Approximate Solutions of Nonsmooth Systems via Generalized Euler-Lagrange and Hamiltonian Equations

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Research Article

Abstract

Recently the traditional calculus of variations has been extended to be applicable for systems containing nonsmooth function. In this paper, we have investigated the generalized derivative of nonsmooth functions. The obtained results were applied to investigate the generalized Euler-Lagrange and Hamilton equations for constrained system. The approach was applied within an illustrative.

Keywords: Generalized Derivative; Calculus of variations; Euler-Lagrange equation; Hamilton equation

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1 Introduction

The calculus of variations is concerned with the problem of extremizing functionals. It has many appliacation in physics, geometry, engineering, dynamics, control theory, and economics [Kalman (1960); Rockafellar, Clarke et al. (1972); Rockafellar (1970, 1975, 1976); Clarke (2009); Clarke et al. (2008); Clarke (1975, 1976); Almeida and Torres (2009); Mordukhovich (1988); Gale (1967)].

The formulation of a problem of the calculus of variations requires two step: the specification of a performance criterion; and then, the statement of physical constraints that should be satisfied. The basic problem is stated as follows:

$$Minimize\ J(x(\cdot)) = \int_{t_0}^{t_f} L(t, x(t), \dot{x}(t))dt \tag{1.1}$$

$$subject\ to \qquad x(t_0) = x_0,\ x(t_f) = x_f, \tag{1.2}$$

where $x(\cdot)$ is the state variable and $L(\cdot, x(\cdot), \dot{x}(\cdot))$ (function L, in short) is Riemann integrable nonsmooth function and x_0, x_f are given vectors.

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The traditional calculus of variation cannot be used to obtain the Euler-Lagrange and Hamilton equations of nonsmooth systems. The main aim of this paper is to obtain the corresponding generalized Euler-Lagrange and Hamilton equations for above nonsmooth variational problem (NSVP), within generalized derivatives which is proposed by Kamyad et.al. [Kamyad et al. (2011)] that utilize in the next section. Other generalized derivatives have been proposed are not practical, for instance the limiting proximal subgradient by Clarke [Clarke (1989)], the approximate subdifferential by Ioff [Ioffe (1984, 1981, 1984); Ioffe and Rockafellar, (1996)] and the subdifferential by Mordukhovich [Rockafellar and Wets (2009); Mordukhovich (1988, 1992, 1995, 1994)]. Hence, theirs results include some restrictions for examples the function L must be locally Lipschitz or convex and may, the set of generalized derivative of L on $[t_0, t_f]$, either is empty. Then we almost can not use them for obtainning the optimal solution of the nonsmooth problem (1.1)-(1.2). It is noteworthy that, these conditions are only criterions for testing the optimality of a given state $x(\cdot)$. We present a different definition to derive generalized derivatives (GDs) for nonsmooth functions, in which the involved functions are Riemann integrable but not necessary locally Lipschitz or continuous. It will be shown that this kind of GDs is particularly helpful, practical and dose not have the above restriction. There are some conditions on function L in problem (1.1) for existence of an optimal solution (see [Kalman (1960); Rockafellar (1970); Clarke (1975)]). For problem (1.1), by assuming differentiability, one way to deal with this problem is to solve the second order differential equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0, \tag{1.3}$$

called the Euler-Lagrange (EL) equation. The two given boundary conditions provide sufficient information to determine the two arbitrary constants. But if there are no boundary constraints, then we need to impose other conditions, called the natural boundary conditions (see [Gelfand and Silverman (2000)]),

$$\left[\frac{\partial L}{\partial \dot{x}}\right]\Big|_{t=t_0} = 0 \qquad and \qquad \left[\frac{\partial L}{\partial \dot{x}}\right]\Big|_{t=t_f} = 0. \tag{1.4}$$

Clearly, such terminal conditions are important in models, the optimal control or decision rules are not unique without these conditions. Here, in non-differentiability (or nonsmoothness) conditions, we compute Euler Lagrange equations for unconstrained and constrained nonsmooth variational problems, by using generalized derivatives in the nonsmooth analysis that was presented by [Kamyad et al. (2011)]. The resulting equations are found to be similar to those for smooth variational problems. In other words, the results of nonsmooth calculus of variations reduce to those obtained from traditional smooth calculus of variations where the derivative is replaced by generalized derivate. Futhermore, we propose necessary optimality conditions for nonsmooth control systems via generalized Hamilton-Jacobi equation.

The plan of this paper is as follows: in section 2, we present a novel generalized derivative for Riemann integrable nonsmooth functions and state the assumptions, notations and the results of the literature needed in the sequel. Section 3, reviews a generalized Euler-Lagrange (GEL) equation for problem (1.1). Our contribution is then given in section 4: we analyze the generalized Hamilton equation for nonsmooth optimal control (NSOC) system. Finally, in section 5, we explain the novelties of our results. Section 6, is devoted to our conclusions.

2 Preliminaries on generalized derivatives

Here, at first, we brifly introduce a practical GD which is proposed by [Skandari et al. (2013, 2014)] and use the new GDs for nonsmooth calculus of variations. Let Ω be a connected and compact set and $L : \Omega \subset \mathbb{R} \to \mathbb{R}$ is a one-variable Riemann integrable nonsmooth function. Assume that $C(\Omega)$ and $C^1(\Omega)$ are the spaces of continuous and continuous differentiable functions on the set Ω, respectively. Assume that $\varphi_j(\cdot)$, $j = 0, 1, 2, \ldots$, are the continuously differentiable basic functions for the space $C(\Omega)$ and $N_\delta(s)$ is the neighbourhood of s with radius δ. Divide Ω into the similar sets Ω_i, $i = 1, 2, \ldots, m$ (m is a sufficiently big number), such that $\Omega = \cup_{i=1}^{m}\Omega_i$ and $int(\Omega_i) \cap int(\Omega_j) = \emptyset$, $i \neq j$. Given $s_i \in int(\Omega_i)$, $i = 1, 2, \ldots, m$, and $\delta > 0$ is sufficiently small number. Now consider the following optimization problem which we use it for the GD of $L(\cdot)$:

$$Minimize \sum_{i=1}^{m} \int_{N_\delta(s_i)} \Big| L(x) - L(s_i) - (x - s_i) \sum_{j=0}^{\infty} a_j \varphi_j(s_i) \Big| dx \tag{2.1}$$

where $a_j \in \mathbb{R}$, $j = 0, 1, 2, \ldots$, are unknown variables of this problem. By assumptions $g(s) = \Sigma_{j=0}^{\infty} a_j \varphi_j(s)$, $s \in \Omega$, the problem (2.1) is equivalent to the following problem:

$$Minimize \sum_{i=1}^{m} \int_{N_\delta(s_i)} \Big| L(x) - L(s_i) - (x - s_i) g(s_i) \Big| dx \tag{2.2}$$

It is obvious that if $g^*(\cdot) \in C(\Omega)$ is an optimal solution for problem (2.2) then there exist a_j^*, $j = 0, 1, 2, \ldots$, such that $g^*(s_i) = \sum_{j=0}^{\infty} a_j^* \varphi_j(s_i)$, $i = 1, 2, \ldots, m$.

Theorem 2.1. *Lef $L \in C^1(\Omega)$ and $\delta > 0$ be a sufficiently small number. the unique optimal solution of the optimization problem (2.2) is the function $L'(\cdot)$.*

Here, we state a Lemma such that we use it for converting the nonsmooth optimization problem (2.2) to the smooth problem.

Lemma 2.2. *Let the pair $(u^*(\cdot), v^*(\cdot))$ be the optimal solution of the following smooth problem:*

$$\begin{aligned} &Minimize \quad v(x) \\ &subject\ to \\ &\quad v(x) \geq u(x), \;\; v(x) \geq -u(x), \\ &\quad u(\cdot), v(\cdot) \in C(\Omega), \;\; x \in \Omega. \end{aligned} \tag{2.3}$$

where Ω is a compact set. Then $u^*(\cdot)$ is the optimal solution of the following nonsmooth problem:

$$\begin{aligned} &Minimize \quad |u(x)| \\ &subject\ to \quad u(\cdot) \in C(\Omega). \end{aligned}$$

Proof. See [Skandari et al. (2014)]. □

Now, using Lemma (2.2), we can approximate the nonsmooth optimization problem (2.2) into a corresponding smooth optimization problem as follows:

$$\begin{aligned} &Minimize \quad \sum_{i=1}^{m} \int_{N_\delta(s_i)} v(x, s_i) dx \\ &subject\ to \\ &\quad -v(x, s_i) \leq L(x) - L(s_i) - (x - s_i) g(s_i), \qquad i = 1, 2, \ldots, m, \\ &\quad -v(x, s_i) \leq -L(x) + L(s_i) + (x - s_i) g(s_i), \qquad i = 1, 2, \ldots, m, \\ &\quad x \in (s_i - \delta, s_i + \delta), \;\; g(\cdot) \in C(\Omega), \\ &\quad v(\cdot, \cdot) \in C(\Omega^2), \;\; v(\cdot, \cdot) \geq 0. \end{aligned} \tag{2.4}$$

Theorem 2.3. *Let $L : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a Riemann integrable nonsmooth function and $g^*(\cdot)$ be the optimal solution of the optimization problem (2.4). The GD of function $L(\cdot)$ denoted by $\partial L(\cdot)$ and is defined as $\partial L(\cdot) = g^*(.)$.*

Proof. See [Skandari et al. (2014)]. □

Remark 2.1. We refer the interested reader to [Skandari et al. (2013, 2014); Kamyad et al. (2011)], where this functional optimization problem is approximated with the corresponding linear programming problem that we can solve it by linear programming methods such as simplex method. Indeed, we can approximate the obtained GDs of nonsmooth function $L(\cdot)$ with Fourier series [Stein and Weiss (1971)]. So for any Riemann integrable nonsmooth function $L(\cdot)$, if a_j^*, $j = 0, 1, 2, \ldots$, is the optimal solution of the problem (2.1), then $g^*(.) = \sum_{j=0}^{\infty} a_j^* \varphi_j(.)$ is an optimal solution for the problem (2.2) and we have the GDs as $\partial L(\cdot) = \sum_{j=0}^{\infty} a_j^* \varphi_j(.)$. For computing applications, we approximate the GDs by a finite Fourier series.

The next section is to write the Euler-Lagrange and the corresponding Hamiltonian equations.

3 Generalized Euler-Lagrange equations

Now, we shall think in terms of finding state trajectories that minimize performance measures. In control problems, trajectories are determined by control histories and initial conditions; however to simplify the discussion it will be assumed initially that there are no such constraints and that the states can be directly and independently varied.

In this line of thought, we consider the functional (1.1) defined on the set of continuous curves $x : [t_0, t_f] \to \mathbb{R}$ that satisfying prescribed boundary conditions (1.2).

Definition 3.1. The functional $J(.)$ is said to have a local minimum (resp. local maximum) at $x(.)$ if there exists a $\delta > 0$ such that $J(x(.)) \leq J(\hat{x}(.))$ (resp. $J(x(.)) \geq J(\hat{x}(.))$) for all $\hat{x}(.)$ satisfying $\|x(.) - \hat{x}(.)\| < \delta$.

It is desired to find the function x^*, among all curves $x(t)$ satisfying the boundary conditions (1.2), for which the functional (1.1) has a relative extermum. Rockafellar [Rockafellar (1970, 1975, 1976, 1970); Rockafellar, Clarke et al. (1972)] and Clarke [Clarke (1975, 2009, 1975, 1976, 1989); Clarke et al. (2008)] began the studies where L for any $t \in [0, 1]$ is convex and Lipschitz continuity function respectively but here, L is a continuous nonsmooth function. Now by obtaining the GD of L from last section, a necessary condition for this problem, is given by the next result.

Lemma 3.1. *Suppose that* $\int_{t_0}^{t_f} \eta(x)g(x)dx = 0$ *for all* $\eta(.) \in C[t_0, t_f]$. *If* $g : [t_0, t_f] \to \mathbb{R}$ *is a continuous function then* $g \equiv 0$ *on the interval* $[t_0, t_f]$.

Proof. See [Kamyad et al. (2011)]. □

Theorem 3.2. *A necessary condition for the differentiable functional* $J(x)$ *to have an extermum for* $x = x^*$ *is that its variation vanishes for* $x = x^*$

$$\delta J = 0.$$

Proof. See [Gelfand and Silverman (2000)]. □

Theorem 3.3. *Let* x *be a extermizer of* J *as in problem (1.1)-(1.2), then, for all* $t \in [t_0, t_f]$, x *is a solution of the GEL equation*

$$\partial_x L(t, x(t), \dot{x}(t)) - \frac{d}{dt}\partial_{\dot{x}} L(t, x(t), \dot{x}(t)) = 0. \tag{3.1}$$

Proof. Suppose that x is an extermizer of J. We can proceed as Lagrange did, by considering the value of J at a nearby function $\tilde{x} = x + \epsilon h$, where $\epsilon \in \mathbb{R}$ be a real number with $|\epsilon| \ll 1$, and h be an admissible variation such that $h(t_0) = h(t_f) = 0$. Now consider the increment of the functional J as follows:

$$\delta J = J(x + \epsilon h) - J(x) = \int_{t_0+\delta t_0}^{t_f+\delta t_f} L(t, x + \epsilon h, \dot{x} + \epsilon \dot{h})dt - \int_{t_0}^{t_f} L(t, x, \dot{x})dt$$

Here we have

$$\int_{t_0+\delta t_0}^{t_f+\delta t_f} L(t, x + \epsilon h, \dot{x} + \epsilon \dot{h})dt = \int_{t_0}^{t_f} L(t, x + \epsilon h, \dot{x} + \epsilon \dot{h})dt$$
$$+ \int_{t_f}^{t_f+\delta t_f} L(t, x + \epsilon h, \dot{x} + \epsilon \dot{h})dt - \int_{t_0}^{t_0+\delta t_0} L(t, x + \epsilon h, \dot{x} + \epsilon \dot{h})dt$$

By using the generalized first order Taylor expanision of the nonsmooth function $L(t, x + \epsilon h, \dot{x} + \epsilon \dot{h})$ at point $(t, x, \dot{x})$ [Skandari et al. (2014)], we obtain

$$\int_{t_0}^{t_f} L(t, x + \epsilon h, \dot{x} + \epsilon \dot{h})dt = \int_{t_0}^{t_f} L(t, x, \dot{x})dt + \int_{t_0}^{t_f} \epsilon h(t)\partial_x L(t, x, \dot{x})dt$$
$$+ \int_{t_0}^{t_f} \epsilon \dot{h}(t)\partial_{\dot{x}} L(t, x, \dot{x})dt + E_{\epsilon,\partial L}(t, x, \dot{x})$$

Where $\lim_{\epsilon \to 0} E_{\epsilon,\partial L}(t, x, \dot{x}) = 0$ [Skandari et al. (2014)]. Now using integration by parts, we obtain

$$\int_{t_0}^{t_f} L(t, x + \epsilon h, \dot{x} + \epsilon \dot{h})dt \simeq \int_{t_0}^{t_f} L(t, x, \dot{x})dt$$
$$+ \int_{t_0}^{t_f} \epsilon h(t)\Big(\partial_x L(t, x, \dot{x}) - \frac{d}{dt}\partial_{\dot{x}} L(t, x, \dot{x})\Big)dt + \Big[\epsilon h(t)\partial_x L(t, x, \dot{x})\Big]\Big|_{t_0}^{t_f}$$

We also have

$$\int_{t_f}^{t_f+\delta t_f} L(t, x + \epsilon h, \dot{x} + \epsilon \dot{h})dt \simeq \delta t_f L(t, x, \dot{x})\Big|_{t=t_f},$$
$$\int_{t_0}^{t_0+\delta t_0} L(t, x + \epsilon h, \dot{x} + \epsilon \dot{h})dt \simeq \delta t_0 L(t, x, \dot{x})\Big|_{t=t_0}.$$

So finally we obtain

$$\delta J \simeq \int_{t_0}^{t_f} \epsilon h(t)\Big(\partial_x L(t, x, \dot{x}) - \frac{d}{dt}\partial_{\dot{x}} L(t, x, \dot{x})\Big)dt + \Big[\epsilon h(t)\partial_x L(t, x, \dot{x})\Big]\Big|_{t_0}^{t_f}$$
$$+ \delta t_f L(t, x, \dot{x})|_{t=t_f} - \delta t_0 L(t, x, \dot{x})\Big|_{t=t_0}$$

Since h, ϵ, δt_0, δt_f are arbitrary, according to theorem (3.2), equating δJ to zero yields the result. □

Now, we present the Euler-Lagrange equation for functionals containing dependent variables.

Theorem 3.4. *Let x be a extermizer of J as in problem (1.1)-(1.2), where $x = (x_1, \ldots, x_n)$, $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)$, and x_k, $k = 1, 2, \ldots, n$, are continuous real-valued functions defined on $[t_0, t_f]$. Then x is a solution of the GEL equation*

$$\partial_{x_k} L(t, x(t), \dot{x}(t)) - \frac{d}{dt}\partial_{\dot{x}_k} L(t, x(t), \dot{x}(t)) = 0$$

$k = 1, 2, \ldots, n$, for all $t \in [t_0, t_f]$.

We give new necessary optimality conditions for: (i) functionals of form (1.1) with free boundary conditions; (ii) nonsmooth isoperimetric problem; and (iii) nonsmooth problem with subsidiary holonomic constraints.

3.1 Natural boundary conditions

Determine continuous curves x such that the functional J, defined in (1.1), has an extermum at x. Note that no boundary conditions are now imposed.

Theorem 3.5. *Let x be a local extermizer to problem (1.1). Then, x satisfies the following GEL equation*

$$\partial_x L(t, x(t), \dot{x}(t)) - \frac{d}{dt}\partial_{\dot{x}} L(t, x(t), \dot{x}(t)) = 0 \tag{3.2}$$

for all $t \in [t_0, t_f]$. Morever, if $x(t_0)$ is not specified, then

$$\Big[\partial_{\dot{x}} L(t, x(t), \dot{x}(t))\Big]\Big|_{t=t_0} = 0 \tag{3.3}$$

and

$$\Big[\partial_{\dot{x}} L(t, x(t), \dot{x}(t))\Big]\Big|_{t=t_f} = 0. \tag{3.4}$$

Proof. If in previous proof picking curves such thet $h(t_0) = 0$ and $h(t_f) \neq 0$, and others such that $h(t_f) = 0$ and $h(t_0) \neq 0$, we deduce the natural boundary conditions (3.3). □

3.2 The Nonsmooth Isoperimetric Problem (NSIP)

We want to find the extermizer of a given functional, when restricted to a prescribed integral constraint. Problems of this type have found many applications in differential geometry, discrete and convex geometry, probability, Banach space theory and multiobjective optimization [Almeida and Torres (2009); Malinowska and Torres (2009)]. We state the NSIP in the following way: find the function x that satisfies the boundary conditions (1.2), the integral constraint

$$I(x(\cdot)) = \int_{t_0}^{t_f} g(t, x(t), \dot{x}(t))dt = \kappa, \qquad \kappa \in \mathbb{R} \tag{3.5}$$

and we obtain a minimum or maximum for (1.1). Similarly as before, we assume that L is a continuous nonsmooth function and suppose that κ is a specified real constant.

Theorem 3.6. *Let x be an extermizer of J given by (1.1) under the condition (1.2) and (3.5). Suppose that x is not an external for I in (3.5), then, there exists a costate variable λ such that x satisfies the GEL equation*

$$\partial_x F - \frac{d}{dt}\partial_{\dot{x}} F = 0$$

for all $t \in [t_0, t_f]$ where $F = L - \lambda g$.

Proof. Let $\epsilon_1, \epsilon_2 \in \mathbb{R}$ be two sufficiently small parameters, such that $\|\epsilon_i\| \ll 1$; $i = 1, 2$. Consider a variation curve of x with two parameters, say $x(t) + \epsilon_1 h_1(t) + \epsilon_2 h_2(t)$ where h_1 and h_2 are two continuous curves satisfying $h_i(t_0) = h_i(t_f) = 0$, $i = 1, 2$. First, we define function $\jmath$ and ℓ by

$$\jmath(\epsilon_1, \epsilon_2) = J(x + \epsilon_1 h_1 + \epsilon_2 h_2)$$

and

$$\ell(\epsilon_1, \epsilon_2) = I(x + \epsilon_1 h_1 + \epsilon_2 h_2) - \kappa$$

Doing calculations as in the proof of theorem (3.3), we obtain

$$\frac{\partial \ell}{\partial \epsilon_2}\Big|_{(0,0)} = \int_{t_0}^{t_f} \Big(h_2 \partial_x g + \dot{h}_2 \partial_{\dot{x}} g\Big)dt = \int_{t_0}^{t_f} \Big(\partial_x g - \frac{d}{dt}\partial_{\dot{x}} g\Big)h_2 dt$$

Since x is not an extermal for I, by the fundamental Lemma of the calculus of variations [Gelfand and Silverman (2000)], there must exist a function h_2 for which

$$\frac{\partial \ell}{\partial \epsilon_2}\Big|_{(0,0)} \neq 0. \tag{3.6}$$

By the implicit function theorem there exist function $\epsilon_2(.)$ defined in an open neighborhood of zero as we may write $\ell(\epsilon_1, \epsilon_2(\epsilon_1)) = 0$. Because $(0,0)$ is an extermum $\jmath$ subject to the constraint $\ell(0,0) = 0$, and $(\partial_{\epsilon_1}\ell, \partial_{\epsilon_2}\ell)(0,0) \neq 0$, by the Lagrange multiplier rule [Gelfand and Silverman (2000)], there exists λ verifying the equation

$$\nabla\Big(\jmath - \lambda\ell\Big)\Big|_{(0,0)} = (0,0)$$

in particular,

$$\frac{\partial \jmath}{\partial \epsilon_1}\Big|_{(0,0)} - \lambda\frac{\partial \ell}{\partial \epsilon_1}\Big|_{(0,0)} = 0 \tag{3.7}$$

Similarly as before, we obtain

$$\frac{\partial \jmath}{\partial \epsilon_1}\Big|_{(0,0)} = \int_{t_0}^{t_f} \Big(h_1\partial_x L + \dot{h}_1\partial_{\dot{x}} L\Big)dt = \int_{t_0}^{t_f} \Big(\partial_x L - \frac{d}{dt}\partial_{\dot{x}} L\Big)h_1 dt$$

and

$$\frac{\partial \ell}{\partial \epsilon_1}\Big|_{(0,0)} = \int_{t_0}^{t_f} \Big(h_1\partial_x g + \dot{h}_1\partial_{\dot{x}} g\Big)dt = \int_{t_0}^{t_f} \Big(\partial_x g - \frac{d}{dt}\partial_{\dot{x}} g\Big)h_1 dt$$

By using this relations in (3.7), we have

$$\int_0^1 \Big[\partial_x L - \frac{d}{dt}\partial_{\dot{x}} L - \lambda\Big(\partial_x g - \frac{d}{dt}\partial_{\dot{x}} f\Big)\Big] h_1(t)dt = 0 \tag{3.8}$$

As (3.8) holds for any function h_1 and by Lemma (3.1), one has

$$\partial_x L - \frac{d}{dt}\partial_{\dot{x}} L - \lambda\Big(\partial_x g - \frac{d}{dt}\partial_{\dot{x}} g\Big) = 0.$$

Introducing $F = L - \lambda g$ we get the desired result. □

3.3 Holonomic constraints

In this section, we consider following problem: find functions x_1 and x_2 for wich the functional

$$J(x_1(\cdot), x_2(\cdot)) = \int_{t_0}^{t_f} L\Big(t, x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)\Big)dt \tag{3.9}$$

has an extermum, when the admissible functions restricted to the boundary conditions

$$(x_1(t_0), x_2(t_0)) = (x_1^0, x_2^0) \qquad and \qquad (x_1(t_f), x_2(t_f)) = (x_1^f, x_2^f) \tag{3.10}$$

$x_i^j \in \mathbb{R}$ for $i = 1, 2, j = t_0, t_f$, and the following holonomic condition satisfy

$$g(t, x_1(t), x_2(t)) = 0. \tag{3.11}$$

where L is as before.

Theorem 3.7. *Let the pair (x_1, x_2) be an extermizer of J as in (3.9), subject to the constraints (3.10)-(3.11). If $\frac{\partial g}{\partial x_2} \neq 0$, then there exists a continuous function $\lambda : [t_0, t_f] \to \mathbb{R}$ such that (x_1, x_2) is a solution of the GEL equations*

$$\partial_{x_k} F - \frac{d}{dt}\partial_{\dot{x}_k} F = 0 \tag{3.12}$$

for all $t \in [t_0, t_f]$ and $k = 1, 2$, where $F = L - \lambda g$.

Proof. Consider a variation of the optimal solution of type

$$(\hat{x_1}(t), \hat{x_2}(t)) = x(t) + \epsilon h(t) = (x_1 + \epsilon h_1, x_2 + \epsilon h_2)$$

where $(h_1(t), h_2(t)$ are continuous curves such that $h_i(t_0) = h_i(t_f) = 0$, $i = 1, 2$ and ϵ is as before. By hypothesis, $\frac{\partial g}{\partial x_2}(t, \hat{x}_1, \hat{x}_2) \neq 0$ therefor it is possible to solve the equation $g(t, \hat{x_1}(t), \hat{x_2}(t)) = 0$ with respect to h_2, $h_2 = h_2(\epsilon, h_1)$. Let $j(\epsilon) = J(\hat{x_1}(t), \hat{x_2}(t))$. Differentiating $j(\epsilon)$ at $\epsilon = 0$, we have

$$\begin{aligned} 0 &= \int_{t_0}^{t_f} \Big(\partial_{x_1} L h_1(t) + \partial_{\dot{x}_1} L \dot{h}_1(t) + \partial_{x_2} L h_2(t) + \partial_{\dot{x}_2} L \dot{h}_2(t)\Big) dt \\ &= \int_{t_0}^{t_f} \Big(\underbrace{\Big(\partial_{x_1} L - \frac{d}{dt}\partial_{\dot{x}_1} L\Big)}_{GEL_1} h_1(t) + \underbrace{\Big(\partial_{x_2} L - \frac{d}{dt}\partial_{\dot{x}_2} L\Big)}_{GEL_2} h_2(t)\Big) \end{aligned} \tag{3.13}$$

Where GEL_1 is the GEL equation respect to x_1 and GEL_2 is the GEL equation respect to x_2. Since $(\hat{x_1}(t), \hat{x_2}(t))$ satisfy the condition (3.11), we have

$$0 = \Big[\frac{d}{d\epsilon} g(t, \hat{x_1}(t), \hat{x_2}(t))\Big]\Big|_{\epsilon=0} = \Big(\partial_{x_1} g\Big) h_1(t) + \Big(\partial_{x_2} g\Big) h_2(t)$$

Getting:

$$h_2(t) = -\frac{\partial_{x_1} g}{\partial_{x_2} g} h_1(t) \tag{3.14}$$

And λ as follows:

$$\lambda(t) = \frac{\partial_{x_2} L - \frac{d}{dt}\partial_{\dot{x}_2} L}{\partial_{x_2} g}. \tag{3.15}$$

Combining (3.14) and (3.15), equation (3.13) can be written as

$$\int_{t_0}^{t_f} \Big(\partial_{x_1} L - \frac{d}{dt}\partial_{\dot{x}_1} L - \lambda(t)\partial_{x_1} g\Big) h_1(t) dt = 0.$$

by Lemma (3.1), and since h_1 is an arbitrary curve, we deduce that

$$\partial_{x_1} L - \frac{d}{dt}\partial_{\dot{x}_1} L - \lambda(t)\partial_{x_1} g = 0. \tag{3.16}$$

Define $F = L - \lambda g$, thus equation (3.12) is obtained. □

We now state (without proof) our previous result in its general form.

Theorem 3.8. *Let J be given by (1.1) where $x = (x_1, \ldots, x_n)$ and $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)$, such that x_k, $k = 1, \ldots, n$, are continuous functions defined on the set of curves that satisfy the boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_1$ and satisfy the constraint $g(t, x) = 0$. If x is an extermizer for J, and if $\frac{\partial g}{\partial x_n} \neq 0$ for all $t \in [t_0, t_f]$, then there exists a continuous function $\lambda(t)$ such that x satisfy the Euler-Lagrange equations*

$$\partial_{x_k} F - \frac{d}{dt}\partial_{\dot{x}_k} F = 0$$

for all $t \in [t_0, t_f]$, where $F = L - \lambda g$.

4 Variational Approach to NSOC Problem

In this section, we formulate the NSOCP for a single input system. Now consider the optimal control of

$$J(x,u) = h(x(t_f),t_f) + \frac{1}{2}\int_{t_0}^{t_f} (x^TQx + u^TRu)dt \tag{4.1}$$

subject to the dynamic constraint

$$\dot{x}(t) = f(x) + g(x)u, \tag{4.2}$$

and the given boundary conditions as $x(t_0) = x_0$, $x(t_f)$ is free and t_f is fixed. Here $x(t)$ and $u(t)$ are n-dimensional state vector and m-dimensional control input, respectively. We make the following assuption:

(A1) The function $f(\cdot)$ is a continuous nonsmooth function,

A2) The functions $h(\cdot)$ and $g(\cdot)$, all of them are differentiable functions,

(A3) $R_{m\times m} > 0$ and $Q_{n\times n} \geq 0$ are constant, symmetric, and respectively positive definite and nonnegative definite matrices.

We say that a state-control pair $(x(\cdot), u(\cdot))$ is admissible if the following conditions hold:

1) The state $x(\cdot)$ is differentiable on $[t_0, t_f]$.
2) The control $u(\cdot)$ is piecewise continuous on $[t_0, t_f]$.
3) The condition $x(t_0) = x_0$ is satisfied.
4) The pair $(x(\cdot), u(\cdot))$ satisfies the differential equation (4.2).

An optimal solution is a pair of functions $(x(.), u(.))$ that minimizes J in (4.1), subject to the nonsmooth dynamic equation (4.2). Using the Lagrange multiplier vector $p(t)$ we introduce the augmented performance index as

$$J_a(u) = h(x(t_f),t_f) + \int_{t_0}^{t_f} \Big\{\frac{1}{2}(x^TQx + u^TRu) + p^T(f(x) + g(x)u - \dot{x})\Big\}dt. \tag{4.3}$$

Taking the first variation of equation (4.3) we obtain

$$\delta J_a(u) = \frac{d}{dx}h(x(t_f),t_f)\delta x(t_f) + \int_{t_0}^{t_f} \Big\{\Big(Qx + p^T(\partial_x f + \dot{g}(x)u)\Big)^T\delta x \tag{4.4}$$
$$-p^T\delta\dot{x} + \Big(Ru + p^Tg(x)\Big)\delta u + \Big(f(x) + g(x)u - \dot{x}\Big)^T\delta p\Big\}dt.$$

Using integration by parts, equation (4.4) can be written as

$$\delta J_a(u) = \frac{d}{dx}h(x(t_f),t_f)\delta x(t_f) - [p\delta x]\Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \Big\{\Big(Qx + p^T\Big[\partial_x f + \tag{4.5}$$
$$\dot{g}(x)u\Big] + \dot{p}\Big)^T\delta x + \Big(Ru + p^Tg(x)\Big)\delta u + \Big(f(x) + g(x)u - \dot{x}\Big)^T\delta p\Big\}dt.$$

Because $x(t_0)$ is specified, we have $\delta x(t_0) = 0$, and since $x(t_f)$ is not specified, we require $p(t_f) = \frac{d}{dx}h(x(t_f),t_f)$. Minimization of $J_a(u)$ requires the coefficients of δp, δx and δu in equation (4.5) be zero. This leads to

$$\dot{x} = f(x) + g(x)u, \tag{4.6}$$

$$\dot{p} = -\Big(Qx + p^T(\partial_x f + \dot{g(x)}u)\Big), \tag{4.7}$$

$$Ru + p^Tg(x) = 0 \tag{4.8}$$

$$x(t_0) = x_0,\ p(t_f) = \frac{d}{dx}h(x(t_f), t_f). \tag{4.9}$$

Equations (4.6)-(4.8) constituate a set of necessary conditions for the optimality of the NSOCP considered here; these conditions are not, in general, sufficient. these equations with the boundary condition (4.9) make a TPBVP. They can be solved using a direct numerical technique.

As a special case, assume that the performance index is an integral of quadratic forms in the state and control,

$$J(x(.)) = \frac{1}{2}\int_0^1 \Big(q(t)x^2(t) + r(t)u^2\Big)dt$$

where $q(t) \geq 0$ and $r(t) > 0$ for $t \in [0, 1]$, and the dynamics of the system is described by the following nonsmooth linear differential equation,

$$\dot{x}(t) = a(t)|x| + b(t)u. \tag{4.10}$$

The EL equations (4.6)-(4.8) and (4.10) lead to equation (4.10) and

$$-\dot{p}(t) = q(t)x + a(t)p^T\partial_x(|x|) \tag{4.11}$$

and

$$r(t)u + b(t)p = 0 \tag{4.12}$$

From equations (4.10) and (4.12), we have

$$\dot{x}(t) = a(t)|x| - r^{-1}(t)b^2(t)p \tag{4.13}$$

The state $x(t)$ and the costate $p(t)$ are obtained by solving the nonsmooth differential equations (4.10) and (4.13) subject to the terminal conditions $x(0) = x_0,\ p(1) = 0$. Once $p(t)$ is known, the control variable $u(t)$ can be obtained using equation (4.12).

4.1 Generalized Hamilton-Jacobi Bellman (GHJB) equation

We once more consider the optimal control problem (4.1)-(4.2), but here, we use an approach via dynamics programming. Let $J^*(x(t), t)$ denote the optimal cost on the interval (t, t_f). Then the basic optimal equation of dynamics programming reads

$$J^*(x, t) = \min_u \Big\{ \int_t^{t+\delta t} \Big(x^TQx + u^TRu\Big)d\tau + J^*(x + \delta x, t + \delta t)\Big\} \tag{4.14}$$

$$= \min_u \Big\{ \Big(x^TQx + u^TRu\Big)\delta t + J^*(x, t) + \frac{d}{dt}J^*(x, t)\delta t\Big\}, \tag{4.15}$$

therefor, (as $J^*(x, t)$ dose not depend upon u),

$$0 = \min_u \Big\{ \Big(x^TQx + u^TRu\Big)\delta t + \frac{d}{dt}J^*(x(t), t)\delta t\Big\} \tag{4.16}$$

and on making $\delta t \downarrow 0$

$$0 = \min_u \Big\{ \Big(x^TQx + u^TRu\Big) + \frac{d}{dt}J^*(x(t), t)\Big\} \tag{4.17}$$

Since

$$\frac{d}{dt}J^*(x(t), t) = \partial_x J^* \dot{x}(t) + \partial_t J^*,$$

Then by substituting this result into (4.17), and since $\partial_t J^*$ is not depended to u, we obtain the GHJB equation

$$-\partial_t J^* = \min_u \Big\{ \Big(x^TQx + u^TRu\Big) + \Big(f(x) + g(x)u\Big)\partial_x J^* \Big\} = 0. \tag{4.18}$$

which yeilds the optimal control law as follows

$$u^* = -\frac{1}{2}R^{-1}g^T(x)\partial_x J^*. \tag{4.19}$$

Substituting (4.19) into (4.18) yields the GHJB equation:

$$\partial_t J^* + \Big(\partial_x J^*\Big)f(x) + Q(x) - \frac{1}{4}\Big(\partial_x J^*\Big)^T g(x)R^{-1}g(x)^T\Big(\partial_x J^*\Big) = 0. \tag{4.20}$$

The GHJB equation in (4.20) and (4.19) provide the solution to fixed-final time NSOCP. However, a closed-form solution for GHJB equation is impossible to fined.

5 Test problems

In this section we employ the new results obtained in the previous sections to solve three examples.

Example 5.1. *Consider the following NSVP:*

$$J(x) = \int_0^1 \Big(sin(\pi|\dot{x} - 0.5|)e^{|\dot{x}-0.8|} - \dot{x}^2(\dot{x} - 0.2)\Big)dt \tag{5.1}$$

subject to the boundary conditions:

$$x(0) = 0 \qquad and \qquad x(1) = 0.5 \tag{5.2}$$

for $L(t, x, \dot{x}) = sin(\pi|\dot{x} - 0.5|)e^{|\dot{x}-0.8|} - \dot{x}^2(\dot{x} - 0.2)$ and by applying theorem 3.3 for this problem, we obtain the GEL equation as follows:

$$\frac{d}{dt}\partial_{\dot{x}} L(t, x, \dot{x}) = 0. \tag{5.3}$$

Hence, by applying Remark 2.1 and using the Fourier series, the corresponding GEL equation (5.3) *will be as follows [Skandari et al. (2013)]:*

$$\frac{d}{dt}\sum_{j=1}^{\infty} a_j^* cos(\pi j\dot{x}) = 0, \qquad t \in [0, 1]. \tag{5.4}$$

Example 5.2. *Let J be given by the eapression*

$$J(x) = \int_0^1 \Big(|x - 0.5| - |x - 0.4| + |\dot{x} - 0.3| - |\dot{x} - 0.1|\Big)dt, \tag{5.5}$$

subject to the boundary conditions:

$$x(0) = 0.65 \qquad and \qquad x(1) = 1. \tag{5.6}$$

By asumption $L(t, x, \dot{x}) = |x - 0.5| - |x - 0.4| + |\dot{x} - 0.3| - |\dot{x} - 0.1|$ and by applying theorem (3.3) for this NSVP, we have:

$$\partial_x L(t, x, \dot{x}) - \frac{d}{dt}\partial_{\dot{x}} L(t, x, \dot{x}) = 0. \tag{5.7}$$

Hence, by applying theorem 2.3, we will have:

$$\sum_{j=1}^{\infty} a_j^* cos(\pi jx) - \frac{d}{dt}\sum_{j=1}^{\infty} b_j^* cos(\pi j\dot{x}) = 0, \qquad t \in [0, 1]. \tag{5.8}$$

In these examples, it is hard to solve GEL equations (5.4) and (5.8), analytically. Using numerical method for solving these problems can be usefull and may be considered in future works. For this purpose, the problems (5.4) and (5.8) are approximated as the finite dimensional problems for $j = 1, 2, \cdots, N$, where $N \in \mathbb{N}$ is a given big number.

Example 5.3. *Consider the following NSOC problem*

$$Minimize\ J = x^2(1) + \int_0^1 u^2(t)dt \tag{5.9}$$
$$subject\ to\ \ \dot{x} = |x| + u$$
$$x(0) = 1$$

For this example, we have:

$$Q(x) = 0,\ r = b(t) = 1,\ f(x) = |x|,\ h(x(1), 1) = x^2(1),\ t_0 = 0,\ t_f = 1.$$

From (4.20), and using $x \geq 0$, the corresponding GHJB equation are given by:

$$-\partial_t J = x\partial_x J - \frac{1}{4}\Big(\partial_x J\Big)^2, \tag{5.10}$$

and the optimal control law is easily obtained by:

$$u^*(t) = -\frac{1}{2}\partial_x J. \tag{5.11}$$

The exact solution of the GHJB equation is:

$$J(x,t) = \frac{2x^2}{1 + e^{2(t-t_f)}}, \tag{5.12}$$

which implies the optimal feedback control law to be $u^*(t) = \dfrac{-2x}{1 + e^{2(t-t_f)}}$. Now, by applying $x < 0$ the exact solution of the GHJB equation:

$$-\partial_t J = -x\partial_x J - \frac{1}{4}\Big(\partial_x J\Big)^2, \tag{5.13}$$

will be as follows:

$$J(x,t) = \frac{-2x^2}{1 - 3e^{2(t_f - t)}}, \tag{5.14}$$

where in $u^*(t) = \dfrac{2x}{1 - 3e^{2(t_f-t)}}$.

We refer the interested reader to [Skandari et al. (2014)] which deals with nonsmooth optimal control problems, providing a direct method for solving such problems without using the GEL equation type.

6 Conclusions

Nonsmooth calculus become a very good candidate to describe the models with non-smooth dynamics. In this paper, we first introduced a novel GD for nonsmooth functions. Then by using the generalized first order taylor expanision for nonsmooth functions that proposed by [Kamyad et al. (2011)], we introduced the GEL equation for nonsmooth problem (1.1). We have given it in a practical equation to obtain an approximate optimal solution for nonsmooth problem (1.1). Then study nonsmooth variational problems via derivative [Skandari et al. (2014)] under the presence of certain constraints. Transversality conditions are optimaly conditions that are used along with EL equations in order to find the optimal paths of dynamical models. Here, we extend our result over four nonsmooth variational problems. Then we obtain explicity the Hamilton equation for a NSOC problem (4.1)-(4.2), and then present necessary conditions for optimality these systems.

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