alpha, \beta, \lambda, \theta, \delta) = \frac{n!}{(j-1)!(n-j)!} f(x)(F(x))^{j-1}(1-F(x))^{n-j},$$

where $f(x)$ and $F(x)$ are the pdf and cdf of KwEUq distribution. we have

$$f_{X_{j:n}}(x; \alpha, \beta, \lambda, \theta, \delta) = \sum_{m=0}^{n-j} \frac{n!(-1)^m}{(j-1)(n-j-m)!m!} f(x)(F(x))^{j+m-1}. \quad (2.12)$$

Binomial expansion of $(F(x))^{j+m-1}$ is

$$(F(x))^{j+m-1} = \sum_{l=0}^{j+m-1} \binom{j+m-1}{l} (-1)^l \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda} \right]^{\delta l}. \quad (2.13)$$

Substituting (2.13) into (2.12), we have

$$f_{X_{j:n}}(x; \alpha, \beta, \lambda, \theta, \delta) = \sum_{m=0}^{n-j} \sum_{l=0}^{j+m-1} \frac{\binom{j+m-1}{l} (-1)^{m+l} n! \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda} \right]^{\delta l}}{(j-1)(n-j-m)!m!} f(x)$$

Upon some algebraic manipulations, we obtain

$$f_{X_{j:n}}(x; \alpha, \beta, \lambda, \theta, \delta) = \sum_{m=0}^{n-j} \sum_{l=0}^{j+m-1} \frac{\binom{j+m-1}{l} n! (-1)^{m+l} f(x; \alpha, \beta, \lambda, \theta, \delta(l+1))}{(l+1)(j-1)!(n-j-m)!m!},$$

where $f(x; \alpha, \beta, \lambda, \theta, \delta(l+1))$ is the pdf of the KwEUq distribution with power $\delta(l+1)$. The moments of order statistics of the KwEUq distribution can be obtained using (2.3).

2.4 Probability Weighted Moments

The PWM of the random variable X with the KwEUq distribution, say $\rho_{r,s}$, is defined by

$$\rho_{r,s} = E[X^r F(X)^s] = \int_a^b x^r F^s(x) f(x) dx, \quad (2.14)$$

where $F(x)$ and $f(x)$ are the cdf and pdf of the KwEUq distribution. By expansion of $F^s(x)$ as

$$F^s(x) = \sum_{u=0}^s \binom{s}{u} (-1)^u \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda} \right]^{\delta u}, \quad (2.15)$$

and inserting (2.15) into (2.14), we have

$$\rho_{r,s} = \sum_{u=0}^s \binom{s}{u} (-1)^u \int_a^b x^r \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda} \right]^{\delta u} f(x) dx. \quad (2.16)$$

Upon Often some algebraic manipulations, we obtain $\rho_{r,s}$ as

$$\rho_{r,s} = \sum_{u=0}^s \binom{s}{u} \frac{(-1)^u}{(u+1)} \int_a^b x^r f(x; \alpha, \beta, \lambda, \theta, \delta(u+1)) dx,$$

hence,

$$\rho_{r,s} = \sum_{u=0}^s \binom{s}{u} \frac{(-1)^u}{(u+1)} \frac{\delta(u+1)\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}} J(r, 2, \theta\lambda-1, \delta(u+1)-1).$$

3 Estimation

The maximum likelihood estimation (MLE) method is one of the most popular and commonly used estimation methods for estimating the unknown parameters of a distribution. However, we found that there is an irregularity in the $\log$ -likelihood function of the KwEUq at the minimum and maximum order statistics, thus, the usual maximum likelihood method fails to exist. Thus, the use of alternative maximum likelihood estimation (AMLE) and modified maximum likelihood estimation (MMLE) are employed.

3.1 Maximum Likelihood Estimation

Let $X_1, X_2, X_3, \dots, X_n$ be an independent and identically distributed random sample from KwEUq distribution and let $X_{1:n} \leq \dots \leq X_{n:n}$, be the order statistics from this sample. Let $\Theta = (\delta, \theta, \lambda)^T$ and let the approximate maximum likelihood estimates be $\hat{\Theta}_{MLE} = (\hat{\delta}, \hat{\theta}, \hat{\lambda})$, then, the the likelihood function $l(\Theta)$ of KwEUq distribution for complete data set is given by

$$\begin{aligned} \log l(\Theta) &= n \log \delta + n \log \lambda + n \log \theta + \theta \lambda n \log \alpha + (1 - \theta \lambda) n \log 3 \\ &+ 2 \sum_{i=1}^n \log(x_i - \beta) + (\theta \lambda - 1) \sum_{i=1}^n \log((x_i - \beta)^3 + (\beta - a)^3) \\ &+ (\delta - 1) \sum_{i=1}^n \log \left[1 - \left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \right]. \end{aligned} \quad (3.1)$$

The first partial derivative of $\log l(\Theta)$ with respect to δ, θ and λ are respectively given by

$$\frac{\partial(\log l(\Theta))}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^n \log \left[1 - \left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \right] \quad (3.2)$$

$$\begin{aligned} \frac{\partial(\log l(\Theta))}{\partial \theta} &= \frac{n}{\theta} + n \log \alpha - \lambda n \log 3 + \lambda \sum_{i=1}^n \log((x_i - \beta)^3 + (\beta - a)^3) \\ &+ (\delta - 1) \sum_{i=1}^n \frac{(-1) \left(\left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \ln \left(\left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \right) \right)}{\left[1 - \left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \right]} \end{aligned} \quad (3.3)$$

$$\begin{aligned} \frac{\partial(\log l(\Theta))}{\partial \lambda} &= \frac{n}{\lambda} - \theta n \log 3 + (\theta - 1) \sum_{i=1}^n \log((x_i - \beta)^3 + (\beta - a)^3) \\ &+ (\delta - 1) \sum_{i=1}^n \frac{(-1) \left(\left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \ln \left(\left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \right) \right)}{\left[1 - \left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \right]}. \end{aligned} \quad (3.4)$$

We can determined $\hat{\delta}$ from (3.2) if the value of $\hat{a}, \hat{\beta}, \hat{\alpha}, \hat{\theta}$ and $\hat{\lambda}$ are known,

$$\delta = \frac{-n}{\sum_{i=1}^n \log \left[1 - \left(\frac{\alpha}{3} \right)^{\theta \lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta \lambda} \right]}. \quad (3.5)$$

This can be computed by solving (3.2), (3.3) and (3.4), but the following proposition describe the irregularity of the MLE method.