Original Research Article

A STUDY ON MEROMORPHIC HARMONIC STARLIKE FUNCTIONS BY USING A NEW GENERALIZED DIFFERENTIAL OPERATOR

ABSTRACT. A new class of meromorphic harmonic starlike functions ,exterior to the unit disc $\widetilde{\mathbb{U}} := \{z\, |z| > 1\}$, were introduced in this study. Coefficient bounds, distortion theorems and extreme points for this functions were also obtained.

1. INTRODUCTION

$f = u + iv$ is a complex harmonic function in a domain $\mathbb{D} \subset \mathbb{C}$ if each of u and v is real continuous harmonic functions in $\mathbb{D}$. In any simply connected domain, f is writen in the form of $h + \overline{g}$ where both h and g are analytic in $\mathbb{D}$ [1].

A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ [1]. The harmonic functions in the exterior of the unit disc $\widetilde{\mathbb{U}} := \{z\, |z| > 1\}$ were investigated by Hengartner and Schober in [2], and they were represented by the following equation of (1.1)

$$f(z) = h(z) + \overline{g(z)} + Alog|z|, \tag{1.1}$$

where $h(z)$ and $g(z)$ are defined by

$$h(z) = \gamma z + \sum_{k=1}^{\infty} a_k z^{-k} \qquad and \qquad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k} \tag{1.2}$$

for $0 \leq |\beta| < |\gamma|$, $A \in \mathbb{C}$ and $z \in \widetilde{\mathbb{U}}$. In addition, different classes of meromorphic harmonic functions have been studied by Jahangiri and Silverman[3], Jahangiri[4] and Murugunsundaramoorthy [5,6]. Since harmonic functions are been used in many fields of sciences, new studies on harmonic functions are still of scientific interest.

In this study, a new operator $\mathfrak{D}$ was defined for meromorphic harmonic functions in $\widetilde{\mathbb{U}}$. The classes $\mathcal{GS}(n)$ and $\overline{\mathcal{GS}(n)}$ were also defined. Some properties of these classes, such as coefficient estimates and a distortion theorem, were then investigated.This new operator $\mathfrak{D}$ is defined as follows:

$$\mathfrak{D}^0 f(z) = f(z)$$

$$\mathfrak{D}^1 f(z) = (\lambda-\alpha)\frac{\overline{\left(z^{\frac{\lambda-\alpha+1}{\lambda-\alpha}} g(z)\right)'}}{z^{\frac{1}{\lambda-\alpha}}} - (\lambda-\alpha) z^{\frac{2\lambda-2\alpha+1}{\lambda-\alpha}} \left(\frac{h(z)}{z^{\frac{\lambda-\alpha+1}{\lambda-\alpha}}}\right)'$$

and for $n = 2, ...,$

$$\mathfrak{D}^n f(z) = \mathfrak{D}(\mathfrak{D}^{n-1} f(z)).$$

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Using this new operator,

$$\mathfrak{D}^n f(z) = \gamma z + \sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n a_k z^{-k}$$

$$+\overline{(2\lambda-2\alpha+1)^n \beta z + (-1)^n \sum_{k=1}^{\infty}[(k-1)(\lambda-\alpha)-1]^n b_k z^{-k}}$$

was obtained for $n = 0, 1, ...,$ and $0 \leq (2\lambda - 2\alpha + 1)^n |\beta| < |\gamma|$.

Let $\mathcal{GS}(n)$ show the class of harmonic functions with sense preserving and univalent functions that consist of functions satisfying for $z \in \widetilde{\mathbb{U}}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\Re\left\{2 - \frac{\mathfrak{D}^{n+1} f(z)}{\mathfrak{D}^n f(z)}\right\} > 0. \tag{1.3}$$

Also, let $\overline{\mathcal{GS}(n)}$ be the subclass of $\mathcal{GS}(n)$ which consists of meromorphic harmonic functions of the form of (1.4)

$$f_n(z) = h(z) + \overline{g_n(z)} = -\gamma z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}} \tag{1.4}$$

where $\gamma > \beta \geq 0, a_k \geq 0, b_k \geq 0$. A necessary and sufficient condition for f functions of the form (1.1) to be starlike in $\widetilde{\mathbb{U}}$ is that

$$\frac{\partial}{\partial\theta} arg(f(re^{i\theta})) = \Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > 0, \qquad z \in \widetilde{\mathbb{U}}. \tag{1.5}$$

for each z, $|z| = r > 1$. This classification (1.5) for harmonic univalent functions was first used by Jahangiri [7].

2. Coefficient Inequalities

In this section, sufficient conditions of coefficient inequalities for $f(z)$ to belongs to the class $\mathcal{GS}(n)$ are obtained.

Theorem 2.1. *If $f(z) = h(z) + \overline{g(z)}$ where $\frac{1}{2} \leq \lambda - \alpha \leq 1$, $\alpha \geq 0, \lambda \geq 0$, $h(z)$ and $g(z)$ are of the form* (1.2) *and the inequality*

$$\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[k(\lambda-\alpha)+(\lambda-\alpha-1)]|a_k|$$

$$+\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+1]|b_k|+|b_1| \leq |\gamma|-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)|\beta| \tag{2.1}$$

is satisfied, then $f(z)$ is univalent, sense preserving and $f(z) \in \mathcal{GS}(n)$ in $\widetilde{\mathbb{U}}$.

Proof. We must show that if the condition (2.1) is satisfied, then $f(z) \in \mathcal{GS}(n)$. Hence, it is sufficient to show that $p_n(z)$ is in the class $\mathcal{GS}(n)$ which is the class of harmonic functions with positive real part.

$$|p_n(z) + 1| > |p_n(z) - 1|, z \in \widetilde{\mathbb{U}}, \tag{2.2}$$

where

$$p_n(z) = \frac{2\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)}{\mathfrak{D}^n f(z)} \tag{2.3}$$

from (2.2) we obtain,

$$\frac{|3\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)|}{|\mathfrak{D}^n f(z)|} - \frac{|\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)|}{|\mathfrak{D}^n f(z)|} > 0. \tag{2.4}$$

Since

$$|3\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)| - |\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)|$$

$$= |2\gamma z + \sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)+(-\lambda+\alpha+2)]a_k z^{-k} + \overline{(2\lambda-2\alpha+1)^n(2-2\lambda+2\alpha)\beta z}$$

$$+ \overline{(-1)^n \sum_{k=1}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+2]b_k z^{-k}}|$$

$$-|\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)-(\lambda-\alpha)]a_k z^{-k} + \overline{(2\lambda-2\alpha+1)^n(-2\lambda+2\alpha)\beta z}$$

$$+ \overline{(-1)^n \sum_{k=1}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)]b_k z^{-k}}|$$

$$\geq 2|\gamma||z| - |\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)-(\lambda-\alpha-2)]a_k z^{-k}| - |(2\lambda-2\alpha+1)^n(2-2\lambda+2\alpha)\beta z| - 2|b_1||z|^{-1}$$

$$-|\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+2]b_k z^{-k}| - |\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)-(\lambda-\alpha)]a_k z^{-k}|$$

$$-|(2\lambda-2\alpha+1)^n(-2\lambda+2\alpha)\beta z| - |\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)]b_k z^{-k}|$$

$$\geq 2|\gamma||z| - \sum_{k=1}^{\infty}|[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)-(\lambda-\alpha-2)]||a_k||z|^{-k} - |(2\lambda-2\alpha+1)^n(2-2\lambda+2\alpha)||\beta||z|$$

$$-\sum_{k=2}^{\infty}|[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+2]||b_k||z|^{-k} - 2|b_1| - \sum_{k=1}^{\infty}|[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[-k(\lambda-\alpha)-(\lambda-\alpha)]||a_k||z|^{-k}$$

$$-|(2\lambda-2\alpha+1)^n(-2\lambda+2\alpha)||\beta||z| - \sum_{k=2}^{\infty}|[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)]||b_k||z|^{-k}$$

$$\geq 2\{|\gamma| - \sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[k(\lambda-\alpha)+(\lambda-\alpha-1)]|a_k| - (2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)|\beta|$$

$$-|b_1| - \sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+1]|b_k|\} \geq 0. \tag{2.5}$$

So the proof of Theorem 2.1 is complete. □

In the following theorem we show that the sufficient coefficient condition given by (2.1) is also necessary for the family $\overline{\mathcal{GS}(n)}$.

Theorem 2.2. *Let $f_n(z) = h(z) + \overline{g_n(z)}$. Then $f_n(z) \in \overline{\mathcal{GS}(n)}$ if and only if*

$$\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[k(\lambda-\alpha)+(\lambda-\alpha-1)]a_k$$

$$+\sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+1]b_k + b_1 \leq \gamma - (2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta. \tag{2.6}$$

Proof. Taking into account of Theorem 2.1, we need to prove the "only if" part, since $\overline{\mathcal{GS}(n)} \subset \mathcal{GS}(n)$. Let $f_n(z) \in \overline{\mathcal{GS}(n)}$, and z be a complex number. If $\Re(z) > 0$ then $\Re(\frac{1}{z}) > 0$. Therefore, we obtained that as follows.

$$0 < \Re\left\{\frac{\mathfrak{D}^n f(z)}{2\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)}\right\} \leq \left|\frac{\mathfrak{D}^n f(z)}{2\mathfrak{D}^n f(z) - \mathfrak{D}^{n+1} f(z)}\right|$$
$$= \left|\frac{-\gamma z - \sum_{k=1}^{\infty} A a_k z^{-k} + \overline{(2\lambda-2\alpha+1)^n \beta z - (-1)^n \sum_{k=1}^{\infty} C b_k z^{-k}}}{-\gamma z + \sum_{k=1}^{\infty} AB a_k z^{-k} + \overline{(2\lambda-2\alpha+1)^n(-2\lambda+2\alpha+1)\beta z - (-1)^n \sum_{k=1}^{\infty} CD b_k z^{-k}}}\right|$$
$$\leq \frac{\gamma|z| + \sum_{k=1}^{\infty} A a_k |z|^{-k} + (2\lambda-2\alpha+1)^n \beta |z| + \sum_{k=2}^{\infty} C b_k |z|^{-k} + b_1 |z|^{-k}}{\gamma|z| - \sum_{k=1}^{\infty} AB a_k |z|^{-k} - (2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta|z| - \sum_{k=2}^{\infty} CD b_k |z|^{-k} - b_1|z|^{-k}}$$
$$< \frac{\gamma + \sum_{k=1}^{\infty} A a_k + (2\lambda-2\alpha+1)^n \beta + \sum_{k=2}^{\infty} C b_k + b_1}{\gamma - \sum_{k=1}^{\infty} AB a_k - (2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta - \sum_{k=2}^{\infty} CD b_k - b_1}. \tag{2.7}$$

where

$A = [k(\lambda-\alpha)+(\lambda-\alpha+1)]^n$, $B = [k(\lambda-\alpha)+(\lambda-\alpha-1)]$, $C = [(k-1)(\lambda-\alpha)-1]^n$, $D = [(k-1)(\lambda-\alpha)+1]$.

The inequation (2.7) leads to the following inequality.

$$\sum_{k=1}^{\infty}[k(\lambda-\alpha)+(\lambda-\alpha+1)]^n[k(\lambda-\alpha)+(\lambda-\alpha-1)]a_k + \sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha)-1]^n[(k-1)(\lambda-\alpha)+1]b_k + b_1$$
$$\leq \gamma - (2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta.$$

So, the proof of the theorem is being completed. □

3. A Distortion Theorem and Extreme Points

In this section we will obtain distortion bounds and extreme points for functions $f(z) \in \overline{\mathcal{GS}(n)}$ which f_n is defined by (1.4).

Theorem 3.1. *Let the function $f_n(z)$ be in the class $\overline{\mathcal{GS}(n)}$. Then for $0 < |z| = r < 1$, we have*

$$(\gamma-\beta)r - [\gamma-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta]r^{-1} \leq |f_n(z)| \leq (\gamma+\beta)r + [\gamma-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta]r^{-1}. \tag{3.1}$$

Proof. Taking into account of Theorem 2.2 ,for $0 < |z| = r < 1$, we obtain

$$|f_n(z)| = \left|-\gamma z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}\right|$$
$$\leq \gamma r + \beta r + \sum_{k=1}^{\infty}(a_k + b_k) r^{-k} \leq \gamma r + \beta r + \sum_{k=1}^{\infty}(a_k + b_k) r^{-1}$$
$$\leq \gamma r + \beta r + r^{-1}\Big(\sum_{k=1}^{\infty}[k(\lambda-\alpha) + (\lambda-\alpha+1)]^n[k(\lambda-\alpha) + (\lambda-\alpha-1)]a_k$$
$$+ \sum_{k=2}^{\infty}[(k-1)(\lambda-\alpha) - 1]^n[(k-1)(\lambda-\alpha) + 1]b_k + b_1\Big)$$
$$\leq (\gamma+\beta)r + [\gamma-(2\lambda-2\alpha+1)^n(2\lambda-2\alpha-1)\beta]r^{-1}$$

by the coefficient inequalitiy in (2.6). □

The distortion bounds given in Theorem 3.1 is valid for functions $f_n = h + \overline{g_n}$ which is given in the form (1.4) and it is also known that the bounds is valid for functions of the form $f = h + \overline{g}$ where h and g are given by (1.2) if the coefficient condition (2.1) is satisfied.

The extreme points of closed convex hulls of $\overline{\mathcal{GS}(n)}$ denoted by $clco\overline{\mathcal{GS}(n)}$ were determined in the next theorem.

Theorem 3.2. *Let $f_n = h + \overline{g_n}$ is given by (1.4).Let be $\lambda - \alpha \geq 1$. Then, $f_n \in clco\overline{\mathcal{GS}(n)}$ if and only if it can be expressed as*

$$f_n(z) = \sum_{k=0}^{\infty} [x_k h_{n,k}(z) + y_k g_{n,k}(z)]$$

where $x_k \geq 0, y_k \geq 0$ and $\sum_{k=0}^{\infty}(x_k + y_k) = \gamma$

$$h_{n,0}(z) = -z, \qquad g_{n,0}(z) = -z + \frac{\overline{z}}{(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)}, \qquad g_{n,1}(z) = -z - (-1)^n \overline{z}^{-1},$$

$$h_{n,k}(z) = -z - \frac{1}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k}, \qquad k \geq 1 \qquad \text{and}$$

$$g_{n,k}(z) = -z - \frac{(-1)^n}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]} \overline{z}^{-k}, \qquad k \geq 2$$

In particular, the extreme points of $\overline{\mathcal{GS}(n)}$ are $\{h_{n,k}\}$ and $\{g_{n,k}\}$.

Proof. For $\lambda - \alpha > 1$, let

$$f_n(z) = \sum_{k=0}^{\infty} [x_k h_{n,k}(z) + y_k g_{n,k}(z)]$$

where $x_k \geq 0, y_k \geq 0$ and $\sum_{k=0}^{\infty}(x_k + y_k) = \gamma$.

Then we have

$$f_n(z) = \sum_{k=0}^{\infty} [x_k h_{n,k}(z) + y_k g_{n,k}(z)]$$

$$= x_0 h_{n,0}(z) + \sum_{k=1}^{\infty} x_k [-z - \frac{1}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k}]$$

$$+ y_0 g_{n,0}(z) + y_1 g_{n,1}(z) + \sum_{k=2}^{\infty} y_k [-z - \frac{(-1)^n}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]} \overline{z}^{-k}]$$

$$= -\sum_{k=0}^{\infty}(x_k + y_k) z - \sum_{k=1}^{\infty} \frac{x_k}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k}$$

$$+ \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]} \overline{z} - (-1)^n y_1 \overline{z}^{-1} - (-1)^n \sum_{k=2}^{\infty} \frac{y_k}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]} \overline{z}^{-k}]$$

$$= -\gamma z - \sum_{k=1}^{\infty} \frac{x_k}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k}$$

$$+ \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]} \overline{z} - (-1)^n y_1 \overline{z}^{-1} - (-1)^n \sum_{k=2}^{\infty} \frac{y_k}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]} \overline{z}^{-k}.$$

Since

$$\sum_{k=1}^{\infty} [k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)] \frac{x_k}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]}$$

$$+ \sum_{k=2}^{\infty} [(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1] \frac{y_k}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]} + y_1$$

$$= \left(\sum_{k=1}^{\infty} x_k + y_1 + \sum_{k=2}^{\infty} y_k \right)$$

$$= \gamma - y_0 - x_0 \leq \gamma - [(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)] \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]}$$

by Theorem (2.2), so $f_n(z) \in clco\overline{\mathcal{GS}(n)}$. Conversely, suppose that $f_n(z) \in clco\overline{\mathcal{GS}(n)}$, then we may write

$$f_n(z) = h(z) + \overline{g_n(z)} = -\gamma z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}$$

where $\gamma > \beta \geq 0, a_k \geq 0, b_k \geq 0$. We set

$$a_k = \frac{x_k}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]}, k = 1, 2, ...,$$

$$\beta = \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]}, \qquad b_1 = y_1,$$

$$b_k = \frac{y_k}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]}, k = 2, ...$$

Hence we obtain

$$f_n(z) = h(z) + \overline{g_n(z)} = -\gamma z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}$$

$$= -\sum_{k=0}^{\infty} (x_k + y_k) z - \sum_{k=1}^{\infty} \frac{x_k}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k}$$

$$+ \frac{y_0}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]} \overline{z} - (-1)^n y_1 \overline{z}^{-1} - \sum_{k=2}^{\infty} \frac{(-1)^n y_k}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]} \overline{z}^{-k}$$

$$= x_0(-z) + \sum_{k=1}^{\infty} x_k \left[-z - \frac{1}{[k(\lambda - \alpha) + (\lambda - \alpha + 1)]^n [k(\lambda - \alpha) + (\lambda - \alpha - 1)]} z^{-k} \right]$$

$$+ y_0 \left(-z + \frac{\overline{z}}{[(2\lambda - 2\alpha + 1)^n (2\lambda - 2\alpha - 1)]} \right) + y_1 (-z - (-1)^n \overline{z}^{-1})$$

$$+ \sum_{k=2}^{\infty} y_k \left[-z - \frac{(-1)^n}{[(k-1)(\lambda - \alpha) - 1]^n [(k-1)(\lambda - \alpha) + 1]} \overline{z}^{-k} \right].$$

Consequently we obtain following relation as required

$$f_n(z) = \sum_{k=0}^{\infty} [x_k h_{n,k}(z) + y_k g_{n,k}(z)].$$

$\square$

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