Cumulative Effects of the Temperature and Damping on the Time Dependent Entropy and Decoherence in the Caldirola-Kanai Harmonic Oscillator

Abstract

The time Dependence of probability and Shannon entropy of the specific damped harmonic oscillator systems is studied by using a prototypical Schrodinger cat state through the Feynman path method. By averaging the probability distribution over the thermal distribution of velocities, we show that, the temperature and the damped coefficient or dissipation as well as the distance separating two consecutive wave functions influence the coherence of the system.

Keywords: damped harmonic oscillator ; Feynman path integral ; Decoherence ; Shannon entropy ; thermal distribution probability

I. Introduction

Quantum decoherence, where coherence in a quantum system is reduced due to interaction with its environment is a fundamental and complicated concept of physics. Decoherence refers to the destruction of a quantum interference pattern and is relevant to the many experiments that depend on achieving and maintaining entangled states. Examples of such efforts are in the areas of quantum teleportation[1] quantum information and computation[2,3], entangled states[4], Schrödinger cats[5], and the quantum-classical interface[6]. For an overview of many of the interesting experiments involving decoherence, we refer to[4,7-11].

Understanding this phenomenon in dissipative harmonic oscillators is of a great interested**.** It is of great physical importance and has found many applications especially in quantum optics.

For example, it plays a central role in the quantum theory of lasers and masers[12-14]. Moreover, nowadays, many of research is dedicated to understand decoherence in harmonic oscillator[15,16]. Isar *et al*[17] determine the degree of quantum decoherence of a harmonic oscillator interacting with a thermal bath using Lindblad theory[18,19]. Other authors[20]use a semi-classical approach to examine decoherence in an anharmonic oscillator coupled to a thermal harmonic bath. Darius *et al*[21] exploit the Feynman path integral to study the memory in a non-locally damped oscillator. Moreover, Ozgur *et al*[16] determine the time dependence of Leipnik's entropy in the damped harmonic oscillator via path integral techniques. Sang Pyo Kim et *al*[22] study decoherence in quantum damped oscillators, G.W. Ford *et al*[23] show that decoherence depend on the temperature through the attenuation coefficient.

The aim of this paper is to study the cumulative effect of temperature and dissipation on the coherence of a damped harmonic oscillator of a particle in thermal equilibrium by using the Caldirola-Kanai model based on the idea of Bateman[24].

This model is known as a popular model used to describe dissipative systems. Here, we focus our attention on the study of decoherence by evaluating quantum Shannon entropy. In the literature for both open and closed quantum systems, different information-theoretical entropy measures have been discussed[25,27]. In contrast, quantum Shannon entropy[28,29] can also be used to characterize the loss of information related to evolving pure quantum states[30].

This paper is organized as follows. In Sec. II, we present the mathematical tools based on the path integral formalism. We discuss the case of a damped harmonic oscillator and build the associated propagator. In Sec. III, we investigate the effects of the damped coefficient and the temperature on the coherence of the system through the Shannon entropy behavior. Discussion and concluding remarks are given in the last section.

II. Model description

We start by presenting the model which consists of a particle of mass m, labeled by the position variable q and the momentum p. Then follows the description of the used mathematical tools which is the path integral formalism introduced by Feynman[31].

These tools suggest that the transformation function called propagator is analogue to $exp\left(\frac{i}{\hbar} S_{cl}\right)$ in which S_{cl} stands for the action, solution of Hamilton-Jacobi equation. On the other way, the transition amplitude of the particle (of mass m) from the position q_a at time t_a to the position q_b at time t_b, known as the propagator, represents the solution of the Schrodinger equation. This lagrangian formulation generalizes the theory of relativity (time and space). Nowadays, several problems of physics are solved via these techniques[16,32]

Next, we consider the Bateman Hamiltonian[24] defined as :

$$H = \bar{p}p - \gamma\left[x\bar{p} - \bar{x}p\right] + \Omega^2 \bar{x}x \qquad (1)$$

where $\bar{p}$ and $\bar{x}$ are the mirror variables corresponding to the coordinate x and the momentum p. The quantities γ and Ω are respectively the damped coefficient and the system frequency. The associated lagrangian is given by

$$L = \dot{\bar{x}}\dot{x} - x\bar{x} + \gamma\left(x\dot{\bar{x}} - x\bar{x}\right) \qquad (2)$$

Using Euler-Lagrange equation, we derive the following two motion equations: [33]

$$\begin{cases} \ddot{x} + \gamma\,\dot{x} + \Omega^2 x = 0 \\ \ddot{\bar{x}} + \gamma\,\dot{\bar{x}} + \Omega^2 \bar{x} = 0 \end{cases} \qquad (3)$$

Bateman's dual Hamiltonian describes classical mechanics correctly, but this model faces some difficulties. It violates Heisenberg's principle for $\gamma \neq 0$. Therefore, to solve quantum mechanical problem, Caldirola-Kanai[34] build a theory based on the idea of Bateman dissipative system by considering the standard Hamiltonian of harmonic oscillator with time dependent mass given by[34] $m(t) = m_0 \, exp(2\gamma t)$. Hence, the Hamiltonian and the lagrangian of the system become respectively:

$$H = \frac{p^2}{2m(t)} + \frac{1}{2} m(t)\, \omega^2 x^2$$
$$L = p\dot{x} - H = exp(2\gamma t)\left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2\right] \qquad (4)$$

From the lagrangian theory and exploiting Eq.(4), the equation of motion takes the form :

$$\ddot{x} + 2\gamma\,\dot{x} + \Omega^2 x = 0 \qquad (5)$$

The classical solution of Eq.(5) is given by

$$x(t) = C_1 \, exp(a_1 t) + C_2 \, exp(a_2 t) \qquad (6)$$

wherein a_1 and a_2 are complex quantities defined as : $a_1 = -\gamma - i\Omega$, $a_2 = -\gamma + i\Omega$ with $\Omega = \sqrt{\omega^2 - \gamma^2}$. The integration constants C_1 and C_2 are evaluated when the particle moves from the position x_a at the time t_a to the position x_b at time t_b. The determination of the propagator is convenient for founding quantum mechanical solution for this Hamiltonian. Therefore, the classical action S_{cl} is defined as :

$$S_{cl}\left(x_a, t_a; x_b, t_b\right) = \int L\left(x, \dot{x}, t\right) dt = \int_{t_a}^{t_b} \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right) dt$$

whose computation for the current study case leads to :

$$S_{cl}(x_a, t_a; x_b, t_b) = \frac{m_0 \Omega \mu_1}{\exp\{-2\gamma(t_a + t_b)\}\sin\{\Omega(t_a - t_b)\}} + \frac{m_0 \gamma\, \mu_2}{\exp\{-2\gamma(t_a + t_b)\}} \qquad (7)$$

In deriving relation (7), we set:

$$\mu_1 = \left[x_b^2\, exp(-2\gamma t_a) + x_a^2\, exp(-2\gamma t_b)\right] - 2x_a x_b\, exp[-\gamma(t_a + t_b)] + cos\{\Omega(t_a - t_b)\}$$

$$\mu_2 = x_b^2\, exp(-2\gamma t_a) - x_a^2\, exp(-2\gamma t_b)$$

From the classical action, the expression of the corresponding propagator of the damped harmonic oscillator is defined below.

$$\aleph(x_a, t_a; x_b, t_b) = \left[\frac{m_0 \Omega}{2\pi i \hbar \exp\{2\gamma(t_a + t_b)\}\sin\{\Omega(t_b - t_a)\}}\right]^{1/2} \exp\left(\frac{i}{\hbar} S_{cl}(x_a, t_a; x_b, t_b)\right) \quad (8)$$

This result is identical to the one establish in[32] using the propagator method developed by Um *et al*[35]. It also appears from Eq.(8) that the propagator $\aleph(x_a, t_a; x_b, t_b)$ depends on the damped coefficient γ that links the system with the environment in which it evolves.

Hereafter, we intend to use the propagator (8) and derive some characteristic parameters (such as the distribution probability and the Shannon entropy) of the system subjected to a specific double Gaussian wave functions. These investigations aim to measure the impact of the environment (temperature and dissipation) on the behavior of the system when the latter progresses.

III. System properties under specific double Gaussian case

One of the specificity of this section deals with the choice of a double Gaussian wave function $(\phi''(x_b, 0))$ that includes the thermal distribution of velocities. Here, our starting point is the prototypical Schrödinger cat state i.e., an initial state corresponding to two separated Gaussian wave packets[22]. The motivation of this choice of wave packet is that it describes accurately the interference pattern arising in Young's two slits experiments[36] or that arising from the quantum measurement involving a pair of "*Gaussian slits*"[37] :

$$\phi''(x_b,0)=\frac{\left(2\pi\sigma^2\right)^{-1/4}}{\sqrt{2\left(1+exp\left(\frac{-d^2}{8\sigma^2}\right)\right)}}\left\{exp\left[\frac{-1}{4\sigma^2}\left(x-\frac{d}{2}\right)^2+\frac{im_0v}{\hbar}x\right]+exp\left[\frac{-1}{4\sigma^2}\left(x+\frac{d}{2}\right)^2+\frac{im_0v}{\hbar}x\right]\right\} \qquad (9)$$

in which σ is the width of each packet, d represents the distance between the top of the two successive waves in the double Gaussian state $\sigma << d$, and v is the particle velocity. To appreciate the impact of this wave packet on the thermodynamic parameters of the system, we seek separately its distribution probability and Shannon entropy.

A. Distribution probability

We determine the distribution probability for a double Gaussian wave packet to find the particle at time t at coordinate x. This probability can be written in the Feynman Hibbs form as[30]:

$$P''(x,t)=\left|\varphi''(x,t)\right|^2=\int dx'_b\int dx_b\,\aleph^*(x_a,x'_b,t)\,\aleph(x_a,x_b,t)\,\varphi(x'_b,0)\,\varphi^*(x_b,0) \qquad (10)$$

The distribution probability P'' is obtained by substituting expression (9) into Eq.(10). The computation yields the upcoming quantity:

$$P''=\frac{16\pi A\sigma^2}{\sqrt{1-16a^2\sigma^4}}e^{u_{11}u_{12}}\left[cosh(u_{11}u_{13})+cosh(u_{11}u_{14})\right] \qquad (11)$$

wherein $u_{11}=\frac{\sigma^2}{1-16a^2\sigma^4}$; $u_{12}=\frac{d^2}{8\sigma^4}+2b^2-\frac{2m_0^2v^2}{\hbar^2}+\frac{4im_0vd}{\hbar}$; $u_{13}=4abd+\frac{4iam_0vb}{\hbar}$

$u_{14}=\frac{bd}{\sigma^2}+\frac{im_0vb}{\hbar\sigma^2}$ and $A=\frac{\left(8\pi\sigma^2\right)^{-1/2}m_0\Omega\exp\left(2\gamma\,t_b-\frac{d^2}{8\sigma^2}\right)}{2\pi\hbar\sin(\Omega t)\left[1+\exp\left(-\frac{d^2}{8\sigma^2}\right)\right]}$

One could note that the distribution probability depends not only on time, position and system frequency, but also on the distance separating the two successive peaks of the double

Gaussian function. In the limit case $d=0$, we recover the probability of a single Gaussian wave function[14].

We consider now the case of a particle in the thermal equilibrium. The principles of statistical mechanics tell us that, we obtain the corresponding probability distribution by averaging distribution Eq.(11) over the thermal distribution of velocities[22] :

$$P_T(x,t)=\sqrt{\frac{m_0}{2\pi K_B T}}\int_{-\infty}^{+\infty} dv\, exp\left(-\frac{m_0 v^2}{2K_B T}\right)P''(x,t) \tag{12a}$$

whose calculation in our case leads to

$$P_T(x,t)=\frac{32\pi^{\frac{3}{2}}A\sigma^2\sqrt{m_0}\,exp(A'C)}{\left(\frac{4\pi m_0^2 K_B T}{\hbar^2}+m_0\pi\right)^{\frac{1}{2}}\sqrt{1-16a^2\sigma^4}}\left[cosh(v_{11})\,exp(-v_{12})+cosh(v_{13})\,exp(-v_{14})\right] \tag{12b}$$

in which $v_{11}=4A'abd-\frac{16K_BT\sigma^2m_0^2b^2}{4K_BTm_0A'+\hbar^2}$; $v_{12}=\frac{8K_BTm_0b^2+8K_BT\sigma^2m_0a^2}{4K_BTm_0A'+\hbar^2}$;

$v_{13}=\frac{A'bd}{\sigma^2}-\frac{4K_BTm_0bd}{\sigma^2\left(4K_BTm_0A'+\hbar^2\right)}$; $v_{14}=\frac{8K_BTm_0^2b^2+8K_BT\sigma^2m_0b^2}{\sigma^4\left(4K_BTm_0A'+\hbar^2\right)}$; $A'=\frac{\sigma^2}{1-16a^2\sigma^4}$

and $C=\frac{d^2}{8\sigma^4}+2b^2$

Furthermore, we intend to deeply measure the loss of information of the system by investigating the influence of the environment on the Shannon entropy.

B. Shannon entropy

It is well known that the major way to appreciate the purity of a system is to study the evolution of its entropy. When this quantity tends to zero, we obtain a pure state. Decoherence stands for the loose of information in the system. This occurs when the exchange between the environment and the system affects the evolution of the concerned system. In this subsection, we investigate the Shannon entropy relates to the double Gaussian wave function for a specific damped harmonic oscillator that interacts with the environment. Owing to the definition, this entropy is Mathematically defined by Boltzmann- Shannon as :

$$S(t) = -K \int_{-\infty}^{+\infty} P_T \, ln \, P_T \, dx \qquad (14)$$

in which the probability P_T is defined by Eq.(12b). Hereafter, we explore the influence of each system characteristics on the evolution of this entropy. Fig.1. presents the behavior of the entropy $S(t)$ for the pure state.We observe that even with temperature,we obtain a coherent state. From Fig .2. and Fig.3. , we observe the increasing of the entropy with the damped factor γ. That means, the decoherence in the case of low temperature is due to the dissipation factor. Fig. 4., Fig.5. and Fig.6 show the high temperature regime where the the dissipation becomes bigger and bigger. We observe that the entropy increases with the temperature for small values of the damped factor γ. Physically, it indicates that the temperature is the main factor responsible of the decoherence.

IV. Conclusion

In this paper, we have studied the thermodynamic properties of a damped harmonic oscillator using the Caldirola-Kanaï model based on the idea of Bateman. The Feynman path method is exploiting to investigate the time dependent Shannon entropy for specific double Gaussian wave functions. We have derived the classical action, the probability distribution and the entanglement entropy. Those quantities are affected by the damped coefficient, the frequency of the system and the temperature. We have observed that the envelope of the Shannon entropy curves with time indicated that the system is losing information with time. Simply, the Shannon entropy show universal scaling that is reminiscent of thermodynamic quantities as in[38,39].

We can use the damped factor and the temperature parameters for two purposes : to favor the decoherence or favor the coherence of the system. We have noted that even if the temperature and dissipation have cumulative effects, the main control parameter is the damped factor. Our system can be very interesting for engineering purposes. To code or decode information, we have to take into consideration the damped coefficient, the frequency of oscillation, and the temperature. For future investigations, we can also take into consideration the physical domain of the damped factor as well as the distance separated the two consecutive wave functions.

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Figure Captions

Fig .1.: Shannon entropy $S(t)$ as function of time for the pure state: harmonic oscillator without dissipation

Fig .2.: Shannon entropy $S(t)$ as function of time for γ = 0.002 ; Ω = 1.7 ; d = 5 ; and T = 0.0000001

Fig .3.: Shannon entropy $S(t)$ as function of time for γ =0.02; Ω =1.7: d=5 and T=0.0000001

Fig. 4.: Shannon entropy $S(t)$ as function of time for γ =0.0007 ; Ω = 1 ; d = 5 ; and T = 0.000001

Fig. 5.: Shannon entropy $S(t)$ as function of time for γ = 0.0007 ; Ω = 1 ; d = 5 ; and T = 0.000003

Fig .6.: Shannon entropy $S(t)$ as function of time for γ = 0.00001 ; Ω = 1 ; d = 5 ; and T = 0.00002

Figures

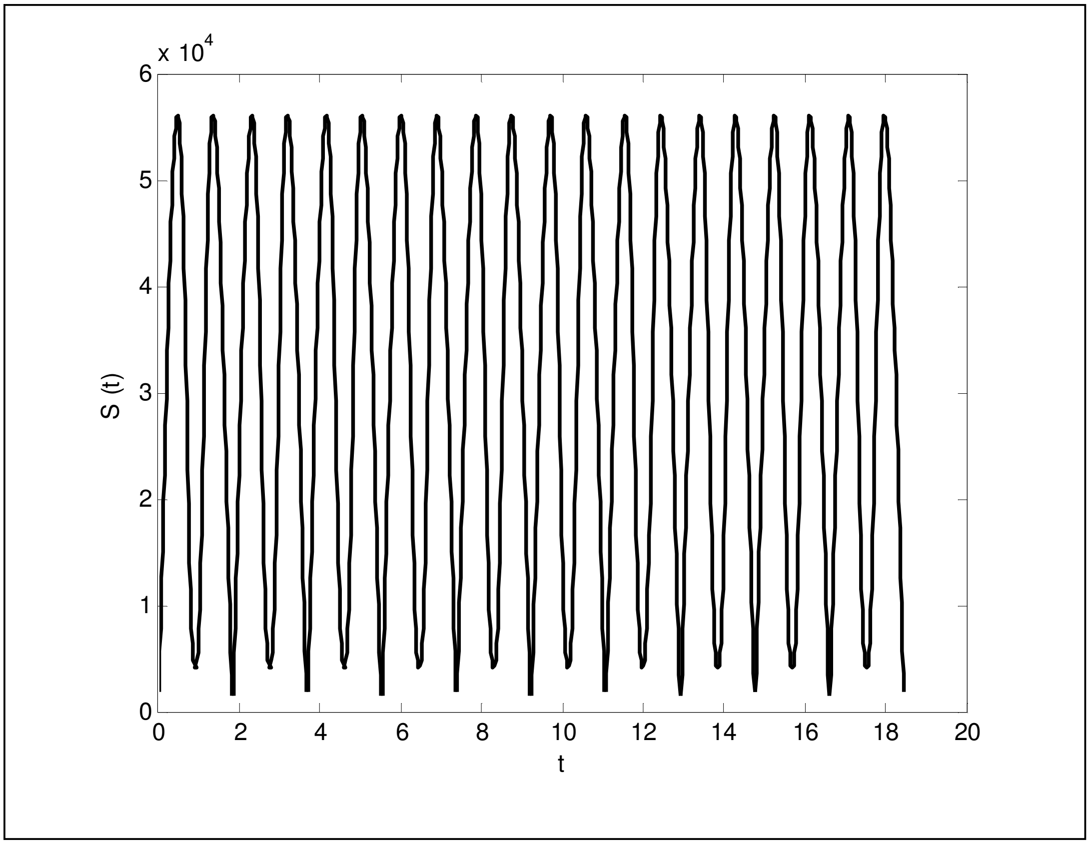
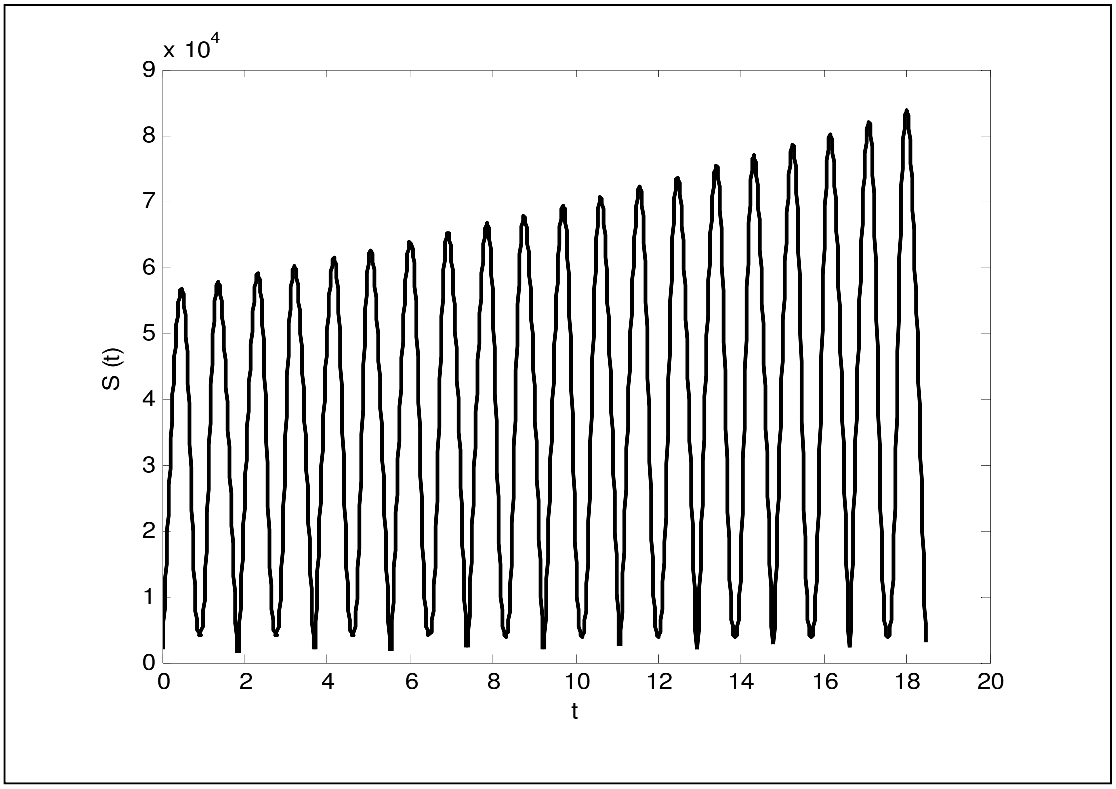


Fig .1: Pelap *et al*



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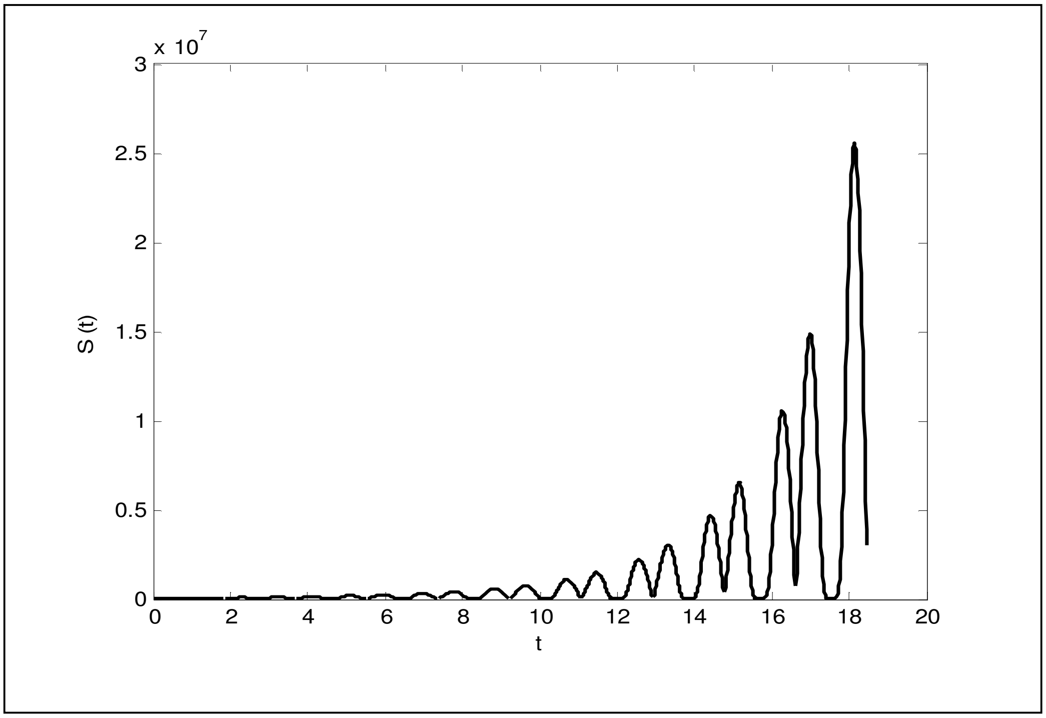

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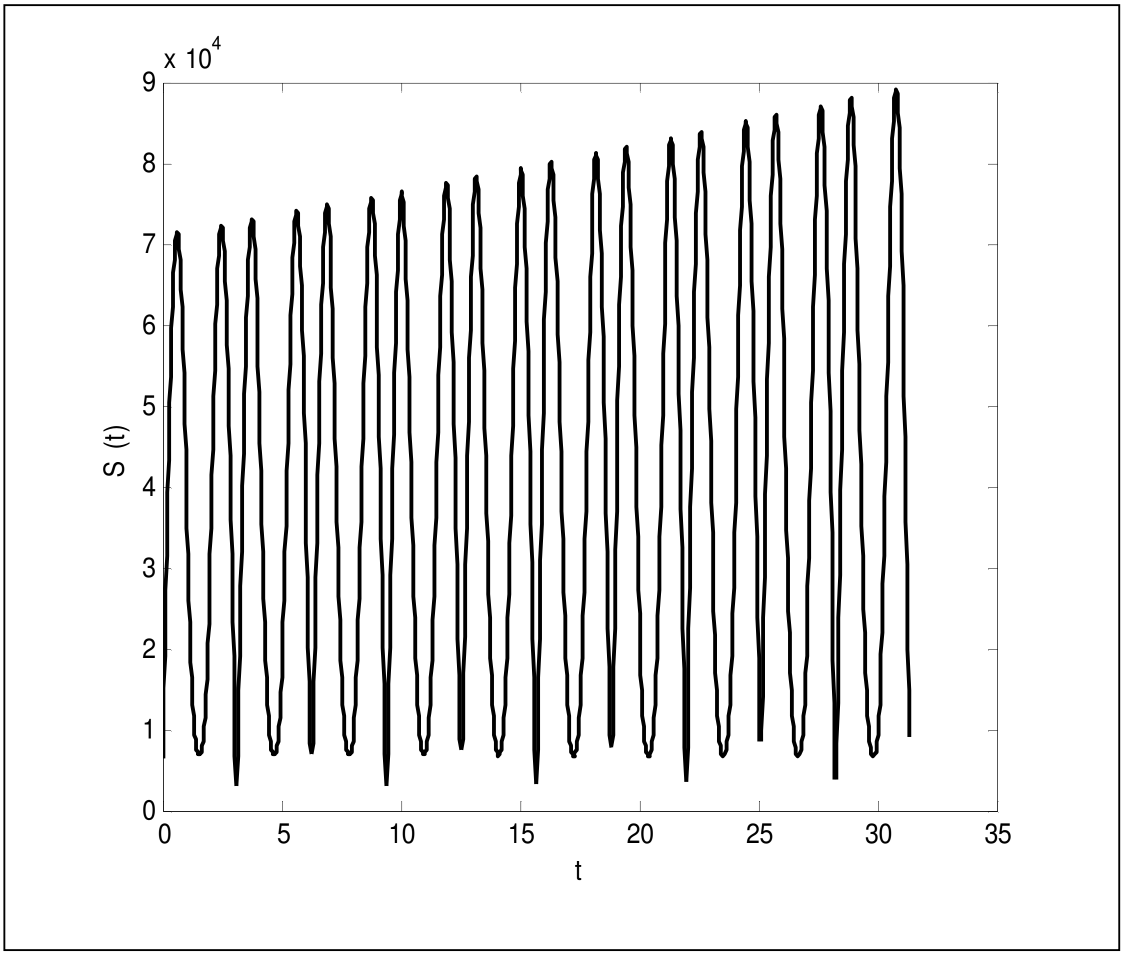

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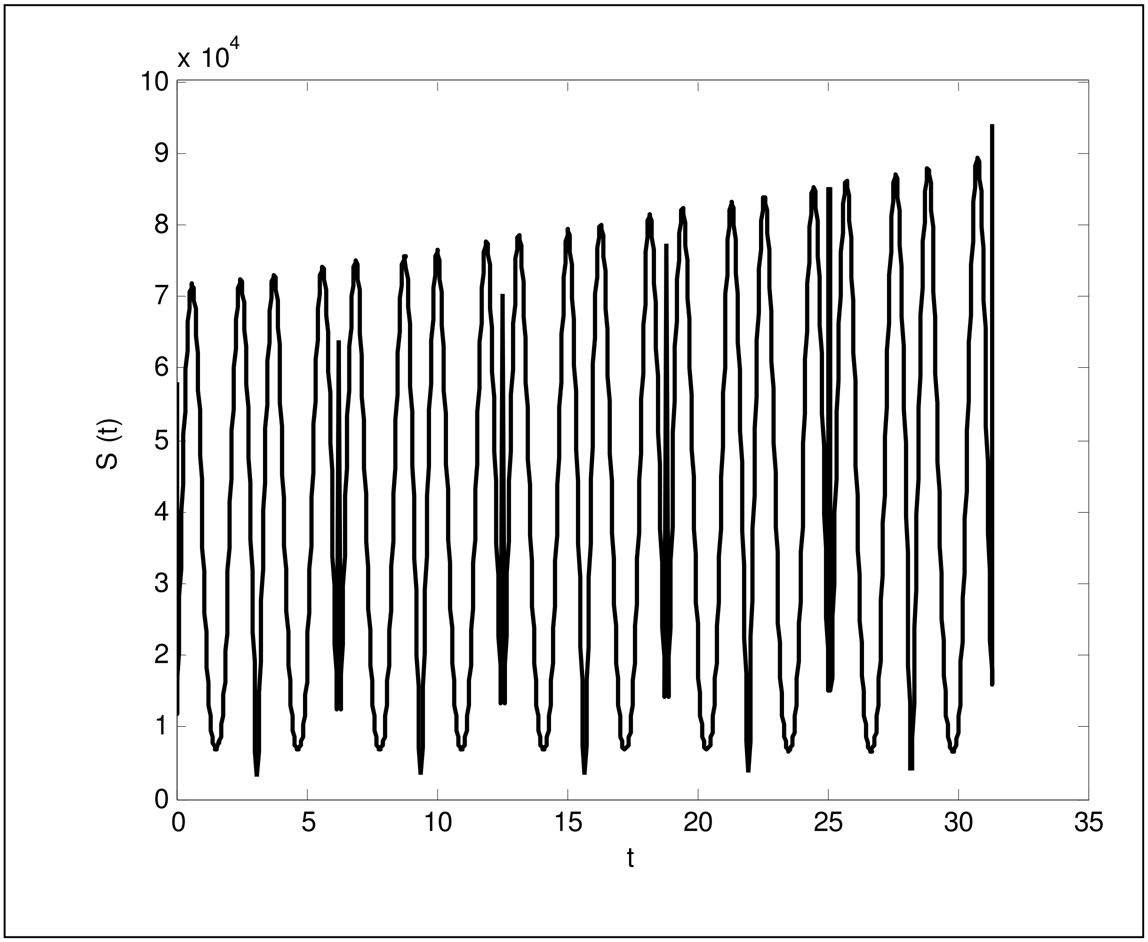

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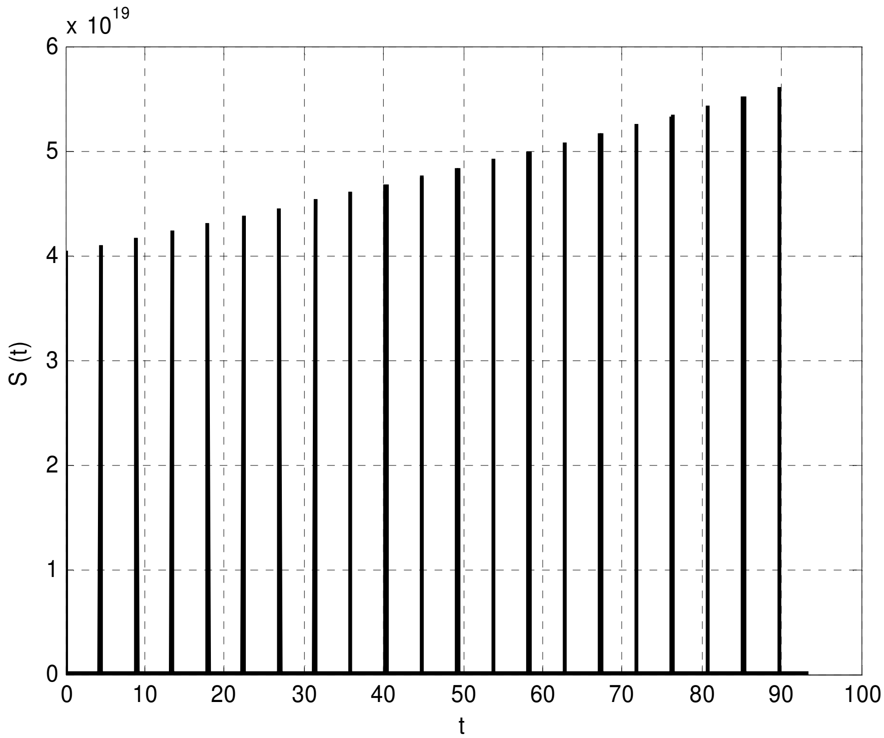

Fig .6: Pelap *et al*