Original Research Article

ON PAIRWISE SINGULAR COMPACTIFICATION

Abstract

By introducing the notion of a pairwise singular compactification for a pairwise hausdorff, pairwise locally compact bitopological space it is proved that a αX is a pairwise singular compactification for X iff αX-X is a pairwise retract of αX.

1. Introduction

By a space we mean a Bitopological space and by a map we mean a pairwise continuous map between Bitopological spaces. Letters X,Y,Z are used for Bitopological spaces and f,g,h etc are used for maps between them.

A Bitopological space is a triple $(X, \Im_1, \Im_2)$ where $\Im_1$ and $\Im_2$ are topologies on a set X.
J.C.Kelly [5] initiated the systematic study of such spaces and several other authors namely Weston, Lane [7], Patty [10] etc. contributed to the development of the theory. Kelly introduced pairwise Housdorff spaces, pairwise regular and pairwise normal spaces in the theory [5].

A cover *U* of a Bitopological space $(X,\Im_1, \Im_2)$ is called pairwise open if $U \subseteq \Im_1 \cup \Im_2$ and *U* contains at least one non-empty member of $\Im_1$ and one non-empty member of $\Im_2$. A Bitopological space $(X,\Im_1, \Im_2)$ is called pairwise compact if every pairwise open cover of $(X,\Im_1,\Im_2)$ has a finite subcover [11]. According to I.L. Reilly [11] a Bitopological space $(X,\Im_1,\Im_2)$ is called a pairwise locally compact if $\Im_1$ is locally compact with respect to $\Im_2$ and $\Im_2$ is locally compact with respect to $\Im_1$. Recall that $\Im_1$ is locally compact with respect to $\Im_2$ if each point of X has a $\Im_1$ open neighborhood whose $\Im_2$ – Closure is pairwise compact.

A Bitopological space $(X,\Im_1,\Im_2)$ is called pairwise Hausdroff if for two distinct points x and y there is a $\Im_1$ neighborhood U of x and $\Im_2$ neighborhood V of y such that $U \cap V = \phi$ [5]

A function f: $(X,\Im_1,\Im_2) \rightarrow (Y, L_1, L_2)$ is called pairwise continuous if the induced function f:$(X, \Im_1) \rightarrow (Y, L_1)$ and f:$(X, \Im_2) \rightarrow (Y, L_2)$ are continuous [11].

Notion of the singular set of a mapping defined by Whyburn and Cain was further investigated by various workers including Cain, Chandler, Tzunng, Magill Jr. Faulkner and Duda etc. Later this concept led to the concept of a singular compactification and this combination of these two independent areas added many steps to the theory of compactifications.

A Compactification αX is a compact, Hausdroff space that contains X as a dense subspace. A Compactification αX is called a singular compactification if it arises out of a singular mapping from X to αX-X.

We recall the construction of a singular compactification given by Chandler etc al.: Consider a map f: $X \to Y$ with X locally compact Hausdroff and Y is a compact Hausdroff space. Equip the disjoint union $X \cup Y$ of X and Y with a topology in which all open sets of X are open in $X \cup Y$ and for $y \in Y$, the family $\{V \cup f^{-1}(v) - K \mid V$ is an open set in Y containing y and K is a compact set in X$\}$ form a neighborhood base. With this topology $X \cup Y$ is easily seen to be compact and Hausdroff. Denote this space by $X+_f Y$. Here compactness of Y gives the compactness of $X+_f Y$, while local compactness of X is responsible for the Hausdroff of $X+_f Y$.

The idea of the pairwise singular map between Bitopological Spaces was introduced in [17]. Recall that a pairwise continuous map

$$f: (X, \Im_1, \Im_2) \to (Y, L_1, L_2)$$

is called a pairwise singular map if it is a L_1 singular with respect to $\Im_2$ and L_2 singular with respect to $\Im_1$.

f is L_1 singular with respect to $\Im_2$ if for each $U \in L_1$, $\Im_2$ cl $f^{-1}(U)$ is not compact and vice-versa.

Continuing our study in this area we have introduced pairwise singular compactification for pairwise locally compact spaces in section 2 of this paper. Following Faulkner [3] a characterization of pairwise singular Compactifications is obtained for Bitopological spaces in terms of Pairwise retracts.

One point compactification for pairwise locally compact, pairwise Hausdroff Bitopological spaces are already introduced by I.L. Reilly in [11]. For concerned definitions we follow Reilly.

2. Pairwise Singular Compactifications

In this section we construct an analogue of pairwise singular compactification for a given pairwise locally compact, pairwise Hausdroff

Bitopological space. The section begins with the following definition of pairwise singular sets [12].

2.1. Definition. Let f: $(X,\Im_1,\Im_2) \to (Y, L_1, L_2)$ be a pairwise continuous map where X,Y are pairwise locally compact pairwise Hausdroff Bitopological spaces. Then a point $y\in Y$ is called L_1 singular point with respect to L_2 if for each open set $U\in L_1$ of Y with $y\in U$, L_2 cl $f^{-1}(U)$ is not compact.

A Point $y\in Y$ is called L_2 singular point with respect to $\Im_1$ if for each open set $V\in L_2$ of Y with $y\in V$, $\Im_1$ cl $f^{-1}(V)$ is not compact.

A Point $p\in Y$ is called a pairwise singular point if it is L_1 singular point with respect to $\Im_2$ and L_2 singular point with respect to $\Im_1$.

The set of all pairwise singular points of f: $X\to Y$ is called the pairwise singular set of f and it is denoted by $S_B(f)$.

2.2.Definition. Let $f:(X,\Im_1,\Im_2) \to (Y, L_1, L_2)$ be a pairwise continuous map with f(x) pairwise dense in (Y, L_1, L_2). Then f is called pairwise singular if $S_B(f)=Y$.

The pairwise singular set of $f:(X,\Im_1,\Im_2) \to (Y, L_1, L_2)$ denoted by $S_B(f)$ is in fact the following:

$$S_B(f) = S\,(f, L_1,\Im_2) \cap S\,(f, L_2, \Im_1).$$

Let X be a pairwise locally compact, pairwise Hausdroff Bitoipological space and K be a pairwise compact Bitopological space. Let f: $X\to Y$ be a pairwise continuous and pairwise singular map with f(X) pairwise dense in Y. Consider the following:

$B_1 = \Im_1 \cup\{U\cup f^{-1}(U) - H \mid U\in L_1$ and H is pairwise compact, H is $\Im_1$ – compact, $\Im_2$ – compact$\}$

$B_2 = \Im_2 \cup\{V\cup f^{-1}(V) - M \mid V\in L_2$ and M is pairwise compact, M is $\Im_1$ – compact, $\Im_2$ – compact$\}$

Then B_1, B_2 form bases for two respective topologies on $X\cup_f Y$, thus making it a Bitopological space. We denote it by $(X\cup_f Y,P_1, P_2)$.

1. $(X\cup_f Y$, P_1, $P_2)$ is pairwise compact : Take a pairwise open cover U of $(X\cup_f Y,P_1, P_2)$. We take U to consist of basic open sets $(X\cup_f Y, P_1, P_2)$. Clearly U forms a pairwise cover of (Y, L_1, L_2). Since Y is pairwise compact, U permits a finite subcover say: $\{U_i U f^{-1}(U_i)-H_i \mid i=1\ldots..n\} \cup \{V_j U f^{-1}(V_j)-M_j \mid j=1\ldots..m\}$.

Note that this family covers the whole of $X \cup Y$ except the union of $\bigcup_{i=1}^{n} H_i$ and $\bigcup_{j=1}^{m} M_j$.
These are pairwise compact and pairwise compact is an absolute property. Hence we get a finite subcover for U, giving the pairwise compactness of $(X \cup_f Y, P_1, P_2)$.

2. $(X \cup_f Y, P_1, P_2)$ is pairwise Hausdroff: To show that $(X \cup_f Y, P_1, P_2)$ is pairwise Hausdroff, there are three cases arises.

1. If $x, y \in X$, then we get $U \in \Im_1, V \in \Im_2$ with $x \in U$, $y \in V$
Such that $U \cap V = \varnothing$ using the pairwise Hausdroff of X.

2. If $x, y \in K$, choose $U \in L_1$, $V \in L_2$ with $U \cap V = \varnothing$.
$U \cup f^{-1}(U)$ and $V \cup f^{-1}(V)$ are the required members of P_1, P_2.

3. If $x \in X$, $y \in K$ then choose $V \in \Im_1$ such that $x \in V$ and $U \in L_2$: $y \in U$
now $V \cap [U \cup f^{-1}(U) - \Im_2 \, clV] = \varnothing$.

Since K is L_1 compact therefore $(X \cup_f Y, P_1, P_2)$ is pairwise Hausdroff.

3. $(X, \Im_1, \Im_2)$ is pairwise dense in $(X \cup_f Y, P_1, P_2)$:
To show that X is a dense subspace of $X \cup Y$, take a non empty open set $U \in P_1$ or $U \in P_2$. If there are non empty members of $\Im_1$ and $\Im_2$, then $U \cap X \neq \varnothing$.

Take $U \cup f^{-1}(U) - H \in P_1$, where $U \in L_1$, then $(U \cup f^{-1}(U) - H) \cap X \neq \varnothing$.
Since $f^{-1}(U)$ is not contained in H.
$\therefore$ H is $\Im_2$ compact.

Similarly if $V \cup f^{-1}(V) - M \in P_2$, where $V \in L_2$.
Then $(V \cup f^{-1}(V) - M) \cap X \neq \varnothing$.

Since $f^{-1}(V)$ is not contained in M.

Gaglielmi [4] obtained that a compactification αX of a locally compact space X is singular iff $\alpha X - X$ is a retract of αX. In this section we obtain a characterization of pairwise singular compactifications for Bitopological spaces in terms of pairwise retracts.

2.3. Theorem: - A pairwise compactification of a pairwise locally compact space X is pairwise singular iff $\alpha X - X$ is a pairwise retract of αX.

Proof: - If αX is a pairwise singular compactification through the map $f: X \to \alpha X\text{-}X$.
Define $r: \alpha X \to \alpha X - X$ by

$$r(x) = \{ \begin{matrix} x;\ \text{if } x \in \alpha X\text{-}X \\ f(x);\ \text{if } x \in X. \end{matrix}$$

We need only show that r is continuous.

If V is an open set in αX-X then $r^{-1}(V)=V \cup f^{-1}(V)$ is obviously open in αX . Thus αX-X is a pairwise retract of αX.
Conversely, if r is a pairwise retraction of αX onto αX-X, then the restriction r/x = f.

If U is an L_1^*open set around $P \in \alpha X$-X.
Since X is dense in αX and r^{-1} (U) is an open neighborhood of p. p is necessarily in

$$L_1^* \text{ cl }_{\alpha X} (X \cap r^{-1}(U)) = L_1^* \text{ cl }_{\alpha X}(f^{-1}(U))$$

Implying that

$$\Im_2 \text{ cl }_{\alpha X}(f^{-1}(U)) \neq L_1^* \text{ cl }_{\alpha X}(f^{-1}(U))$$

This shows that $\Im_2 \text{ cl }_{\alpha X}(f^{-1}(U))$ is not compact and hence $p \in S_B(f)$.

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