Original research paper

Multiple Similarity Solutions for Surface-Tension Driven Flows in a Slot with an Insulated Bottom

Abstract

In this paper, a two-point boundary values problem

$$f''' + Q(Aff'' - f'^2) = \beta, \qquad Q, \beta \in \Re; A \geq 1,$$

subject to the boundary conditions

$$f(0) = f(1) = f''(0) + 1 = f'(1) = 0$$

is considered. The given problem arises from a study of similarity transformation for surface-tension driven flows of low Prandtl number fluids in a slot with an insulated bottom. Existence properties of solutions are examined and all possible solutions for the problem are classified using mathematical analysis for $A \geq 1$. Numerical computation of bifurcation diagrams is conducted to verify the results obtained by mathematical analysis. Multiple solutions occur for a range of Q values when $1 < A < 2$ and a unique solution exists for each Q when $A \geq 2$.

Keywords: Similarity solutions; Classification; Shooting scheme; Homogeneity method

2010 Mathematics Subject Classification: 34C15; 34E10

1 Introduction

The Navier-Stokes equations are the basic equations governing the motion of viscous fluid. Since these equations are necessarily nonlinear and complicated when applied to realistic problems, analytical results are often restricted to particular models with special properties. However, in some certain flows, the Navier-Stokes equations are reduced to nonlinear ordinary differential equations through a similarity transform for studying the solution properties [Aziz (2009); Wang (2009); Gorder et al. (2010); Xu et al. (2014); Costin et al. (2014)].

In this paper, a two point boundary value problem(TPBVP)

$$f''' + Q(Aff'' - f'^2) = \beta, \qquad Q, \beta \in \Re; A \geq 1, \tag{1.1}$$

subject to the boundary conditions

$$f(0) = f(1) = f''(0) + 1 = f'(1) = 0 \tag{1.2}$$

is studied. The given problem arises from a similarity reduction of boundary layer approximation of Navier-Stokes system in a microgravity environment [Gill et al. (1984)]. The Navier-Stokes system was applied to describe the steady state for the distributions of velocity in a low Prandtl(Pr) number fluid in a slot with an insulated bottom. Here Q is related to the Prandtl number, β is an integrable constant, $f(y)$ is related to the stream function, and $y = 1$ denotes the insulated bottom of the slot. For the derivation of equation (1.1) and (1.2) we refer to [Chen et al. (1993); Hwang et al. (1989)]. Numerical solutions of the TPBVP for $A = 1$ and $A = 2$ were studied by Hwang et al. [Hwang et al. (1989)] using a multiple shooting code BVPSOL. Hwang et al. also proved the existence properties for a portion of the solutions for $A = 1$ and $A = 2$ using mathematical analysis. The rest of the existence properties of the solutions for $A = 1$ and $A = 2$ were proved by Hwang and Wang [Hwang and Wang (1990)]. It is our purpose to study the TPBVP for $A \geq 1$. To provide details, mathematical analysis of the existence properties of the solutions for $A \geq 1$ is given in Sec. 2. Numerical computation of bifurcation diagrams and discussion is given in Sec. 3. Sec. 4 provides a brief conclusion.

2 Mathematical analysis of existence properties of solutions

Note that for each A, if $Q = 0$, the TPBVP has a unique solution $f(\eta) = \eta(1-\eta)^2/4$ for $\beta = 3/2$. Therefore, $Q \neq 0$ is assumed in our study. Let $y = b(1-\eta)$ and $g(y) = Qf(\eta)/b$ for $Q \neq 0$ and $b > 0$. The TPBVP is equivalent to

$$g''' + g'^2 - Agg'' = -Q\beta/b^4, \tag{2.1}$$

subject to the conditions

$$g(0) = g(b) = g'(0) = g''(b) + (Q/b^3) = 0. \tag{2.2}$$

Denote $g''(0)$ and $-Q\beta/b^4$ by α and B, respectively. By assuming values α and B, Eqs. (2.1) and (2.2) become the initial value problem:

$$g''' + g'^2 - Agg'' = B, \tag{2.3}$$

$$g(0) = g'(0) = g''(0) - \alpha = 0. \tag{2.4}$$

Suppose that the solution $g(y; \alpha, B, A)$ to Eqs. (2.3) and (2.4) meets the $y - axis$ at a positive value y^*. By setting $b = y^*$, the initial value problem in Eqs. (2.3) and (2.4) has a solution when $Q = -(y^*)^3 g''(y^*)$ and $\beta = -B(y^*)^4/Q$.
Given $A \geq 0$, we denote $g(y; \alpha, B) = g(y; \alpha, B, A)$. $g(y; \alpha, B)$ can be extended to the maximal interval $[0, M)$, where $M = M(\alpha, B) \leq \infty$. In fact, g tends to ∞ or $-\infty$ as y approaches M if $M < \infty$. Therefore, the classification of positive zeroes of g is given by (α, B) chosen from the following quadrants:

$$D_1 = \{(\alpha, B) \mid \alpha \geq 0, B > 0\},$$

$$D_2 = \{(\alpha, B) \mid \alpha < 0, B > 0\},$$

$$D_3 = \{(\alpha, B) \mid \alpha \leq 0, B < 0\},$$

and

$$D_4 = \{(\alpha, B) \mid \alpha > 0, B < 0\}.$$

We shall classify all possible solutions of Eqs. (1.1) and (1.2) by assuming values of α, B, A in Eqs. (2.1) and (2.3). It is clear that $g(y; 0, 0, A) = 0$ for all $A > 0$, $g(y; 0, B, 3/2) = \frac{B}{6}\eta^3$ for all $B \in \Re$, and $g(y; \alpha, 0, 2) = \frac{1}{2}\alpha\eta^2$ for all $\alpha \in \Re$. Thus, $(\alpha, B) \neq (0, 0)$, $(\alpha, A) \neq (0, \frac{3}{2})$, and $(B, A) \neq (0, 2)$ are assumed in the following discussions. Moreover, let $(0, M)$ be the corresponding maximal interval of $g(y; \alpha, B, A)$, where $M = M(\alpha, B, A)$. Note that g can only blow up to ∞ or $-\infty$ if $M < \infty$. The following expressions are used frequently in the mathematical analysis:

$$g''' = B - (g')^2 + Agg'', \tag{2.5}$$

$$g^{(iv)} = (A-2)g'g'' + Agg''', \tag{2.6}$$

$$g^{(v)} = (A-2)(g'')^2 + (2A-2)g'g''' + Agg^{(iv)}. \tag{2.7}$$

2.1 $A \geq 1$ and $B \leq 0$

Lemma 2.1. *For $A \geq 0$ and $B \leq 0$, $g''(y; \alpha, B, A)$ has at most one zero for all $\alpha \in \Re$.*

Proof. Assume that y_1 and y_2 are the first and second zero of g'', respectively. By (2.3), $g'''(y_i) = B - g'(y_i)^2 \leq 0$ for i=1,2. If the equality holds for i = 1 or 2, then $B = 0$, $g'(y_i) = 0$, and $g''(y_i) = 0$. Consider Eq. (2.3) together with the initial condition $g(y_i)$, $g'(y_i) = 0$, and $g''(y_i) = 0$. Then $g(y) \equiv g(y_i)$, $y \in [y_i, M)$ is the solution. In fact, the solution $g(y) \equiv g(y_i)$ can be extended to the maximal interval $[0, M)$. Therefore $g(y) \equiv 0$. This contradicts the assumption that $(\alpha, B) \neq (0, 0)$. Therefore $g'''(y_i) = B - g'(y_i)^2 < 0$ for i=1,2. This implies that g'' has a zero in (y_1, y_2), which is a contradiction. ■

Theorem 2.2. *For $A > 0$, $B \leq 0$, and $\alpha \leq 0$, $g(y; \alpha, B, A) < 0$ on $(0, M)$.*

Proof. Since $g'''(0) = B \leq 0$, $g''(0) = \alpha \leq 0$, and $(\alpha, B) \neq (0, 0)$, g'' is negative initially. Assume that g'' has a zero on $(0, M)$ and let $\bar{y}$ be the first positive zero. This implies that $g'''(\bar{y}) \geq 0$ and g' is negative on $(0, \bar{y})$, but $g'''(\bar{y}) = B - g'(\bar{y})^2 < 0$ is a contradiction. Therefore, $g'' < 0$ on $(0, M)$. This, together with the initial conditions $g'(0) = 0$ and $g(0) = 0$, gives the result $g(y; \alpha, B, A) < 0$ on $(0, M)$. ■

Lemma 2.3. *For $A \in [0, 2)$, $B \leq 0$, and $\alpha > 0$, $g''(y; \alpha, B, A)$ has exactly one zero.*

Proof. Assume $g'' > 0$ on $(0, M)$ and then $g > 0$ and $g' > 0$ on $(0, M)$. Let $\mu(y) = \exp(-A\int_0^y g)$. We have $(\mu g''')' = (A-2)\mu g'g'' < 0$ and thus $g''' \leq B\exp(A\int_0^y g) \leq 0$. Thus, $g^{(iv)} < 0$ implying that g'' is concave downward on $(0, M)$ which contradicts to $g'' > 0$ on $(0, M)$. Hence, $g''(y; \alpha, B, A)$ has at least one zero. From Lemma 2.1, $g''(y; \alpha, B, A)$ has exactly one zero. ■

Theorem 2.4. *For $A \in [0, 2)$, $\alpha > 0$, and $B \leq 0$, $g(y; \alpha, B, A)$ has exactly one zero.*

Proof. Let y_2 be the zero of g'' and assume that $g' > 0$ on $(0, M)$ which leads to $g > 0$ on $(0, M)$. By the proof of Lemma 2.3, $g'''(y) \leq B \leq 0$ on $(0, y_2)$. Thus, $g''' = B - g'^2 + Agg'' < 0$ on (y_2, M), and g' is concave downward on $(0, M)$. This contradicts to $g' > 0$ on $(0, M)$, and g' has exactly one positive zero. Similarly, g has exactly one zero. ■

Theorem 2.5. *For $A > 2$, $\alpha > 0$, and $B \leq 0$, $g(y; \alpha, B, A)$ has either one or no zero.*

Proof. It is easy to prove that if $g'' > 0$ on (0,M), then g has no zero, and if g'' has exactly one zero on $(0, M)$, then g has exactly one zero on $(0, M)$. By Theorem 2.2, $g(y; 0, B, A) < 0$ on $(0, M)$ for all $B < 0$. By continuous dependence on the initial data, if α is sufficiently small, then g has exactly one zero. ■

2.2 $A \geq 1$ and $B > 0$

Lemma 2.6. *For $A > 2$ and $\alpha \geq 0$, $g^{(iv)}(y;\alpha, B, A) > 0$ on $(0, M)$.*

Proof. Assume that $B > 0$. From Eq. (2.4), we have $g'''(0) = B > 0$. Therefore, all of g, g', and g'' are increasing and positive initially. When $g^{(k)}(t) > 0$ for all $0 \leq k \leq 3$ and $A \geq 2$, we have $g^{(iv)}(t) > 0$. So, $g^{(k)}(t) > 0$ is increasing at t for all $0 \leq k \leq 3$. Therefore, $g^{(k)}(t) > 0$ on $(0, M)$ for all $0 \leq k \leq 4$ if $B > 0$ and $A \geq 2$. Now if $B = 0$, we may assume $\alpha > 0$. Since $g^{(iv)}(0) = 0$ and $g^{(v)}(0) > 0$ for $A > 2$, $g^{(iv)}$ is increasing and positive initially. Therefore, $g^{(k)}(t) > 0$ on $(0, M)$ for all $0 \leq k \leq 4$ by similar arguments as stated above. ■

Theorem 2.7. *For $A > 2$, $\alpha \geq 0$, $g(y;\alpha, B, A) > 0$ on $(0, M)$.*

Proof. Note that $g(t) = \frac{\alpha}{2}t^2$ is the solution for $A = 2$ and $B = 0$. This fact, together with Lemma 2.6, completes the proof of this theorem. ■

Lemma 2.8. *For $A > 2$, $\alpha \leq 0$, and $B > 0$, $g''(y;\alpha, B, A)$ has at most one zero.*

Proof. If $g''' > 0$ on $(0, M)$, the Lemma is clear. Suppose that g''' has a positive zero and let y_0 be the first zero of g'''. It follows that $g^{(iv)}(y_0) \leq 0$ from Eq. (2.6). Hence, $g'(y_0)g''(y_0) \leq 0$ if $A > 2$. Because $g''' > 0$ on $(0, y_0)$, $g''(y_0) > 0$ and $g'(y_0) \leq 0$. Now, we divide the proof into two cases. Case (i): $g'(y_0) = 0$. In this case, $g'''(y_0) = g^{(iv)}(y_0) = 0$ and $g^{(v)}(y_0) > 0$. Thus, $g''' > 0$ on $(y_0, y_0 + \delta)$ for some $\delta > 0$. Suppose that $g''' > 0$ on (y_0, y_1) and $g'''(y_1) = 0$. This implies that both g'' and g' are positive and increasing on (y_0, y_1). So $g^{(iv)}(y_0) > 0$. This is a contradiction. Therefore, y_0 is the unique zero of g''', and g'' has exactly one zero. Case (ii): $g'(y_0) < 0$. This leads that $g'(y_0)^2 < B$. Let $y_* \in (0, y_0)$ be the first zero of g''. Thus, g and g' are negative on $[y_*, y_0]$. This implies $g^{(iv)} \leq 0$ on (y_*, y_0). Now, we claim that $g'' > 0$ for all $y > y_0$. Assume that y^* is the second zero of g''. Since g'' and g''' cannot both be zero at y^*, we have $g'''(y^*) < 0$. In fact, $g''' \leq 0$ on $[y_0, y^*]$. Otherwise, there exists y_2 in (y_0, y^*) such that $g'''(y_2) = 0$ and $g^{(iv)}(y_2) > 0$. This implies $g'(y_2) > 0$. By similar arguments as in Case (i), $g'' > 0$ for $y > y_0$. This contradicts the assumption that g'' has a second zero at y^*. Let $y_3 > y_0$ be a zero of g satisfying $g < 0$ on $[y_0, y_3)$. Furthermore, let $\bar{y} = \min\{y_3, y^*\}$. Thus $g'''(\bar{y}) \leq 0$ and $g^{(iv)} > [A-2]g'g''$ on $[y_0, \bar{y}]$. Then,

$$\begin{aligned}
\int_{y_0}^{\bar{y}} g^{(iv)}(y)dy &> \int_{y_0}^{\bar{y}} [A-2]g'(y)g''(y)dy, \\
g'''(\bar{y}) &> \frac{A-2}{2}(g'(\bar{y}))^2 - \frac{A-2}{2}(g'(y_0))^2.
\end{aligned}$$

Next,

$$\begin{aligned}
g'''(\bar{y}) - \frac{A-2}{2}[B + Ag(\bar{y})g''(\bar{y}) - g'''(\bar{y})] &= g'''(\bar{y}) - \frac{A-2}{2}(g'(\bar{y}))^2 \\
&\geq -\frac{A-2}{2}g'^2(y_0) \\
&> -\frac{A-2}{2}B.
\end{aligned}$$

Thus, $g''' - (A-2)gg'' > 0$ at $y = \bar{y}$. It contradicts the sign of $g'''(\bar{y})$. Thus, $g'' > 0$ for all $y > y_0$ and therefore, the proof is complete. ■

The following theorem is obtained immediately.

Theorem 2.9. *For $A > 2$, $\alpha < 0$, and $B > 0$, $g(y;\alpha, B, A)$ has at most one zero.*

For the mathematical analysis of the rest of the cases, a notation containing the sign of $g^{(k)}$, where $k = 0, 1, \cdots, 5$, is defined with (sign g, sign g', $\cdots$, sign $g^{(v)}$). We use “+”“-”,“0”,“+0”,“-0”, and "*" to indicate positive, negative, zero, positive or zero, negative or zero, and indeterminate or unimportant, respectively. For example, $(+,-,0,+0,-0,*)$ at y means that $g(y)>0$, $g'(y)<0$, $g''(y)=0$, $g'''(y)\geq 0$, $g^{(iv)}(y)\leq 0$, and the sign of $g^{(v)}(y)$ is indeterminate or it does not affect the result of the analysis.

Lemma 2.10. *For $A\in(1,2)$, $\alpha\geq 0$, and $B>0$, $g''(y;\alpha,B,A)$ has at most one zero.*

Proof. From the initial condition Eq. (2.4), we have $(+,+,+,+,*,*)$ on $(0,\delta)$ for some $\delta>0$. Let y_* be the first zero of g'', then we have $(+,+,0,-,-,-)$ at y_* because g'' and g''' cannot be zero simultaneously. Consequently, we have $(+,+,-,-,-,-)$ on $(y_*,y_*+\delta_1)$ for some $\delta_1>0$. Suppose that $g^{(4)}(y_1)=0$ for some $y_1>y_*$ and $g^{(iv)}(y)<0$ for $y\in(y_*,y_1)$. Because $g''(y)<0$ and $g'''(y)<0$ for $y\in(y_*,y_1]$, there are three possible cases of g and g' values at y_1: (i) $(-0,-,-,-,0,+0)$, (ii) $(+,-,-,-,0,+0)$, or (iii) $(+,+0,-,-,0,+0)$.

For case (i), if $g'''(y_2)=0$ for some $y_2>y_1$ and $g'''(y)<0$ for $y\in(y_1,y_2)$, then $g^{(iv)}(y_2)\geq 0$. Now, $g'''(y_2)=0$ and $g^{(iv)}(y_2)=0$ imply $g'(y_2)=0$ or $g''(y_2)=0$, which cannot hold in this case. Therefore, $g^{(iv)}(y_2)>0$. Thus, we have $(-,-,-,0,+,*)$ at y_2. However, this contradicts the sign of $g^{(iv)}(y_2)$ determined by Eq. (2.6).

For, case (ii), $g^{(iv)}(y_1)<0$ from Eq. (2.6). This contradicts with the assumption $g^{(iv)}(y_1)=0$.

For case (iii), $g^{(v)}(y_1)<0$ from Eq. (2.7). This contradicts with the sign of $g^{(v)}(y_1)$ in this case.

The above three cases give the conclusion that $g'''<0$ for $y\in(y_*,M)$. Therefore, $g''<0$ for $y\in(y_*,M)$. ∎

Lemma 2.10 proves the following theorem.

Theorem 2.11. *For $A\in(1,2)$, $\alpha\geq 0$ and $B>0$, $g(y;\alpha,B,A)$ has at most one zero.*

Theorem 2.12. *For $A\in(1,2)$, $\alpha<0$ and $B>0$, $g(y;\alpha,B,A)$ has at most two zeroes.*

Proof. From the initial condition in Eq. (2.4), we have $(-,-,-,+,-,-)$ on $(0,\delta)$ for some $\delta>0$. We let $y_1>0$ such that there are two possible cases: (i) y_1 is the first zero of g''', and g, g', and g'' do not change their sign in $(0,y_1]$. (ii) y_1 is the first zero of g'', and g, g', and g''' do not change their sign in $(0,y_1]$.

For case (i), $g^{(iv)}(y_1)<0$ from Eq. (2.6). From case (i) in the proof of Lemma 2.10, $g'''(y)<0$ for $y\in(y_1,M)$ and $g(y;\alpha,B,A)$ has no zero on $(0,M)$.

For case (ii), $g^{(iv)}(y_1)<0$ and we have $(-,-,+,+,-,*)$ on $(y_1,y_1+\delta_1)$ for some $\delta_1>0$. We let $y_2>y_1$ such that there are two possible cases: (a) $g'''(y_2)=0$, $g'''(y)>0$ for $y\in(y_1,y_2)$, and we have $(-,-,+,+0,*,*)$ on $(y_1,y_2]$. (b) $g'(y_2)=0$, $g'(y)<0$ for $y\in(y_1,y_2)$, and we have $(-,-0,+,+,*,*)$ on $(y_1,y_2]$.

Case (a) is impossible because $g^{(iv)}(y_2)>0$ from Eq. (2.6).

For case (b), we have $(-,+,+,+,-,*)$ on $(y_2,y_2+\delta_2)$ for some $\delta_2>0$. We let $y_3>y_2$ such that there are two possible cases: (1) $g(y_3)=0$, $g(y)<0$ for $y\in(y_2,y_3)$, and we have $(-0,+,+,+,*,*)$ on $(y_2,y_3]$. (2) $g'''(y_3)=0$, $g'''(y)>0$ for $y\in(y_2,y_3)$, and we have $(-,+,+,+0,*,*)$ on $(y_2,y_3]$.

For case (1), $y^{(iv)}(y_3)<0$ from Eq. (2.6), and we have $(+,+,+,+,-,*)$ on $(y_3,y_3+\delta_3)$ for some $\delta_3>0$. From the proof of Lemma 2.10, g has at most one zero in (y_3,M), and thus, g has at most two zeroes in $(0,M)$.

For case (2), $y^{(iv)}(y_3)<0$, and we have $(-,+,+,-,-,*)$ on $(y_3,y_3+\delta_4)$ for some $\delta_4>0$. We let $y_4>y_3$ such that there are three possible cases: (A) $g'''(y_4)=0$, $g'''(y)<0$ for $y\in(y_3,y_4)$, and we have $(-,+,+,-0,*,*)$ on $(y_3,y_4]$. (B) $g(y_4)=0$, $g(y)<0$ for $y\in(y_3,y_4)$, and we have $(-0,+,+,-,*,*)$ on $(y_3,y_4]$. (C) $g''(y_4)=0$, $g''(y)>0$ for $y\in(y_3,y_4)$, and we have $(-,+,+0,-,*,*)$ on $(y_3,y_4]$.

Case (A) is impossible because $g^{(iv)}(y_4)<0$ from Eq. (2.6).

For case (B), we have $(+,+,+,-,-,*)$ on $(y_4,y_4+\delta_5)$ for some $\delta_5>0$. We may apply the proof of Lemma 2.3 and Theorem 2.4, and g has exactly one zero in (y_4,M). Therefore, g has exactly two zeroes in $(0,M)$.

For case (C), $y^{(iv)}(y_4) > 0$, from Eq. (2.6), implies that there is $\bar{y}$, where $y_3 < \bar{y} < y_4$, such that $y^{(iv)}(\bar{y}) = 0$ and $y^{(iv)}(y) > 0$ for $y \in (\bar{y}, y_4]$. However, $g^{(v)}(\bar{y}) < 0$ from Eq. (2.7). Therefore, case (C) is impossible.

From all the cases discussed above, we have concluded that $g(y; \alpha, B, A)$ has at most two zeroes. ∎

From the above lemmas and theorems and the cases studied by Hwang et al. [Hwang et al. (1989)] for $A = 1$ and $A = 2$, the existence properties of solutions for $A \geq 1$ are summarized as follows:

(i) For $B \leq 0$, $\alpha \leq 0$, and $A \geq 1$, g has no zero.

(ii) For $B \leq 0$ and $\alpha > 0$, g has one zero if $A \in [1, 2)$, and g has at most one zero if $A \geq 2$.

(iii) For $B > 0$ and $\alpha < 0$, g has at most two zero if $A \in [1, 2)$, and g has at most one zero if $A \geq 2$.

(iv) For $B > 0$ and $\alpha \geq 0$, g has at most one zero if $A \in [1, 2)$, and g has no zero if $A \geq 2$.

3 Numerical simulations and discussion

As in [Chen et al. (1993); Hwang and Wang (1992); Lu and Kazarinoff (1989)] $g(y; \alpha, B, A)$ satisfies the following property.

Proposition 3.1. $g^{(i)}(y; \alpha, B) = \lambda^{i+1} g^{(i)}(\lambda y; \alpha/\lambda^3, B/\lambda^4)$, *for* $\lambda > 0$ *and* $i = 0, 1, 2, \ldots$.

Now, let a_{ij} be the j-th positive zero, if there is any, of $g^{(i)}(y; \alpha, B)$. Then a_{ij}, Q and β satisfy the following homogeneity property.

Proposition 3.2. *For* λ,

$$a_{ij}(\alpha, B) = a_{ij}(\alpha/\lambda^3, B/\lambda^4)/\lambda,$$

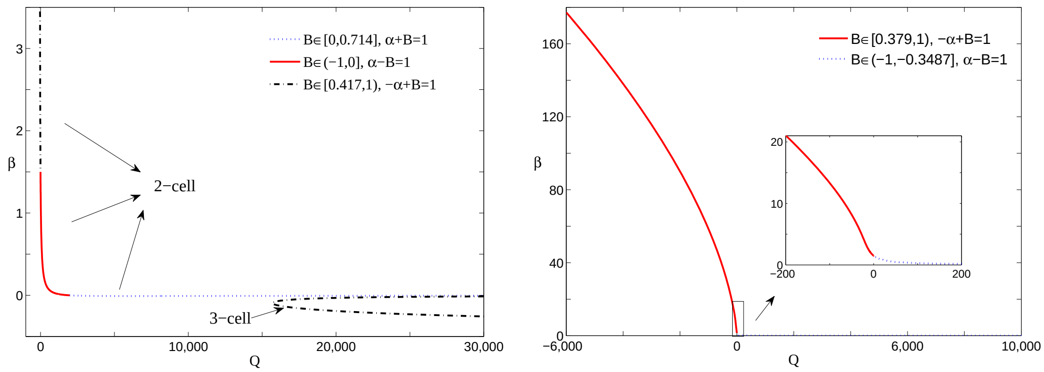
$$Q(\alpha, B) = Q(\alpha/\lambda^3, B/\lambda^4),$$

and

$$\beta(\alpha, B) = \beta(\alpha/\lambda^3, B/\lambda^4).$$

It is easy to verify that a_{ij} and β are C^1 functions in terms of α and B. The homogeneity property in Proposition 3.2 also implies that the corresponding parameter (Q, β) is unique if (α, B) is chosen arbitrarily on a certain curve $B = k\alpha^{4/3}$, where k is a constant.

By the homogeneity, one may locate possible zeroes of $g(\eta; \alpha, B, A)$ by simply applying an initial value problem code SDRIV2 [Kahaner et al. (1989)] for choosing (α, B) along a simple closed curve around the origin in the (α, B) plane for every A. That is, we may pick the shooting parameter (α, B) along the curve, $|\alpha| + |B| = 1$. As defined in [Chen et al. (1993)], a solution f of Eqs. (1.1) and (1.2) is called two-cell if f' has exactly one zero in $(0, 1)$ and it is called three-cell if f' has exactly two zeros in $(0, 1)$. The reader is referred to [Chen et al. (1993); Hwang et al. (1989)] for the physical meanings of two-cell and three-cell solutions. In our first numerical example, we let $A = 1.5$, and Fig. 1(a) shows the bifurcation diagram for $(Q, \beta) \in [-1000, 30000] \times [0.5, 3.5]$. Along the curve $|\alpha| + |B| = 1$, the zeroes of g are computed for $(\alpha, B) \in D_1$, D_2, and D_4. When $(\alpha, B) \in D_1$ and D_4, g has one zero. As in [Gill et al. (1984)], the corresponding TPBVP possesses 2-cell solutions. When $(\alpha, B) \in D_2$, g has two zeroes for $B \in (0.381, 1)$. The corresponding TPBVP possesses 2-cell or 3-cell solutions. Next, we let $A = 4$, and Fig. 1(b) shows the bifurcation diagram for $(Q, \beta) \in [-6000, 10000] \times [0, 180]$. From the analytical results in Sec. 2, g has no zero when $(\alpha, B) \in D_1$ and D_3. The zeroes of g are computed for $(\alpha, B) \in D_2$ and D_4. The function g has one zero for $B \in (-1, -0.348)$ and $B \in (0.348, 1)$, and the corresponding TPBVP possesses 2-cell solutions.



(a) (b)

Figure 1: Bifurcation diagram for (a) $A = 1.5$ and (b) A = 4.

4 Conclusion

In this paper, the existence of solutions for the TPBVP, given by Eqs. (1.1) and (1.2), is studied. The TPBVP is first transformed into an IVP which is presented by Eqs. (2.3) and (2.4). Solving the zeroes of the solution g to Eqs. (2.3) and (2.4) is equivalent to solving the TPBVP. In this paper, the existence properties of the zeroes of g have been proven for $A \geq 1$. From the mathematical analysis, we conclude that the TPBVP possesses only 2-cell solutions when $A \geq 2$, and it may possess 2-cell and 3-cell solutions when $1 \leq A < 2$. A homogeneity property of parameters is proven so that the numerical computation on the parameter ($\alpha - B$) plane is reduced to the perimeter of the square $|\alpha| + |B| = 1$. This greatly improves the computational efficiency. Numerical simulation is then conducted to verify the existence property of solutions for the TPBVP.

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