

A COMPARISON BETWEEN MAXIMUM LIKELIHOOD AND BAYESIAN ESTIMATION METHODS FOR THE SCALE PARAMETER OF THE WEIBULL-EXPONENTIAL DISTRIBUTION

ABSTRACT

We considered the Bayesian analysis of the scale parameter of Weibull-Exponential distribution in this paper. We assumed a class of non-informative priors in deriving the corresponding posterior distributions. In particular, the Bayes estimators and associated risks were calculated under three different loss functions. The performance of the Bayes estimators have been evaluated and compared to the method of maximum likelihood under a comprehensive simulation study. It was discovered that for the said parameters to be estimated, the quadratic loss function under both uniform and Jeffrey's priors should be used for decreasing parameter values while the use of precautionary loss function can be preferred for increasing parameter values irrespective of the variations in sample size.

Keywords: Weibull Exponential, Bayesian Estimation, Mathematical Statistics, Maximum Likelihood Estimation, Simulation

1. Introduction

The Exponential distribution is regarded as *memoryless* and has a constant failure rate; this latter property makes the distribution unsuitable for real life problems and hence there is need to generalize the Exponential distribution in order to increase its flexibility and capability to model some other real life problems, Singh *et al.* (2013). Some of recent generalizations of the exponential distribution include the transmuted exponential distribution by Owoloko *et al.* (2015), transmuted inverse exponential distribution by Oguntunde and Adejumo (2015) and the Weibull-Exponential distribution (WED) by Oguntunde *et al.* (2015).

According to Oguntunde *et al.*, (2015), if X is a Weibull-exponential random variable, then the probability density function (*pdf*) and the cumulative distribution function (*cdf*) of X are respectively given by;

$$f(x) = \alpha\beta\lambda e^{\lambda\beta x} \left(1 - e^{-\lambda x}\right)^{\beta-1} e^{-\alpha(e^{\lambda x}-1)^{\beta}} \quad (1.1)$$

and

$$F(x) = 1 - e^{-\alpha(e^{\lambda x}-1)^{\beta}} \quad (1.2)$$

where; α and β are scale and shape parameters respectively while λ is a exponential parameter.

The classical and non-classical or Bayesian methods of estimation have gained wider applications in statistical theory and analysis. In classical scenario, the parameters are considered to be fixed while in the Bayesian concept, the parameters are viewed as unknown random variables. It is true that in many

real life phenomena which are represented by life time models, the parameters cannot be treated as constant throughout the life testing period (Ibrahim *et al.*, 2001; Martz and Waller, 1982; Singpurwalla, 2006) hence the need for Bayesian estimation for life time models.

Al-Aboud (2009) has considered Bayesian estimation for the extreme value distribution using progressive censored data and asymmetric loss, Himanshu and Arun (2009) estimated the shape parameter of the Generalized Pareto Distribution (GPD) using quasi, inverted gamma and uniform prior distributions under the LINEX, precautionary and entropy loss functions, Sanku (2010) estimated the shape parameter of Generalized Exponential distribution using extended Jeffrey's prior under the quadratic loss function, squared error loss function and general entropy loss function, Sanku and Tanujit (2011) also estimated the parameters of Rayleigh distribution using Bayesian approach, Sajid *et al.*, (2012) estimated the scale parameter of Laplace model using different asymmetric loss functions comprising precautionary, weighted squared, modified (quadratic) squared loss functions and Ahmed *et al.* (2011) considered Bayesian Survival Estimator for Weibull distribution with censored data.

One of the important things that cannot be ignored in Bayesian approach is choosing the appropriate prior(s) for the parameters, so also is the choice of loss function. Pandey *et al.* (2011), Al-Athari (2011), Ahmad and Ahmad (2013), Ahmad *et al.* (2015), Ahmad *et al.* (2016), Gupta and Singh (2017) and Gupta (2017) and many others have however shown a number of symmetric and asymmetric loss functions to be functional in several applications.

The objective of this study is to introduce a statistical comparison between the Bayesian and Maximum Likelihood estimation (MLE) procedures for estimating the scale parameter of the Weibull-Exponential distribution. The resulting estimators are obtained by using squared error, Quadratic and precautionary loss functions.

The layout of the paper is as follows; in Section 2, Maximum likelihood estimate of the scale parameter are obtained, in Section 3, the posterior distributions are obtained under the two different prior distributions while the Bayes estimates and corresponding risks are obtained in section 4. Finally, comparison between MLE and Bayes estimates under the two priors and loss functions are made using simulation study in Section 5. Some concluding remarks are given in Section 6.

2. Maximum Likelihood Estimation

This section presents the estimation of the scale parameter of the Weibull-Exponential distribution using the method of maximum likelihood estimation. Let $X_1, X_2, \dots, X_n$ be a random sample from the WED with unknown parameter vector $\theta = (\alpha, \beta, \lambda)^T$. The total log-likelihood function for θ is obtained from $f(x)$ as follows:

$$L(X_1, X_2, \dots, X_n / \alpha, \beta, \lambda) = (\alpha\beta\lambda)^n e^{\lambda\beta \sum_{i=1}^n X_i} \sum_{i=1}^n \left(1 - e^{-\lambda X_i}\right)^{\beta-1} \exp\left\{-\alpha \sum_{i=1}^n (e^{\lambda X_i} - 1)^\beta\right\} \quad (1)$$

The likelihood function for the scale parameter, α , is given by;

$$L(\alpha | \underline{X}) \propto (\alpha)^n e^{-\alpha \sum_{i=1}^n (e^{\lambda X_i} - 1)^\beta} \quad (2)$$

Let the log-likelihood function, $l = \log L(\alpha | \underline{X})$, therefore

$$l = n \log \alpha - \alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \quad (3)$$

77

78 Differentiating l partially with respect to α , the scale parameter, equating to zero and solving for $\hat{\alpha}$
 79 gives;

$$\begin{aligned} \frac{\partial l}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \\ \Rightarrow \hat{\alpha} &= \frac{n}{\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} \end{aligned} \quad (4)$$

82

83 Hence, equation (4) is the estimator for the scale parameter of the Weibull-Exponential distribution
 obtained by the method of maximum Likelihood estimation.

84 3. Posterior Distributions

85 To obtain the posterior distribution of a parameter once the data has been observed, we apply bayes'
 86 Theorem which is given as:

$$p(\alpha | \underline{X}) = \frac{p(\alpha) L(\alpha | \underline{X})}{\int_0^\infty p(\alpha) L(\alpha | \underline{X}) d\alpha} \quad (5)$$

88 where $p(\alpha)$ and $L(\alpha | \underline{X})$ are the prior distribution and the Likelihood function respectively.

89 Here, Posterior distributions are derived by using uniform and Jeffrey's prior.

90 3.1 Posterior Distributions under the assumption of Uniform Prior

91 The uniform prior as a non-informative prior relating to the scale parameter α is defined as:

$$p(\alpha) \propto 1 \quad ; \quad 0 < \alpha < \infty \quad (6)$$

93 The posterior distribution of the scale parameter α for a given data under uniform prior is obtained
 94 from equation (5) using integration by substitution method as

$$p(\alpha | \underline{X}) = \frac{\alpha^n e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta}}{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^{-(n+1)} \Gamma(n+1)} \quad (7)$$

96 3.2 Posterior Distributions under the assumption of Uniform Prior

97 Also, the Jeffrey's prior as a non-informative prior relating to the scale parameter α of the WED
 98 distribution is defined as:

$$p(\alpha) \propto \frac{1}{\alpha} \quad ; \quad 0 < \alpha < \infty \quad (8)$$

100 The posterior distribution of the scale parameter α for a given data under Jeffrey prior is obtained
 101 from equation (5) using integration by substitution method as

$$p(\alpha | \underline{X}) = \frac{\alpha^{n-1} \left(\sum_{i=1}^n (e^{\lambda x_i} - 1) \right)^\beta e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta}}{\Gamma(n)} \quad (9)$$

103

104 4 Bayes Estimators and their respective corresponding Risks

105 Here, we estimate the scale parameter of the *WED* using three loss functions under the posterior
106 distributions obtained from both the uniform and Jeffrey's priors.

107 The Bayes estimators and their corresponding Bayes posterior risks using uniform prior are as follows:

108

109 Using Squared Error Loss Function (*SELF*) under uniform prior

110 From Legendre (1805) and Gauss (1810), the squared error loss function relating to the scale
111 parameter α is defined as

$$L(\alpha, \alpha_{SELF}) = (\alpha - \alpha_{SELF})^2 \quad (10)$$

113 where α_{SELF} is the estimator of the parameter α under *SELF*.

114 The derivation of Bayes estimator using *SELF* under uniform prior is given below:

$$115 \alpha_{SELF} = E(\alpha) = E(\alpha | \underline{X})$$

$$116 E(\alpha | \underline{X}) = \int_0^\infty \alpha p(\alpha | \underline{X}) d\alpha$$

117 (11)

118 Substituting for $p(\alpha | \underline{X})$ in equation (3.21); we have:

$$119 E(\alpha | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1) \right)^\beta}{\Gamma(n+1)} \int_0^\infty \alpha^{n+1} e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} d\alpha \quad (12)$$

120 Now, using integration by substitution method in equation (12) and simplification, we obtained the
121 Bayes estimator using *SELF* under uniform prior as:

$$122 \alpha_{SELF} = \frac{\Gamma(n+2)}{\Gamma(n+1) \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} \quad (13)$$

123 Using the Squared error loss function (*SELF*), the posterior risk, $p(\alpha_{SELF})$ is defined as:

$$124 P(\alpha_{SELF}) = E(\alpha^2 | \underline{X}) - \{E(\alpha | \underline{X})\}^2 \quad (14)$$

125 And it is obtained as

$$126 P(\alpha_{SELF}) = \frac{\Gamma(n+3)\Gamma(n+1) - (\Gamma(n+2))^2}{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^2 (\Gamma(n+1))^2} \quad (15)$$

127 Using Quadratic Loss Function (*QLF*) under uniform prior

128 The Quadratic loss function (*QLF*) is defined as

$$L(\alpha, \alpha_{QLF}) = \left(\frac{\alpha - \alpha_{QLF}}{\alpha} \right)^2 \quad (16)$$

where α_{QLF} is the estimator of the parameter α under QLF .

The derivation of Bayes estimator using QLF under uniform prior is as follows:

$$\alpha_{QLF} = \frac{E(\alpha^{-1})}{E(\alpha^{-2})} = \frac{E(\alpha^{-1} | \underline{X})}{E(\alpha^{-2} | \underline{X})}$$

$$E(\alpha^{-1} | \underline{X}) = \int_0^{\infty} \alpha^{-1} p(\alpha | \underline{X}) d\alpha \quad (17)$$

Substituting for $p(\alpha | \underline{X})$ in equation (17), we have:

$$E(\alpha^{-1} | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^{\beta} \right)^{n+1}}{\Gamma(n+1)} \int_0^{\infty} \alpha^{n-1} e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^{\beta}} d\alpha \quad (18)$$

Using integration by substitution method in equation (18) and simplifying, we obtained the Bayes estimator using QLF under uniform prior as:

$$\alpha_{QLF} = \frac{\Gamma(n)}{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^{\beta} \right) \Gamma(n-1)} \quad (19)$$

Using the Quadratic loss function (QLF), the posterior risk, $p(\alpha_{QLF})$ is defined as:

$$P(\alpha_{QLF}) = 1 - \frac{\left\{ E(\alpha^{-1} | \underline{X}) \right\}^2}{E(\alpha^{-2} | \underline{X})} \quad (20)$$

Therefore, the posterior risk under uniform prior using the Quadratic loss function is given as:

$$P(\alpha_{QLF}) = \frac{\Gamma(n+1)\Gamma(n-1) - [\Gamma(n)]^2}{\Gamma(n+1)\Gamma(n-1)} \quad (21)$$

145

146 Using Precautionary Loss Function (PLF) under uniform prior

147 The precautionary loss function (PLF) introduced by Norstrom (1996) is an asymmetric loss function
148 and is defined as

$$L(\alpha_{PLF}, \alpha) = \frac{(\alpha_{PLF} - \alpha)^2}{\alpha} \quad (22)$$

151 where α_{PLF} is the estimator of the parameter α under PLF .

152 Similarly, the derivation of Bayes estimator under PLF using uniform prior is given below:

$$\alpha_{PLF} = \left\{ E(\alpha^2) \right\}^{\frac{1}{2}} = \left\{ E(\alpha^2 | \underline{X}) \right\}^{\frac{1}{2}} = \sqrt{E(\alpha^2 | \underline{X})}$$

$$E(\alpha^2 | \underline{X}) = \int_0^{\infty} \alpha^2 p(\alpha | \underline{X}) d\alpha \quad (23)$$

Substituting for $p(\alpha | \underline{X})$ in equation (19); we have:

$$E(\alpha^2 | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^{n+1}}{\Gamma(n+1)} \int_0^{\infty} \alpha^{n+2} e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} d\alpha$$

(24)

Again, using integration by substitution method in equation (24) and simplifying, we obtained the Bayes estimator using *PLF* under uniform prior as:

$$\alpha_{PLF} = \sqrt{\frac{\Gamma(n+3)}{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^2 \Gamma(n+1)}}$$

(25)

Using the Precautionary loss function (*PLF*), the posterior risk, $p(\alpha_{PLF})$ is defined as:

$$P(\alpha_{PLF}) = 2 \{ \alpha_{PLF} - E(\alpha | \underline{X}) \}$$

(26)

and derived to be:

$$P(\alpha_{PLF}) = 2 \left\{ \frac{\left\{ \Gamma(n+3) \Gamma(n+1) \right\}^{\frac{1}{2}} - \Gamma(n+2)}{\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \Gamma(n+1)} \right\} \quad (27)$$

167

In a similar way, the Bayes estimators and posterior risks of the scale parameter of the *WED* using three loss functions under the posterior distribution obtained from Jeffrey's prior are as follows:

170

Using Squared Error Loss Function (*SELF*) under Jeffrey's prior

The derivation of Bayes estimator under *SELF* using Jeffrey's prior is given below:

$$\alpha_{SELF} = E(\alpha) = E(\alpha | \underline{X})$$

$$E(\alpha | \underline{X}) = \int_0^{\infty} \alpha p(\alpha | \underline{X}) d\alpha \quad (28)$$

Substituting for $p(\alpha | \underline{X})$ in equation (28); we have:

$$E(\alpha | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta \right)^n}{\Gamma(n)} \int_0^{\infty} \alpha^n e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^\beta} d\alpha$$

(29)

Using integration by substitution method in equation (29) and simplifying, we obtained the Bayes estimator using *SELF* under Jeffrey prior as:

$$\alpha_{SELF} = \frac{\Gamma(n+1)}{\sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta \Gamma(n)} \quad (30)$$

Using the Squared error loss function (*SELF*), the posterior risk, $p(\alpha_{SELF})$ is defined as:

$$P(\alpha_{SELF}) = E(\alpha^2 | \underline{X}) - \{E(\alpha | \underline{X})\}^2 \quad (31)$$

Therefore, the posterior risk under Jeffrey's prior using the squared error loss function is:

$$P(\alpha_{SELF}) = \frac{\Gamma(n+2)\Gamma(n) - (\Gamma(n+1))^2}{\left(\sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta\right)^2 (\Gamma(n))^2} \quad (32)$$

186

187 Using Quadratic Loss Function (*QLF*) under Jeffrey's prior

188 The derivation of Bayes estimator under *QLF* using Jeffrey's prior is given below:

$$\alpha_{QLF} = \frac{E(\alpha^{-1})}{E(\alpha^{-2})} = \frac{E(\alpha^{-1} | \underline{X})}{E(\alpha^{-2} | \underline{X})} \quad (33)$$

$$E(\alpha^{-1} | \underline{X}) = \int_0^\infty \alpha^{-1} p(\alpha | \underline{X}) d\alpha$$

191 Substituting for $p(\alpha | \underline{X})$ in equation (3.29); we have:

$$E(\alpha^{-1} | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta\right)^n}{\Gamma(n)} \int_0^\infty \alpha^{n-2} e^{-\alpha \sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta} d\alpha$$

193 (34)

194 Using integration by substitution method in equation (34) and simplifying, we obtained the Bayes estimator using *QLF* under Jeffrey prior as:

$$\alpha_{QLF} = \frac{\Gamma(n-1)}{\left(\sum_{i=1}^n (e^{\lambda_{x_i}} - 1)^\beta\right) \Gamma(n-2)}$$

197 (35)

198 Using the Quadratic loss function (*QLF*), the posterior risk, $p(\alpha_{QLF})$ is defined as:

$$P(\alpha_{QLF}) = 1 - \frac{\{E(\alpha^{-1} | \underline{X})\}^2}{E(\alpha^{-2} | \underline{X})} \quad (36)$$

200 Hence, it is obtained as:

$$P(\alpha_{QLF}) = \frac{\Gamma(n)\Gamma(n-2) - [\Gamma(n-1)]^2}{\Gamma(n)\Gamma(n-2)} \quad (37)$$

202

203 Using Precautionary Loss Function (*PLF*) under Jeffrey's prior

204 Similarly, the derivation of Bayes estimator under *PLF* using Jeffrey's prior is given below:

$$\begin{aligned} \alpha_{PLF} &= \left\{ E(\alpha^2) \right\}^{\frac{1}{2}} = \left\{ E(\alpha^2 | \underline{X}) \right\}^{\frac{1}{2}} = \sqrt{E(\alpha^2 | \underline{X})} \\ E(\alpha^2 | \underline{X}) &= \int_0^{\infty} \alpha^2 p(\alpha | \underline{X}) d\alpha \end{aligned} \quad (38)$$

Substituting for $p(\alpha | \underline{X})$ in equation (38); we have:

$$E(\alpha^2 | \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^{\beta} \right)^n}{\Gamma(n)} \int_0^{\infty} \alpha^{n+1} e^{-\alpha \sum_{i=1}^n (e^{\lambda x_i} - 1)^{\beta}} d\alpha$$

(39)

Using integration by substitution method in equation (39) and simplifying, we obtained the Bayes estimator using *PLF* under Jeffrey prior as:

$$\alpha_{PLF} = \sqrt{\frac{\Gamma(n+2)}{\left(\sum_{i=1}^n (e^{\lambda x_i} - 1)^{\beta} \right)^2 \Gamma(n)}}$$

(40)

Using the Precautionary loss function (PLF), the posterior risk, $p(\alpha_{PLF})$ is defined as:

$$P(\alpha_{PLF}) = 2 \{ \alpha_{PLF} - E(\alpha | \underline{X}) \}$$

(41)

Hence, obtained as:

$$P(\alpha_{PLF}) = 2 \left\{ \frac{\{ \Gamma(n+2) \Gamma(n) \}^{\frac{1}{2}} - \Gamma(n+1)}{\sum_{i=1}^n (e^{\lambda x_i} - 1)^{\beta} \Gamma(n)} \right\} \quad (42)$$

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220

221

222 5 Simulation Study

223 We used a package in R software to generate random samples of size $n = (25, 35, 75, 125)$
 224 from *WED* by using different values of α , λ and β . Tables 1 to 3 present the results of our
 225 simulation study by listing the estimates of the scale parameter under the appropriate
 226 estimation methods such as the Maximum Likelihood Estimation (*MLE*), Squared Error Loss
 227 Function (*SELF*), Quadratic Loss Function (*QLF*), and Precautionary Loss Function (*PLF*)
 228 under both Uniform and Jeffrey priors. The Maximum Likelihood estimates and Bayesian
 229 estimates were obtained under a reasonably high number of replications, different sample
 230 sizes were used to investigate the performance of the estimators in relation to their biases and
 231 mean squared errors as well as the sample sizes.

232 The performance of the two methods was evaluated using the following performance
 233 measures:

234 Bias = $\frac{1}{N} \sum_{i=1}^N (\hat{\alpha}_i - \alpha)$ and Mean Square Error, $MSE = \frac{1}{N} \sum_{i=1}^N (\hat{\alpha}_i - \alpha)^2$

235 Where $\hat{\alpha}$ is the estimate of α (the scale parameter) from the i th simulated data and N is the
236 number of Monte Carlo samples.

237 Note: the estimate with the minimum bias and MSE will be considered as the best.

238 Table 1: Simulation of Bayes Estimates, their Biases and Mean Squared Errors based on the
239 replications and sample sizes where $\alpha = 1.0$, $\lambda = 0.5$ and $\beta = 1.5$.

Sample sizes	Measures	MLE	Uniform Prior			Jeffrey's Prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	3.3233	3.4562	3.1904	3.5221	3.3233	3.0575	3.3891
	BIAS	2.7930	2.9447	2.6413	3.0199	2.7930	2.4896	2.8681
	MSE	8.5275	9.4573	7.6460	9.9357	8.5275	6.8129	8.9818
35	Estimate	3.9508	4.0637	3.8379	4.1197	3.9508	3.7250	4.0069
	BIAS	3.3942	3.5197	3.2686	3.5821	3.3942	3.1431	3.4565
	MSE	12.2102	13.1182	11.3349	13.5813	12.2102	10.4922	12.6570
75	Estimate	6.0459	6.1265	5.9653	6.1667	6.0459	5.8847	6.0861
	BIAS	5.2692	5.3528	5.1856	5.3945	5.2692	5.1020	5.3109
	MSE	28.4087	29.3138	27.5177	29.7702	28.4087	26.6409	28.8580
125	Estimate	7.5596	7.6201	7.4992	7.6503	7.5596	7.4387	7.5898
	BIAS	7.0219	7.0860	6.9577	7.1180	7.0219	6.8935	7.0539
	MSE	49.9465	50.8622	49.0392	51.3222	49.9465	48.1402	50.4024

240

241 Table 2: Simulation of Bayes Estimates, their Biases and Mean Squared Errors based on the
242 replications and sample sizes where $\alpha = 1.0$, $\lambda = 2.5$ and $\beta = 3.5$.

Sample sizes	Measures	MLE	Uniform Prior			Jeffrey's Prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	0.0095	0.0099	0.0091	0.0100	0.0095	0.0087	0.0097
	BIAS	0.9889	0.9884	0.9893	0.9882	0.9889	0.9898	0.9886
	MSE	0.9779	0.9770	0.9787	0.9765	0.9779	0.9796	0.9774
35	Estimate	0.0059	0.0061	0.0057	0.0061	0.0059	0.0056	0.0060
	BIAS	0.9933	0.9932	0.9935	0.9931	0.9933	0.9937	0.9932
	MSE	0.9867	0.9864	0.9871	0.9862	0.9867	0.9875	0.9865
75	Estimate	0.0019	0.0019	0.0019	0.0019	0.0019	0.0018	0.0019
	BIAS	0.9979	0.9979	0.9979	0.9979	0.9979	0.9980	0.9979
	MSE	0.9958	0.9958	0.9959	0.9957	0.9958	0.9959	0.9958
125	Estimate	0.0009	0.0009	0.0009	0.0009	0.0009	0.0009	0.0009
	BIAS	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990
	MSE	0.9981	0.9980	0.9981	0.9980	0.9981	0.9981	0.9981

243

244 Table 3: Simulation of Bayes Estimates, their Biases and Mean Squared Errors based on the
245 replications and sample sizes where $\alpha = 1.0$, $\lambda = 2.5$ and $\beta = 0.5$.

Sample sizes	Measures	MLE	Uniform Prior			Jeffrey's Prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	0.0095	0.0099	0.0091	0.0100	0.0095	0.0087	0.0097
	BIAS	0.9889	0.9884	0.9893	0.9882	0.9889	0.9898	0.9886
	MSE	0.9779	0.9770	0.9787	0.9765	0.9779	0.9796	0.9774
35	Estimate	0.0059	0.0061	0.0057	0.0061	0.0059	0.0056	0.0060

	BIAS	0.9933	0.9932	0.9935	0.9931	0.9933	0.9937	0.9932
	MSE	0.9867	0.9864	0.9871	0.9862	0.9867	0.9875	0.9865
75	Estimate	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
	BIAS	0.9979	0.9979	0.9979	0.9979	0.9979	0.9980	0.9979
	MSE	0.9958	0.9958	0.9959	0.9957	0.9958	0.9959	0.9958
125	Estimate	0.0009	0.0009	0.0009	0.0009	0.0009	0.0009	0.0009
	BIAS	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990	0.9990
	MSE	0.9981	0.9980	0.9981	0.9980	0.9981	0.9981	0.9981

6 Conclusions

We have obtained the Bayesian estimators of the scale parameter of Weibull-Exponential distribution in this study, the Posterior distributions of this parameter are derived by using Uniform and Jeffrey's priors. Bayes estimators and their associated risks have been obtained by using three loss functions under the two prior distributions. The three loss functions used in this study are the Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*) and Precautionary Loss Function (*PLF*). The biases and mean square errors based on all the priors and for all the loss functions relating to the shape parameter of the Weibull-Exponential distribution expectedly increase with increase in sample size. For $\alpha = 1.0$, $\lambda = 0.5$ and $\beta = 1.5$, the result indicates that using *QLF* under Jeffrey prior produces the best estimator with minimum bias and *MSE* followed by *QLF* under Uniform prior and these performances are found to be consistent irrespective of the different sample sizes used.

For $\alpha = 1.0$, $\lambda = 2.5$ and $\beta = 3.5$, the result indicates that using *PLF* under uniform prior produces the best estimator with minimum bias and *MSE* followed by *PLF* under Jeffrey's prior and these performances are found to be consistent irrespective of the different sample sizes used and this generally means that increasing the exponential or shape parameter has a very huge impact on the estimate of the scale parameter of the Weibull-Exponential distribution.

Also, for $\alpha = 1.0$, $\lambda = 2.5$ and $\beta = 0.5$, our result shows that using *PLF* under uniform prior produces the best estimator with minimum bias and *MSE* followed by *PLF* under Jeffrey's prior and again these performances are found to be consistent irrespective of the different sample sizes considered and this proves that increasing the exponential parameter alone with a constant shape and scale parameter has a very large effect on the estimate of the scale parameter of the Weibull-Exponential distribution. Most importantly, we found that using the *QLF* under both priors is a better approach for estimating the scale parameter of the *WED* when the exponential and shape parameters take smaller values whereas using the *PLF* under both priors serves better when the exponential parameter takes a bigger value irrespective of other values of the parameters and the different sample sizes.

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