A STUDY ON MEROMORPHIC HARMONIC STARLIKE FUNCTIONS BY USING A NEW GENERALIZED DIFFERENTIAL OPERATOR

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ABSTRACT. In this study, a new generalized differential operator \mathfrak{D} was obtained first. Then this new differential operator was used to introduce a new class $\mathcal{GS}(n)$ of univalent meromorphic harmonic starlike functions exterior to the unit disc $\overline{U} := \{z \mid z \mid > 1\}$. Especially, coefficient bounds for this class have been examined. This coefficient condition is also necessary for the class $\overline{\mathcal{GS}(n)}$ which is subclass of meromorphic harmonic functions. Furthermore, some other properties such as distortion theorems and extreme points for these classes were obtained and relevant results were given.

1. INTRODUCTION

f = u + iv is a complex harmonic function in a domain $\mathbb{D} \subset \mathbb{C}$ if each of u and v is real continuous harmonic functions in \mathbb{D} . In any simply connected domain, f is written in the form of $h + \overline{g}$ where both h and g are analytic in \mathbb{D} [2].

A necessary and sufficient condition for f to be locally univalent and orientation preserving in \mathbb{D} is that |h'(z)| > |g'(z)| in \mathbb{D} [2]. The harmonic functions in the exterior of the unit disc $\widetilde{\mathbb{U}} := \{z | z| > 1\}$ were investigated by Hengartner and Schober in [3], and they were represented by the following equation of (1.1)

$$f(z) = h(z) + \overline{g(z)} + A \log |z|, \qquad (1.1)$$

where h(z) and g(z) are defined by

$$h(z) = \gamma z + \sum_{k=1}^{\infty} a_k z^{-k}$$
 and $g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}$ (1.2)

for $0 \leq |\beta| < |\gamma|$, $A \in \mathbb{C}$ and $z \in \widetilde{\mathbb{U}}$. In addition, different classes of meromorphic harmonic functions have been studied by Jahangiri and Silverman[5], Jahangiri[6] and Murugunsundaramoorthy [7,8]. Since harmonic functions are been used in many fields of sciences, new studies on harmonic functions are still of scientific interest.

In this study, a new operator \mathfrak{D} was defined for meromorphic harmonic functions in $\widetilde{\mathbb{U}}$. The classes $\mathcal{GS}(n)$ and $\overline{\mathcal{GS}(n)}$ were also defined. Some properties of these classes, such as coefficient estimates and a distortion theorem, were then investigated. This new operator \mathfrak{D} is defined as follows:

$$\mathfrak{D}^0 f(z) = f(z)$$

$$\mathfrak{D}^{1}f(z) = (\lambda - \alpha) \frac{\overline{\left(z^{\frac{\lambda - \alpha + 1}{\lambda - \alpha}}g(z)\right)'}}{z^{\frac{1}{\lambda - \alpha}}} - (\lambda - \alpha) z^{\frac{2\lambda - 2\alpha + 1}{\lambda - \alpha}} \left(\frac{h(z)}{z^{\frac{\lambda - \alpha + 1}{\lambda - \alpha}}}\right)'$$

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and for n = 2, ...,

$$\mathfrak{D}^n f(z) = \mathfrak{D}(\mathfrak{D}^{n-1} f(z)).$$

Using this new operator,

$$\mathfrak{D}^{n}f(z) = \gamma z + \sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} a_{k} z^{-k} + (2\lambda - 2\alpha + 1)^{n} \beta z + (-1)^{n} \sum_{k=1}^{\infty} [(k - 1)(\lambda - \alpha) - 1]^{n} b_{k} z^{-k}$$

was obtained for $n = 0, 1, ..., \text{ and } 0 \le (2\lambda - 2\alpha + 1)^n |\beta| < |\gamma|.$

Let $\mathcal{GS}(n)$ show the class of harmonic functions with sense preserving and univalent functions that consist of functions satisfying for $z \in \widetilde{\mathbb{U}}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\Re\left\{2 - \frac{\mathfrak{D}^{n+1}f(z)}{\mathfrak{D}^n f(z)}\right\} > 0.$$
(1.3)

Also, let $\overline{\mathcal{GS}(n)}$ be the subclass of $\mathcal{GS}(n)$ which consists of meromorphic harmonic functions of the form of (1.4)

$$f_n(z) = h(z) + \overline{g_n(z)} = -\gamma z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}$$
(1.4)

where $\gamma > \beta \ge 0, a_k \ge 0, b_k \ge 0$. A necessary and sufficient condition for f functions of the form (1.1) to be starlike in $\widetilde{\mathbb{U}}$ is that

$$\frac{\partial}{\partial \theta} arg(f(re^{i\theta})) = \Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > 0, \qquad z \in \widetilde{\mathbb{U}}.$$
(1.5)

for each z, |z| = r > 1. This classification (1.5) for harmonic univalent functions was first used by Jahangiri [4].

2. Coefficient Inequalities

In this section, sufficient conditions of coefficient inequalities for f(z) to belongs to the class $\mathcal{GS}(n)$ are obtained.

Theorem 2.1. If $f(z) = h(z) + \overline{g(z)}$ where $\frac{1}{2} \leq \lambda - \alpha \leq 1$, $\alpha \geq 0$, $\lambda \geq 0$, h(z) and g(z) are of the form (1.2) and the inequality

$$\sum_{k=1}^{\infty} [k(\lambda-\alpha) + (\lambda-\alpha+1)]^{n} [k(\lambda-\alpha) + (\lambda-\alpha-1)] |a_{k}| + \sum_{k=2}^{\infty} [(k-1)(\lambda-\alpha) - 1]^{n} [(k-1)(\lambda-\alpha) + 1] |b_{k}| + |b_{1}| \le |\gamma| - (2\lambda-2\alpha+1)^{n} (2\lambda-2\alpha-1) |\beta|$$
(2.1)

is satisfied, then f(z) is univalent, sense preserving and $f(z) \in \mathcal{GS}(n)$ in $\widetilde{\mathbb{U}}$.

Proof. We must show that if the condition (2.1) is satisfied, then $f(z) \in \mathcal{GS}(n)$. Hence, it is sufficient to show that $p_n(z)$ is in the class $\mathcal{GS}(n)$ which is the class of harmonic functions with positive real part.

$$|p_n(z) + 1| > |p_n(z) - 1|, z \in \widetilde{\mathbb{U}},$$
(2.2)

where

$$p_n(z) = \frac{2\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)}{\mathfrak{D}^n f(z)}$$
(2.3)

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from (2.2) we obtain,

$$\frac{|3\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)|}{|\mathfrak{D}^n f(z)|} - \frac{|\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)|}{|\mathfrak{D}^n f(z)|} > 0.$$
(2.4)

Since

$$\begin{split} |3\mathfrak{D}^{n}f(z) - \mathfrak{D}^{n+1}f(z)| - |\mathfrak{D}^{n}f(z) - \mathfrak{D}^{n+1}f(z)| \\ &= |2\gamma z + \sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} [-k(\lambda - \alpha) + (-\lambda + \alpha + 2)]a_{k}z^{-k} + \overline{(2\lambda - 2\alpha + 1)^{n}(2 - 2\lambda + 2\alpha)\beta z} \\ &+ (-1)^{n} \sum_{k=1}^{\infty} [(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha) + 2]b_{k}z^{-k}| \\ &- |\sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} [-k(\lambda - \alpha) - (\lambda - \alpha)]a_{k}z^{-k} + \overline{(2\lambda - 2\alpha + 1)^{n}(-2\lambda + 2\alpha)\beta z} \\ &+ (-1)^{n} \sum_{k=1}^{\infty} [(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha)]b_{k}z^{-k}| \\ &= 2|\gamma||z| - |\sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} [-k(\lambda - \alpha) - (\lambda - \alpha - 2)]a_{k}z^{-k}| - |(2\lambda - 2\alpha + 1)^{n}(2 - 2\lambda + 2\alpha)\beta z| - 2|b_{1}||z|^{-1} \\ &- |\sum_{k=2}^{\infty} [(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha) + 2]b_{k}z^{-k}| - |\sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} [-k(\lambda - \alpha) - (\lambda - \alpha)]a_{k}z^{-k}| \\ &- |(2\lambda - 2\alpha + 1)^{n}(-2\lambda + 2\alpha)\beta z| - |\sum_{k=2}^{\infty} [(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha)]b_{k}z^{-k}| \\ &\geq 2|\gamma||z| - \sum_{k=1}^{\infty} [[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} [-k(\lambda - \alpha) - (\lambda - \alpha - 2)]||a_{k}||z|^{-k} - |(2\lambda - 2\alpha + 1)^{n}(2 - 2\lambda + 2\alpha)||\beta||z| \\ &- \sum_{k=2}^{\infty} [[(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha) + 2]||b_{k}||z|^{-k} - 2|b_{1}| - \sum_{k=1}^{\infty} [[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} [-k(\lambda - \alpha) - (\lambda - \alpha)]||a_{k}||z|^{-k} \\ &- |(2\lambda - 2\alpha + 1)^{n}(-2\lambda + 2\alpha)||\beta||z| - \sum_{k=2}^{\infty} [[(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha)]||a_{k}||z|^{-k} \\ &- |(2\lambda - 2\alpha + 1)^{n}(-2\lambda + 2\alpha)||\beta||z| - \sum_{k=2}^{\infty} [[(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha)]||a_{k}||z|^{-k} \\ &\geq 2\{|\gamma| - \sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} [k(\lambda - \alpha) + (\lambda - \alpha - 1)]|a_{k}| - (2\lambda - 2\alpha + 1)^{n}(2\lambda - 2\alpha - 1)|\beta| \\ &- |b_{1}| - \sum_{k=2}^{\infty} [[(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha) + 1]|b_{k}|] \geq 0. \quad (2.5) \\ \text{ So the proof of Theorem 2.1 is complete.} \Box$$

So the proof of Theorem 2.1 is complete.

In the following theorem we show that the sufficient coefficient condition given by (2.1)is also necessary for the family $\overline{\mathcal{GS}(n)}$.

Theorem 2.2. Let
$$f_n(z) = h(z) + \overline{g_n(z)}$$
. Then $f_n(z) \in \overline{\mathcal{GS}(n)}$ if and only if

$$\sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)] a_k$$

$$+ \sum_{k=2}^{\infty} [(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1] b_k + b_1 \leq \gamma - (2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)\beta.$$
(2.6)

Proof. Taking into account of Theorem 2.1, we need to prove the "only if" part, since $\overline{\mathcal{GS}(n)} \subset \mathcal{GS}(n)$. Let $f_n(z) \in \overline{\mathcal{GS}(n)}$, and z be a complex number. If $\Re(z) > 0$ then $\Re(\frac{1}{z}) > 0$. Therefore, we obtained that as follows.

$$0 < \Re \left\{ \frac{\mathfrak{D}^{n} f(z)}{2\mathfrak{D}^{n} f(z) - \mathfrak{D}^{n+1} f(z)} \right\} \leq \left| \frac{\mathfrak{D}^{n} f(z)}{2\mathfrak{D}^{n} f(z) - \mathfrak{D}^{n+1} f(z)} \right|$$

$$= \left| \frac{-\gamma z - \sum_{k=1}^{\infty} Aa_{k} z^{-k} + \overline{(2\lambda - 2\alpha + 1)^{n} \beta z - (-1)^{n} \sum_{k=1}^{\infty} Cb_{k} z^{-k}}}{-\gamma z + \sum_{k=1}^{\infty} ABa_{k} z^{-k} + \overline{(2\lambda - 2\alpha + 1)^{n} (-2\lambda + 2\alpha + 1)\beta z - (-1)^{n} \sum_{k=1}^{\infty} CDb_{k} z^{-k}}} \right|$$

$$\leq \frac{\gamma |z| + \sum_{k=1}^{\infty} Aa_{k} |z|^{-k} + (2\lambda - 2\alpha + 1)^{n} \beta |z| + \sum_{k=2}^{\infty} Cb_{k} |z|^{-k} + b_{1} |z|^{-k}}{\gamma |z| - \sum_{k=1}^{\infty} ABa_{k} |z|^{-k} - (2\lambda - 2\alpha + 1)^{n} (2\lambda - 2\alpha - 1)\beta |z| - \sum_{k=2}^{\infty} CDb_{k} |z|^{-k} - b_{1} |z|^{-k}} < \frac{\gamma + \sum_{k=1}^{\infty} Aa_{k} + (2\lambda - 2\alpha + 1)^{n} \beta + \sum_{k=2}^{\infty} Cb_{k} + b_{1}}{\gamma - \sum_{k=1}^{\infty} ABa_{k} - (2\lambda - 2\alpha + 1)^{n} (2\lambda - 2\alpha - 1)\beta - \sum_{k=2}^{\infty} CDb_{k} - b_{1}}.$$

$$(2.7)$$

where

$$A = [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n, B = [k(\lambda - \alpha) + (\lambda - \alpha - 1)], C = [(k - 1)(\lambda - \alpha) - 1]^n, D = [(k - 1)(\lambda - \alpha) + 1].$$

The inequation (2.7) leads to the following inequality.

$$\sum_{k=1}^{\infty} [k(\lambda-\alpha)+(\lambda-\alpha+1)]^n [k(\lambda-\alpha)+(\lambda-\alpha-1)] a_k + \sum_{k=2}^{\infty} [(k-1)(\lambda-\alpha)-1]^n [(k-1)(\lambda-\alpha)+1] b_k + b_1 \\ \leq \gamma - (2\lambda-2\alpha+1)^n (2\lambda-2\alpha-1)\beta.$$

So, the proof of the theorem is being completed. \Box

So, the proof of the theorem is being completed.

3. A DISTORTION THEOREM AND EXTREME POINTS

In this section we will obtain distortion bounds and extreme points for functions $f(z) \in$ $\overline{\mathcal{GS}(n)}$ which f_n is defined by (1.4).

Theorem 3.1. Let the function $f_n(z)$ be in the class $\overline{\mathcal{GS}(n)}$. Then for 0 < |z| = r < 1, we have

$$(\gamma - \beta)r - [\gamma - (2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)\beta]r^{-1} \le |f_n(z)| \le (\gamma + \beta)r + [\gamma - (2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)\beta]r^{-1}.$$
(3.1)

Proof. Taking into account of Theorem 2.2 , for 0 < |z| = r < 1, we obtain

$$|f_{n}(z)| = \left| -\gamma z - \sum_{k=1}^{\infty} a_{k} z^{-k} + \overline{\beta z - (-1)^{n}} \sum_{k=1}^{\infty} b_{k} z^{-k} \right|$$

$$\leq \gamma r + \beta r + \sum_{k=1}^{\infty} (a_{k} + b_{k}) r^{-k} \leq \gamma r + \beta r + \sum_{k=1}^{\infty} (a_{k} + b_{k}) r^{-1}$$

$$\leq \gamma r + \beta r + r^{-1} (\sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} [k(\lambda - \alpha) + (\lambda - \alpha - 1)] a_{k}$$

$$+ \sum_{k=2}^{\infty} [(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha) + 1] b_{k} + b_{1})$$

$$\leq (\gamma + \beta) r + [\gamma - (2\lambda - 2\alpha + 1)^{n} (2\lambda - 2\alpha - 1)\beta] r^{-1}$$

by the coefficient inequality in (2.6).

The distortion bounds given in Theorem 3.1 is valid for functions $f_n = h + \overline{g_n}$ which is given in the form (1.4) and it is also known that the bounds is valid for functions of the form $f = h + \overline{g}$ where h and g are given by (1.2) if the coefficient condition (2.1) is satisfied.

The extreme points of closed convex hulls of $\overline{\mathcal{GS}(n)}$ denoted by $clco\overline{\mathcal{GS}(n)}$ were determined in the next theorem.

Theorem 3.2. Let $f_n = h + \overline{g_n}$ is given by (1.4). Let be $\lambda - \alpha \ge 1$. Then, $f_n \in clco\overline{\mathcal{GS}(n)}$ if and only if it can be expressed as

$$f_n(z) = \sum_{k=0}^{\infty} [x_k h_{n,k}(z) + y_k g_{n,k}(z)]$$

where $x_k \ge 0, y_k \ge 0$ and $\sum_{k=0}^{\infty} (x_k + y_k) = \gamma$

$$h_{n,0}(z) = -z, \qquad g_{n,0}(z) = -z + \frac{\overline{z}}{(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)}, \qquad g_{n,1}(z) = -z - (-1)^n \overline{z}^{-1}$$

$$h_{n,k}(z) = -z - \frac{1}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k}, \qquad k \ge 1 \qquad and$$

$$g_{n,k}(z) = -z - \frac{(-1)^n}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]} \overline{z}^{-k}, \qquad k \ge 2$$

In particular, the extreme points of $\overline{\mathcal{GS}(n)}$ are $\{h_{n,k}\}$ and $\{g_{n,k}\}$.

Proof. For $\lambda - \alpha > 1$, let

$$f_n(z) = \sum_{k=0}^{\infty} [x_k h_{n,k}(z) + y_k g_{n,k}(z)]$$

where $x_k \ge 0, y_k \ge 0$ and $\sum_{k=0}^{\infty} (x_k + y_k) = \gamma$.

Then we have

$$f_n(z) = \sum_{k=0}^{\infty} [x_k h_{n,k}(z) + y_k g_{n,k}(z)]$$

$$\begin{split} &= x_0 h_{n,0}(z) + \sum_{k=1}^{\infty} x_k [-z - \frac{1}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k}] \\ &+ y_0 g_{n,0}(z) + y_1 g_{n,1}(z) + \sum_{k=2}^{\infty} y_k [-z - \frac{(-1)^n}{[(k - 1)(\lambda - \alpha) - 1]^n [(k - 1)(\lambda - \alpha) + 1]} \overline{z}^{-k}] \\ &= -\sum_{k=0}^{\infty} (x_k + y_k) z - \sum_{k=1}^{\infty} \frac{x_k}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k} \\ &+ \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]} \overline{z}^{-(-1)^n y_1} \overline{z}^{-1} - (-1)^n \sum_{k=2}^{\infty} \frac{y_k}{[(k - 1)(\lambda - \alpha) - 1]^n [(k - 1)(\lambda - \alpha) + 1]} \overline{z}^{-k}] \\ &= -\gamma z - \sum_{k=1}^{\infty} \frac{x_k}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k} \\ &+ \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]} \overline{z}^{-(-1)^n y_1} \overline{z}^{-1} - (-1)^n \sum_{k=2}^{\infty} \frac{y_k}{[(k - 1)(\lambda - \alpha) - 1]^n [(k - 1)(\lambda - \alpha) + 1]} \overline{z}^{-k}. \end{split}$$
Since
$$\sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)] \frac{x_k}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} \\ &+ \sum_{k=2}^{\infty} [(k - 1)(\lambda - \alpha) - 1]^n [(k - 1)(\lambda - \alpha) + 1] \frac{y_k}{[(k - 1)(\lambda - \alpha) - 1]^n [(k - 1)(\lambda - \alpha) + 1]} + y_1 \end{split}$$

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$$= \left(\sum_{k=1}^{\infty} x_k + y_1 + \sum_{k=2}^{\infty} y_k\right)$$

= $\gamma - y_0 - x_0 \le \gamma - [(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)] \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]}$

by Theorem (2.2), so $f_n(z) \in clco\overline{\mathcal{GS}(n)}$. Conversely, suppose that $f_n(z) \in clco\overline{\mathcal{GS}(n)}$, then we may write

$$f_n(z) = h(z) + \overline{g_n(z)} = -\gamma z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}$$

where $\gamma > \beta \geq 0, a_k \geq 0, b_k \geq 0$. We set

$$a_{k} = \frac{x_{k}}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n}[k(\lambda - \alpha) + (\lambda - \alpha - 1)]}, k = 1, 2, ...,$$

$$\beta = \frac{y_{0}}{[(2\lambda - 2\alpha + 1)^{n}(2\lambda - 2\alpha - 1)]}, \qquad b_{1} = y_{1},$$

$$b_{k} = \frac{y_{k}}{[(k - 1)(\lambda - \alpha) - 1]^{n}[(k - 1)(\lambda - \alpha) + 1]}, k = 2, ...$$

Hence we obtain

Hence we obtain

$$f_{n}(z) = h(z) + \overline{g_{n}(z)} = -\gamma z - \sum_{k=1}^{\infty} a_{k} z^{-k} + \overline{\beta z - (-1)^{n} \sum_{k=1}^{\infty} b_{k} z^{-k}}$$

$$= -\sum_{k=0}^{\infty} (x_{k} + y_{k}) z - \sum_{k=1}^{\infty} \frac{x_{k}}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k}$$

$$+ \frac{y_{0}}{[(2\lambda - 2\alpha + 1)^{n} (2\lambda - 2\alpha - 1)]} \overline{z} - (-1)^{n} y_{1} \overline{z}^{-1} - \sum_{k=2}^{\infty} \frac{(-1)^{n} y_{k}}{[(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha) + 1]} \overline{z}^{-k}$$

$$= x_{0}(-z) + \sum_{k=1}^{\infty} x_{k} \left[-z - \frac{1}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^{n} [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k} \right]$$

$$+ y_{0} \left(-z + \frac{\overline{z}}{[(2\lambda - 2\alpha + 1)^{n} (2\lambda - 2\alpha - 1)]} \right) + y_{1}(-z - (-1)^{n} \overline{z}^{-1})$$

$$+ \sum_{k=2}^{\infty} y_{k} \left[-z - \frac{(-1)^{n}}{[(k - 1)(\lambda - \alpha) - 1]^{n} [(k - 1)(\lambda - \alpha) + 1]} \overline{z}^{-k} \right].$$

Consequently we obtain following relation as required

$$f_n(z) = \sum_{k=0}^{\infty} [x_k h_{n,k}(z) + y_k g_{n,k}(z)].$$

For $\lambda - \alpha = 1$, we obtain the results of Theorem (6) in [1]. So we avoid giving proof of Theorem (3.2), for this case.

This completes the proof of Theorem (3.2).

4. CONCLUSION

Some new classes of meromorphic harmonic starlike functions involving a new generalized differential operator exterior to the unit disc $\widetilde{\mathbb{U}} := \{z | z | > 1\}$ were introduced. Some characteristic properties of these classes are also investigated. Furthermore, For $\lambda = 1$ and $\alpha = 0$ we obtain the results in [1]. Also for n = 0 coincides with that given by [5]. Therefore, this work may be considered as a generalization form of earlier works.

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