

Refinements of Gaussian Tail Inequality

Abstract

In this paper, we first prove a theorem which gives considerably better bound for $0 \leq t \leq 1/2$ than Gaussian tail inequality (or tail bound for normal density) and thus is a refinement of Gaussian tail inequality in this case. Next we present an interesting result which provides a refinement of Gaussian tail inequality for $t > \sqrt{3}$. Besides, we also prove an improvement of Gaussian tail inequality for $0 < t \leq 1/2$. Finally, we present a more general result which includes a variety of interesting results as special cases.

Key words: Probability Inequalities, Gaussian tail inequality, Probability density function.

1 Introduction

The extremes of Gaussian random fields have wide applications in finance, spatial analysis, physical oceanography, and many other disciplines (Adler, Muller, and Rozovski 1996, Adler, Taylor, and Worsley 2010). Tail probabilities of the extremes have been extensively studied in the literature, with its focus mostly on the development of approximations and bounds for the suprema (Borell 1975, Tsirelson, Ibragimov, and Sudakov 1976, Piterbarg

1996, Sun 1993, Azais and Wschebor 2008, Adler and Taylor 2007). Tail probabilities of other convex functions of Gaussian random process have also been studied. For instance, Liu (2012) and Liu and Xu (2012b) derived the asymptotic approximations of the tail probabilities of the exponential integrals of Gaussian random fields; see also Liu and Xu (2013). Most of the sharp theoretical approximations developed in the literature hold only for constant variance fields, which also need certain smoothness conditions of the Gaussian random fields (Adler and Taylor 2007, Adler, Blanchet, and Liu 2012). For the case of less smooth fields, the approximations involve the unknown Pickands constants (Piterbarg 1996). Therefore, to evaluate the tail probabilities, rare-event simulation serves as an appealing alternative from a computational point of view. In particular, the design and the analysis do not require very sharp approximations of the tail probabilities. Importance sampling based efficient simulation procedures have been proposed in the literature to estimate the tail probabilities. Numerical methods for rare-event analysis of the suprema are studied in Adler, Blanchet, and Liu (2008) and more thoroughly in Adler, Blanchet, and Liu (2012); see also Azais and Wschebor (2009), Li and Liu (2013). Simulation study for the exponential integrals of the Gaussian random fields has been studied in Liu and Xu (2012a), Liu and Xu (2013), Liu and Xu (2014).

2 Refinement of Gaussian Tail Inequality

Let X be a random variable having normal distribution with mean 0 and variance 1. The following result known as Gaussian Tail inequality (GTI) or the tail bound for normal density (for reference see, Gordon R.D(1941), Fan.P(2006)).

Theorem A. If $X \sim N(0, 1)$, then for $t > 0$,

$$P(|X| > t) \leq \sqrt{\frac{2}{\pi}} \left(\frac{1}{t} \right) e^{-\frac{t^2}{2}}.$$

Here we first prove the following result which gives considerably better bound for $0 \leq t \leq \frac{1}{2}$ than the Gaussian Tail inequality (GTI) and thus is a refinement of the Gaussian Tail Inequality (GTI) in this case.

Theorem 1. If $X \sim N(0, 1)$, then for $t > 0$,

$$P(|X| > t) \leq 1 - \sqrt{\frac{2}{\pi}} (t) e^{-\frac{t^2}{2}}.$$

Proof of Theorem 1. The probability density function (p.d.f) of x is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

we have,

$$\begin{aligned} P(X > t) &= \int_t^{\infty} \phi(x) dx \\ &= \int_0^{\infty} \phi(x) dx - \int_0^t \phi(x) dx \end{aligned} \quad (2.1)$$

Since $\phi(x)$ is an even function and

$$\int_{-\infty}^{\infty} \phi(x) dx = 1,$$

therefore,

$$\int_0^{\infty} \phi(x) dx = \int_{-\infty}^0 \phi(x) dx,$$

and hence

$$\begin{aligned} \int_{-\infty}^0 \phi(x) dx + \int_0^{\infty} \phi(x) dx &= \int_{-\infty}^{\infty} \phi(x) dx = 1, \\ \int_0^{\infty} \phi(x) dx + \int_0^{\infty} \phi(x) dx &= 1, \\ 2 \int_0^{\infty} \phi(x) dx &= 1. \end{aligned}$$

This gives,

$$\int_0^{\infty} \phi(x) dx = \frac{1}{2}.$$

Hence from (2.1),

$$P(X > t) \leq \frac{1}{2} - \int_0^t \left(\frac{1}{\sqrt{2\pi}} \right) \left(\frac{1}{e^{\frac{x^2}{2}}} \right) dx. \quad (2.2)$$

Since, $x \leq t$,

therefore,

$$\frac{x^2}{2} \leq \frac{t^2}{2},$$

which implies,

$$t^{\frac{x^2}{2}} \leq e^{\frac{t^2}{2}}.$$

This gives,

$$\begin{aligned} \int_0^t \frac{1}{e^{\frac{x^2}{2}}} dx &\geq \int_0^t \frac{1}{e^{\frac{t^2}{2}}} dx \\ &\geq \frac{1}{e^{\frac{t^2}{2}}} \int_0^t dx = \frac{t}{e^{\frac{t^2}{2}}}. \end{aligned}$$

Using this in (2.2), we get

$$P(X > t) \leq \frac{1}{2} - \left(\frac{1}{\sqrt{2\pi}} \right) \left(\frac{t}{e^{\frac{t^2}{2}}} \right).$$

Hence by symmetry,

$$\begin{aligned} P(|X| > t) &= 2P(X > t) \\ &\leq 2 \left[\frac{1}{2} - \int_0^t \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \right) \right] \\ &= 1 - \sqrt{\frac{2}{\pi}} t e^{-\frac{t^2}{2}}. \end{aligned}$$

This completes the proof of Theorem 1.

Remark 1. To verify that the Theorem 1 gives a refinement of Gaussian Tail inequality (GTI) for $0 < t \leq \frac{1}{2}$, we show

$$1 - \sqrt{\frac{2}{\pi}} \left(t e^{-\frac{t^2}{2}} \right) < \sqrt{\frac{2}{\pi}} \left(\frac{e^{-\frac{t^2}{2}}}{t} \right). \quad (2.3)$$

Since, $e^{\frac{t^2}{2}} \leq e^{\frac{1}{8}}$ for $0 < t \leq \frac{1}{2}$,

and it can be easily verified that $e^{\frac{1}{8}} < \frac{6}{5}$,

so that

$$e^{\frac{t^2}{2}} < \frac{6}{5}.$$

This implies,

$$e^{-\frac{t^2}{2}} > \frac{5}{6}, \quad \text{for } 0 < t \leq \frac{1}{2}.$$

Now,

$$\begin{aligned} \sqrt{\frac{2}{\pi}} \left(t e^{-\frac{t^2}{2}} \right) + \sqrt{\frac{2}{\pi}} \left(\frac{e^{-\frac{t^2}{2}}}{t} \right) &= \sqrt{\frac{2}{\pi}} \left(e^{-\frac{t^2}{2}} \right) \left(t + \frac{1}{t} \right) \\ &> \sqrt{\frac{1}{2}} \left(\frac{5}{6} \right) (2) \\ &= \frac{5}{3\sqrt{2}} = 1. \end{aligned}$$

Hence,

$$\sqrt{\frac{2}{\pi}} \left(\frac{e^{-\frac{t^2}{2}}}{t} \right) > 1 - \sqrt{\frac{2}{\pi}} \left(t e^{-\frac{t^2}{2}} \right),$$

or

$$1 - \sqrt{\frac{2}{\pi}} \left(t e^{-\frac{t^2}{2}} \right) < \sqrt{\frac{2}{\pi}} \left(\frac{e^{-\frac{t^2}{2}}}{t} \right),$$

which proves (2.3) for $0 < t \leq \frac{1}{2}$.

Next we prove the following interesting result which provides a refinement of Gaussian Tail inequality (GTI) for $t > \sqrt{3}$.

Theorem 2. If $X \sim N(0, 1)$, then for $t > 0$,

$$P(|X| > t) \leq \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}} \left(\frac{t^4 - t^2 + 3}{t^5} \right).$$

Proof of Theorem 2. The probability density function (p.d.f) of x is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty,$$

which on differentiation gives,

$$\begin{aligned} \phi'(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (-x) \\ &= -x \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) \\ &= -x\phi(x). \end{aligned} \tag{2.4}$$

Now

$$\begin{aligned} P(X > t) &= \int_t^\infty \phi(x) dx \\ &= \int_t^\infty \frac{x\phi(x)}{x} dx \\ &= - \int_t^\infty \frac{\phi'(x)}{x} dx. \quad (\text{by (2.4)}). \end{aligned}$$

Integrating by parts, keeping $\phi'(x)$ for integration and $\frac{1}{x}$ for differentiation, we get

$$P(X > t) = - \left[\frac{1}{x} \phi(x) \right]_t^\infty + \int_t^\infty \frac{-\phi(x)}{x^2} dx$$

$$= \frac{\phi(t)}{t} - \int_t^{\infty} \frac{\phi(x)}{x^2} dx. \quad (2.5)$$

Now

$$\begin{aligned} \int_t^{\infty} \frac{\phi(x)}{x^2} dx &= \int_t^{\infty} \frac{x\phi(x)}{x^3} dx \\ &= - \int_t^{\infty} \frac{\phi'(x)}{x^3} dx \quad (\text{by (3.2.4)}) \\ &= - \int_t^{\infty} \left(\frac{1}{x^3} \right) \phi'(x) dx. \end{aligned}$$

Integrating by parts as before, we get

$$\begin{aligned} \int_t^{\infty} \frac{\phi(x)}{x^2} dx &= \left[\left(\frac{1}{x^3} \right) \phi(x) \right]_t^{\infty} - \int_t^{\infty} \phi(x) \left(\frac{-3}{x^4} \right) dx \\ &= -\frac{\phi(t)}{t^3} + 3 \int_t^{\infty} \frac{\phi(x)}{x^4} dx. \end{aligned}$$

Thus from (2.5), we have

$$P(X > t) = \frac{\phi(t)}{t} + \frac{\phi(t)}{t^3} - 3 \int_t^{\infty} \frac{\phi(x)}{x^4} dx. \quad (2.6)$$

Again,

$$\begin{aligned}
 \int_t^\infty \frac{\phi(x)}{x^4} dx &= \int_t^\infty \frac{x \phi(x)}{x^5} dx \\
 &= - \int_t^\infty \frac{\phi'(x)}{x^5} dx, \quad (\text{by (3.2.4)}) \\
 &= - \left[\frac{\phi(x)}{x^5} \right]_t^\infty + \int_t^\infty \phi(x) \left(\frac{-5}{x^6} \right) dx \\
 &= \frac{\phi(t)}{t^5} - 5 \int_t^\infty \frac{\phi(x)}{x^6} dx.
 \end{aligned}$$

Hence from (2.6), we have

$$\begin{aligned}
 P(X > t) &= \frac{\phi(t)}{t} - \frac{\phi(t)}{t^3} + 3 \frac{\phi(t)}{t^5} - 15 \int_t^\infty \frac{\phi(x)}{x^6} dx \\
 &= \phi(t) \left(\frac{1}{t} - \frac{1}{t^3} + \frac{3}{t^5} \right) - 15 \int_t^\infty \frac{\phi(x)}{x^6} dx. \tag{2.7}
 \end{aligned}$$

Since $\int_t^\infty \frac{\phi(x)}{x^6} dx \geq 0$, therefore, we have from (2.7),

$$\begin{aligned}
 P(X > t) &\leq \phi(t) \left(\frac{1}{t} - \frac{1}{t^3} + \frac{3}{t^5} \right) \\
 &= \frac{\phi(t)}{t^5} (t^4 - t^2 + 3).
 \end{aligned}$$

By symmetry,

$$\begin{aligned}
 P(|X| > t) &= 2P(X > t) \\
 &\leq 2 \frac{\phi(t)}{t^5} (t^4 - t^2 + 3) \\
 &= \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}} \left(\frac{t^4 - t^2 + 3}{t^5} \right).
 \end{aligned}$$

This completes the proof of Theorem 2.

Remark 2. To verify that the Theorem 2 is a refinement of Gaussian Tail inequality (GTI) for $t > \sqrt{3}$, we note that $t > \sqrt{3}$ implies $t^2 > 3$, so that, $3 - t^2 < 0$, which gives,

$$\left(\frac{t^4 - t^2 + 3}{t^5} \right) < \frac{t^4}{t^5} = \frac{1}{t}.$$

This implies,

$$\sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}} \left(\frac{t^4 - t^2 + 3}{t^5} \right) < \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{t^2}{2}}}{t},$$

which shows that the bound obtain in Theorem 2 is better than the bound given by Gaussian Tail inequality (GTI).

Our next Theorem gives an improvement of Theorem 1 and therefore Gaussian Tail inequality (GTI) for $0 < t \leq \frac{1}{2}$.

Theorem 3. If $X \sim N(0, 1)$, then for $t > 0$,

$$P(|X| > t) \leq 1 - \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}} \left(t + \frac{t^3}{3} \right).$$

Proof of Theorem 3. The probability density function (p.d.f) of x is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty,$$

we have,

$$\begin{aligned} P(X > t) &= \int_t^{\infty} \phi(x) dx \\ &= \int_0^{\infty} \phi(x) dx - \int_0^t \phi(x) dx. \end{aligned} \quad (2.8)$$

Integrating the 2nd integral in (2.8) by parts and using the fact that

$\phi'(x) = -x\phi(x)$, we get

$$\begin{aligned}
 \int_0^t \phi(x)dx &= \int_0^t 1 \cdot \phi(x)dx \\
 &= \left[x \phi(x) \right]_0^t - \int_0^t x \phi'(x)dx \\
 &= t\phi(t) - \int_0^t -x^2 \phi(x)dx \\
 &= t\phi(t) + \int_0^t x^2 \phi(x)dx.
 \end{aligned} \tag{2.9}$$

Since $\phi(x)$ is a decreasing function, it follows that $\phi(x) \geq \phi(t)$ for $x \leq t$. Hence from (2.9), we get

$$\begin{aligned}
 \int_0^t \phi(x)dx &\geq t\phi(t) + \phi(t) \int_0^t x^2 dx \\
 &= t\phi(t) + \phi(t) \left[\frac{x^3}{3} \right]_0^t \\
 &= t\phi(t) + \phi(t) \left[\frac{t^3}{3} \right] \\
 &= \phi(t) \left[t + \frac{t^3}{3} \right].
 \end{aligned} \tag{2.10}$$

Using (2.10) in (2.8) and the fact that

$$\begin{aligned}
 \int_0^\infty \phi(x)dx &= P(X > t) = \frac{1}{2}, \text{ we get} \\
 P(X > t) &\leq \frac{1}{2} - \phi(t) \left[t + \frac{t^3}{3} \right].
 \end{aligned}$$

By symmetry, this implies

$$\begin{aligned} P(|X| > t) &= 2P(X > t) \\ &\leq 1 - 2\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right)\left(t + \frac{t^3}{3}\right) \\ &= 1 - \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}}\left(t + \frac{t^3}{3}\right). \end{aligned}$$

This completes the proof of Theorem 3.

Remark 3. From (2.10), we have

$$\begin{aligned} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right)\left(t + \frac{t^3}{3}\right) &\leq \int_0^t \phi(x)dx \\ &\leq \int_0^\infty \phi(x)dx = \frac{1}{2}, \end{aligned}$$

so that,

$$\sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}}\left(t + \frac{t^3}{3}\right) \leq 1.$$

Finally, in this chapter we present the following more general result, which include a variety of interesting results as special cases.

Theorem 4. If $X \sim N(0, 1)$, then for $t > 0$,

$$P(|X| > t) = \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}} \left(\frac{1}{t} - \frac{1}{t^3} + \frac{1 \cdot 3}{t^5} - \frac{1 \cdot 3 \cdot 5}{t^7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{t^9} - \dots \right).$$

For the proof of this theorem, we need the following lemma.

Lemma 1. If $\phi(x)$ is the p.d.f of the random variable $X \sim N(0, 1)$, then for $t > 0$,

$$\int_t^\infty \frac{\phi(x)}{x^{2n}} = \frac{\phi(t)}{t^{2n+1}} - (2n+1) \int_t^\infty \frac{\phi(x)}{x^{2n+2}} dx, \text{ for } n = 0, 1, 2, \dots$$

Proof of Lemma 1. The probability density function (p.d.f) of X is $\phi(x)$. Integrating by parts, keeping $\frac{1}{x^{2n+1}}$ for differentiation and $\phi'(x)$ for integration and using the fact that $\phi'(x) = -x\phi(x)$, we get for any integer $n \geq 0$, and $t > 0$.

$$\begin{aligned} \int_t^\infty \frac{\phi(x)}{x^{2n}} &= \int_t^\infty \frac{x\phi(x)}{x^{2n+1}} dx \\ &= - \int_t^\infty \frac{\phi'(x)}{x^{2n+1}} dx \\ &= - \left[\frac{1}{x^{2n+1}} \phi(x) \right]_t^\infty + \int_t^\infty - \frac{(2n+1)\phi(x)}{x^{2n+2}} dx \\ &= \frac{\phi(t)}{t^{2n+1}} - \int_t^\infty (2n+1) \frac{\phi(x)}{x^{2n+2}} dx \\ &= \frac{\phi(t)}{t^{2n+1}} - (2n+1) \int_t^\infty \frac{\phi(x)}{x^{2n+2}} dx. \end{aligned}$$

This proves the Lemma 1.

Proof of Theorem 4. The probability density function (p.d.f) of X is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.$$

We have,

$$P(X > t) = \int_t^\infty \phi(x) dx. \quad (2.11)$$

Using above lemma with $n = 0$, it follows that

$$\int_t^\infty \phi(x) dx = \frac{\phi(t)}{t} - \int_t^\infty \frac{\phi(x)}{x^2} dx. \quad (2.12)$$

Using above lemma with $n = 1$, it follows that

$$\int_t^\infty \frac{\phi(x)}{x^2} dx = \frac{\phi(t)}{t^3} - 3 \int_t^\infty \frac{\phi(x)}{x^4} dx. \quad (2.13)$$

From (2.12) and (2.13), we get

$$\int_t^\infty \phi(x) dx = \phi(t) \left(\frac{1}{t} - \frac{1}{t^3} \right) + 3 \int_t^\infty \frac{\phi(x)}{x^4} dx. \quad (2.14)$$

Again using above lemma with $n = 2$, we obtain

$$\int_t^\infty \frac{\phi(x)}{x^4} dx = \frac{\phi(t)}{t^5} - 5 \int_t^\infty \frac{\phi(x)}{x^6} dx.$$

Hence from (2.14), we get

$$\int_t^\infty \phi(x) dx = \phi(t) \left(\frac{1}{t} - \frac{1}{t^3} + \frac{1 \cdot 3}{t^5} \right) - (3 \cdot 5) \int_t^\infty \frac{\phi(x)}{x^6} dx. \quad (2.15)$$

On proceeding in this way, a repeated application of the above lemma yields,

$$\begin{aligned} \int_t^\infty \phi(x) dx &= \phi(t) \left(\frac{1}{t} - \frac{1}{t^3} + \frac{1 \cdot 3}{t^5} - \frac{1 \cdot 3 \cdot 5}{t^7} + \dots \right), \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \left(\frac{1}{t} - \frac{1}{t^3} + \frac{1 \cdot 3}{t^5} - \frac{1 \cdot 3 \cdot 5}{t^7} + \dots \right), \end{aligned} \quad (2.16)$$

which in conjunction with (2.11) gives

$$P(X > t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \left(\frac{1}{t} - \frac{1}{t^3} + \frac{1 \cdot 3}{t^5} - \frac{1 \cdot 3 \cdot 5}{t^7} + \dots \right).$$

By symmetry

$$\begin{aligned} P(|X| > t) &= 2P(X > t) \\ &= \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}} \left(\frac{1}{t} - \frac{1}{t^3} + \frac{1 \cdot 3}{t^5} - \frac{1 \cdot 3 \cdot 5}{t^7} + \dots \right). \end{aligned}$$

This completes the proof of Theorem 4.

Some deductions.

1. A repeated application of the lemma above shows that

$$\begin{aligned} 0 &\leq \int_t^\infty \frac{\phi(x)}{x^2} dx = \frac{\phi(t)}{t^3} - 3 \int_t^\infty \frac{\phi(x)}{x^4} dx, \\ &= \phi(t) \left[\frac{1}{t^3} - \frac{3}{t^5} \right] + \int_t^\infty \frac{\phi(x)}{x^6} dx, \\ &\vdots \\ &= \phi(t) \left(\frac{1}{t^3} - \frac{1 \cdot 3}{t^5} + \frac{1 \cdot 3 \cdot 5}{t^7} - \dots \right), \end{aligned}$$

which implies,

$$\phi(t) \left(-\frac{1}{t^3} + \frac{1 \cdot 3}{t^5} - \frac{1 \cdot 3 \cdot 5}{t^7} + \dots \right) \leq 0.$$

Using this in Theorem 4, we get

$$P(|X| > t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{t^2}{2}}}{t},$$

which is Gaussian Tail inequality (GTI).

2. As before, a repeated application of the lemma above, shows that

$$0 \leq \int_t^\infty \frac{\phi(x)}{x^6} dx = \phi(t) \left(\frac{1 \cdot 3 \cdot 5}{t^7} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{t^9} + \dots \right),$$

which gives,

$$\left(-\frac{1 \cdot 3 \cdot 5}{t^7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{t^9} - \dots \right) \leq 0.$$

Hence from Theorem 4, we obtain,

$$\begin{aligned} P(|X| > t) &\leq \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}} \left(\frac{1}{t} - \frac{1}{t^3} + \frac{1}{t^5} \right), \\ &= \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}} \left(\frac{t^4 - t^2 + 3}{t^5} \right) \text{ for } t > 0, \end{aligned}$$

which is the conclusion of Theorem 2.

Many other interesting results can be deduced from Theorem 4 in a similar fashion.

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