Orthofermion Algebra and Fractional Supersymmetry I

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Abstract

Based on several previous works on fractional statistics using the symmetry $U_q(sl(2))$, we focus in this work to present some properties of the fractional supersymmetry by using the Orthofermion algebra.

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1. Introduction

In the last years, there has been some interest in studying 2d field theoretical models having fractional supersymmetries [1]. The latter are special subsymmetries of the infinite dimensional parafermionic invariance of 2d conformal coset models [2]. Furthermore, the fractional supersymmetries may also be viewed as finite dimensional global symmetries extending the usual 2d supersymmetric algebra

$$\left(Q_{-\frac{1}{2}}\right)^2 = P_{-1} \qquad ; \qquad \left(Q_{\frac{1}{2}}\right)^2 = P_1 \tag{1}$$

generated by the supersymmetric charges $Q_{\pm\frac{1}{2}}$ and the energy momentum vector $P_{\pm 1}$.

The Fractional supersymmetries are related to the dimensional periodic representation of $U_q(sl_2)$; $q^2 = 1$; for which the momentum vector $P_{\pm 1}$ is proportional to the centre of the group representation.
For the k-th root of unity; $q^k = 1$, equation (1) extend as

$$\left(Q_{-\frac{1}{k}}\right)^k = P_{-1} \qquad ; \qquad \left(Q_{\frac{1}{k}}\right)^k = P_1 \tag{2}$$

where $Q_{\pm\frac{1}{k}}$ are new charge operators carrying fractional spins.

Basing on the Zamalodchicov and Fateev (ZF) Z_3 parafermionic symmetry that we can schematize as

$$|0\rangle \xrightarrow{Q^+_{-\frac{2}{3}}} |1\rangle \xrightarrow{Q^+_0} |2\rangle \xrightarrow{Q^+_{-\frac{1}{3}}} |0\rangle$$
$$|0\rangle \xleftarrow{Q^-_{-\frac{1}{3}}} |1\rangle \xleftarrow{Q^-_0} |2\rangle \xleftarrow{Q^-_{-\frac{2}{3}}} |0\rangle$$

Scheme 1

and knowing that the operators charge is depending of the type of the state, the authors of [1] have postulated that the equation (2) must be rewriting as follow

$$\begin{aligned} P_{-1} = &\left(Q^-_{-\frac{1}{3}}Q^-_0Q^-_{-\frac{2}{3}} + Q^+_{-\frac{1}{3}}Q^+_0Q^+_{-\frac{2}{3}}\right)S_0 \\ &+\left(Q^-_0Q^-_{-\frac{2}{3}}Q^-_{-\frac{1}{3}} + Q^+_{-\frac{2}{3}}Q^+_{-\frac{1}{3}}Q^+_0\right)S_2 \\ &+\left(Q^-_{-\frac{2}{3}}Q^-_{-\frac{1}{3}}Q^-_0 + Q^+_0Q^+_{-\frac{2}{3}}Q^+_{-\frac{1}{3}}\right)S_{-2} \end{aligned} \tag{3}$$

where $Q^{\pm}_{-r}$; $r = 0,1,2$ are the charge operators and S_0 , S_2 , S_{-2} are projectors on the |0>, |1>, |2> states respectively.
In addition, the article suggested that the operators charge should check the following identities:

$$Q^{\pm}_{-\frac{1}{3}} = Q^{\mp}_{-\frac{1}{3}}Q^{\mp}_0 \qquad ; \qquad Q^{\mp}_{-\frac{2}{3}} = Q^{\pm}_0Q^{\pm}_{-\frac{2}{3}} \tag{4}$$

The first attempt to find a representation of fractional supersymmetric Z_3 was matrix representation in [4], but beyond of the Z_3 , the article didn't give a representation. The Wk algebra [4] has allowed the authors of [5] to find and generalise the equation (3) to Z_k representation where $k \geq 3$. But although this success, the representation was unable to find the equation (4).
The aim of this paper is to prove that the representations (3) and (4) can be found by using the orthofermion

algebra [6]. The content of this paper is as follow: In section 2, we introduce the Orthofermion Algebra. In section 3, we study the Representation of the Fractional supersymmetric Algebra for k = 3. In the section 4, we generalise the previous representation of the FSA for k > 3. A discussion and a conclusion are giving in section 5.

2. The Orthofermion Algebra

2.1. Orthofermion formulation

The statistics of orthofermions of order k is given by the following equations.

$$c_\alpha c_\beta^\dagger + \delta_{\alpha\beta} \sum_{\gamma=1}^{p} c_\gamma^\dagger c_\gamma = \delta_{\alpha\beta} I \tag{5}$$

$$c_\alpha c_\beta = 0 \tag{6}$$

Knowing that the operators charge is depending of the type of the state, it is possible to represent the fractional supersymmetric algebra (FSA) operators Q with Orthofermion algebra generators. For the

where c_α and $c_\alpha^\dagger$ are annihilation and creation operators respectively and I stands for the identity operator. The above algebra is a generalization of fermions in the sense that for *k = 1* we get the fermionic algebra

$$c_1 c_1^\dagger + c_1^\dagger c_1 = I \qquad ; \qquad c_1^2 = 0 \tag{7}$$

The representation of c_α is given by the *(k + 1) × (k + 1)* matrices

$$[c_\alpha]_{ij} = \delta_{i,1}\delta_{j,\alpha+1} \qquad ; \qquad \forall\, i, j \in \{1, 2, ..., k+1\} \tag{8}$$

Setting $\Pi = I - \sum_{\alpha=1}^{k} c_\alpha^\dagger c_\alpha$, we can write the equation (4) as

$$c_\alpha c_\beta^\dagger = \delta_{\alpha\beta} \Pi \tag{9}$$

It is not difficult to show that Π is a Hermitian projection operator

$$\Pi^2 = \Pi = \Pi^\dagger \tag{10}$$

It follows that for all *α ∈{1, 2, ..., k}*,

$$\Pi c_\alpha = c_\alpha \qquad ; \qquad c_\alpha^\dagger \Pi = c_\alpha^\dagger \tag{11}$$

$$c_\alpha \Pi = 0 \qquad ; \qquad \Pi c_\alpha^\dagger = 0 \tag{12}$$

2.2. Representation of the Orthofermion Algebra

Let *F* be the Fock space on which the generators of Orthofermion Algebra act

$$F = \{|0\rangle, |\alpha\rangle\} \qquad ; \qquad \alpha \in \{1,2,....,k\} \tag{13}$$

The action of projection operator and orthofermion generator on the vectors yields

$$\Pi|0\rangle = |0\rangle \qquad ; \qquad \Pi|\alpha\rangle = 0 \tag{14}$$

$$c_\alpha|0\rangle = 0 \qquad ; \qquad c_\beta|\alpha\rangle = \delta_{\beta\alpha}|0\rangle \tag{15}$$

$$c_\beta^\dagger|0\rangle = |\beta\rangle \qquad ; \qquad c_\beta^\dagger|\alpha\rangle = 0 \tag{16}$$

Therefore, *Π* is the projection onto the "vacuum" state vector $|0\rangle$.

3. Representation of the Fractional supersymmetric Algebra (FSA) (k = 3) :

3.1. Supercharges Representations

Knowing that the operators charge is depending of the type of the state, it is possible to represent the fractional supersymmetric algebra (FSA) operators Q with Orthofermion algebra generators. For the supercharge operators $Q_0^\pm, Q_{-\frac{1}{3}}^\pm$ and $Q_{-\frac{2}{3}}^\pm$ the representations are:

$$\begin{aligned} Q_{-\frac{2}{3}}^- &= c_2^\dagger \quad ; \quad Q_{-\frac{1}{3}}^- = c_1 \quad ; \quad Q_0^- = c_1^\dagger c_2 \\ Q_{-\frac{2}{3}}^+ &= c_1^\dagger \quad ; \quad Q_{-\frac{1}{3}}^+ = c_2 \quad ; \quad Q_0^+ = c_2^\dagger c_1 \end{aligned} \tag{17}$$

such as

$$Q^-_{-\frac{1}{3}}Q^-_0Q^-_{-\frac{2}{3}}|0\rangle = c_1c_1^\dagger c_2c_2^\dagger|0\rangle = |0\rangle$$
$$Q^-_0Q^-_{-\frac{2}{3}}Q^-_{-\frac{1}{3}}|1\rangle = c_1^\dagger c_2c_2^\dagger c_1|1\rangle = |1\rangle \qquad (18)$$
$$Q^-_{-\frac{2}{3}}Q^-_{-\frac{1}{3}}Q^-_0|2\rangle = c_2^\dagger c_1c_1^\dagger c_2|2\rangle = |2\rangle$$

In the same way

$$Q^+_{-\frac{1}{3}}Q^+_0Q^+_{-\frac{2}{3}}|0\rangle = c_2c_2^\dagger c_1c_1^\dagger|0\rangle = |0\rangle$$
$$Q^+_{-\frac{2}{3}}Q^+_{-\frac{1}{3}}Q^+_0|1\rangle = c_1^\dagger c_2c_2^\dagger c_1|1\rangle = |1\rangle \qquad (19)$$
$$Q^+_0Q^+_{-\frac{2}{3}}Q^+_{-\frac{1}{3}}|2\rangle = c_2^\dagger c_1c_1^\dagger c_2|2\rangle = |2\rangle$$

3.2. Remarks and Properties:

The representation of the fractional supersymmetry with the orthofermion algebra allows us to deduct some interesting properties, which are in agreement with the parafermionic algebra.

1. As in parafermionic case, the same operator cannot act twice on the same state:

$Q^-_{-\frac{2}{3}}$ and $Q^+_{-\frac{2}{3}}$ depend on $|0\rangle$ moreover $\left(Q^-_{-\frac{2}{3}}\right)^2 = 0$ and $\left(Q^+_{-\frac{2}{3}}\right)^2 = 0$

$Q^-_{-\frac{1}{3}}$ and Q^+_0 depend on $|1\rangle$ moreover $\left(Q^-_{-\frac{1}{3}}\right)^2 = 0$ and $\left(Q^+_0\right)^2 = 0$

Q^-_0 and $Q^+_{-\frac{1}{3}}$ depend on $|2\rangle$ moreover $\left(Q^-_0\right)^2 = 0$ and $\left(Q^+_{-\frac{1}{3}}\right)^2 = 0$

2. Following [3], every operator charge $Q^\pm_{-x}$ of spin (x) has an adjoint operator $Q^\mp_{-1+x}$ of spin $(-1 + x)$

$$\left(Q^\pm_{-\frac{2}{3}}\right)^\dagger = Q^\mp_{-\frac{1}{3}}$$
$$\left(Q^\pm_{-\frac{1}{3}}\right)^\dagger = Q^\mp_{-\frac{2}{3}} \qquad (20)$$
$$\left(Q^\pm_0\right)^\dagger = Q^\mp_{-1}$$

where $Q^\mp_{-1} = P_{-1}Q^\mp_0$.

3. Only the allowed combinations are no null:

$$Q^-_{-\frac{1}{3}}Q^-_0Q^-_{-\frac{2}{3}} = \Pi \quad ; \quad Q^-_0Q^-_{-\frac{2}{3}}Q^-_{-\frac{1}{3}} = c_1^\dagger c_1 \quad ; \quad Q^-_{-\frac{2}{3}}Q^-_{-\frac{1}{3}}Q^-_0 = c_2^\dagger c_2$$
$$Q^-_{-\frac{1}{3}}Q^-_{-\frac{2}{3}}Q^-_0 = 0 \quad ; \quad Q^-_0Q^-_{-\frac{1}{3}}Q^-_{-\frac{2}{3}} = 0 \quad ; \quad Q^-_{-\frac{2}{3}}Q^-_0Q^-_{-\frac{1}{3}} = 0 \qquad (21)$$

$$Q^+_{-\frac{1}{3}}Q^+_0Q^+_{-\frac{2}{3}} = \Pi \quad ; \quad Q^+_{-\frac{2}{3}}Q^+_{-\frac{1}{3}}Q^+_0 = c_1^\dagger c_1 \quad ; \quad Q^+_0Q^+_{-\frac{2}{3}}Q^+_{-\frac{1}{3}} = c_2^\dagger c_2$$
$$Q^+_{-\frac{1}{3}}Q^+_{-\frac{2}{3}}Q^+_0 = 0 \quad ; \quad Q^+_{-\frac{2}{3}}Q^+_0Q^+_{-\frac{1}{3}} = 0 \quad ; \quad Q^+_0Q^+_{-\frac{1}{3}}Q^+_{-\frac{2}{3}} = 0 \qquad (22)$$

4. If we put

$$Q^- = Q^-_{-\frac{1}{3}} + Q^-_0 + Q^-_{-\frac{2}{3}}$$
$$Q^+ = Q^+_{-\frac{1}{3}} + Q^+_{-\frac{2}{3}} + Q^+_0 \qquad (23)$$

And

$$Q'^- = Q^-_{-\frac{1}{3}} + Q^-_{-1} + Q^-_{-\frac{2}{3}}$$
$$Q'^+ = Q^+_{-\frac{1}{3}} + Q^+_{-\frac{2}{3}} + Q^+_{-1} \qquad (24)$$

We will find that

- $$\left(Q^\pm\right)^\dagger = Q'^\mp \qquad (25)$$

- $$\left(Q^-\right)^3 = P^-_{-1} = Q^-_{-\frac{1}{3}}Q^-_0Q^-_{-\frac{2}{3}} + Q^-_0Q^-_{-\frac{2}{3}}Q^-_{-\frac{1}{3}} + Q^-_{-\frac{2}{3}}Q^-_{-\frac{1}{3}}Q^-_0 = \Pi + c_1^\dagger c_1 + c_2^\dagger c_2 \tag{26}$$

$$\left(Q^+\right)^3 = P^+_{-1} = Q^+_{-\frac{1}{3}}Q^+_0Q^+_{-\frac{2}{3}} + Q^+_{-\frac{2}{3}}Q^+_{-\frac{1}{3}}Q^+_0 + Q^+_0Q^+_{-\frac{2}{3}}Q^+_{-\frac{1}{3}} = \Pi + c_1^\dagger c_1 + c_2^\dagger c_2 \tag{27}$$

and

- $$\left(Q'^-\right)^3 = \left(P^-_{-1}\right)^2 = Q^-_{-\frac{1}{3}}Q^-_{-1}Q^-_{-\frac{2}{3}} + Q^-_{-1}Q^-_{-\frac{2}{3}}Q^-_{-\frac{1}{3}} + Q^-_{-\frac{2}{3}}Q^-_{-\frac{1}{3}}Q^-_{-1} = \left(\Pi + c_1^\dagger c_1 + c_2^\dagger c_2\right)^2 \tag{28}$$

$$\left(Q'^+\right)^3 = \left(P^+_{-1}\right)^2 = Q^+_{-\frac{1}{3}}Q^+_{-1}Q^+_{-\frac{2}{3}} + Q^+_{-\frac{2}{3}}Q^+_{-\frac{1}{3}}Q^+_{-1} + Q^+_{-1}Q^+_{-\frac{2}{3}}Q^+_{-\frac{1}{3}} = \left(\Pi + c_1^\dagger c_1 + c_2^\dagger c_2\right)^2 \tag{29}$$

Both P^+_{-1} and P^+_{-1} verifies:

$$P^-_{-1} = P^+_{-1} \tag{30}$$

5. The Q^- and Q^+ components check the identity of Jacobi

$$\left[\left[Q^-_{-\frac{1}{3}}, Q^-_0\right], Q^-_{-\frac{2}{3}}\right] + \left[\left[Q^-_0, Q^-_{-\frac{2}{3}}\right], Q^-_{-\frac{1}{3}}\right] + \left[\left[Q^-_{-\frac{2}{3}}, Q^-_{-\frac{1}{3}}\right], Q^-_0\right] = 0$$
$$\left[\left[Q^+_{-\frac{1}{3}}, Q^+_0\right], Q^+_{-\frac{2}{3}}\right] + \left[\left[Q^+_{-\frac{2}{3}}, Q^+_{-\frac{1}{3}}\right], Q^+_0\right] + \left[\left[Q^+_0, Q^+_{-\frac{2}{3}}\right], Q^+_{-\frac{1}{3}}\right] = 0 \tag{31}$$

6. If we use eqs (17), we will find the following relations of commutations:

$$\left[Q^-_{-\frac{2}{3}}, Q^-_{-\frac{1}{3}}\right] = Q^+_0 \qquad \left[Q^-_{-\frac{1}{3}}, Q^-_{-\frac{1}{3}}\right] = 0 \qquad \left[Q^-_0, Q^-_{-\frac{1}{3}}\right] = -Q^+_{-\frac{1}{3}}$$
$$\left[Q^-_0, Q^-_0\right] = 0$$
$$\left[Q^-_{-\frac{2}{3}}, Q^-_0\right] = -Q^+_{-\frac{2}{3}} \qquad \left[Q^-_{-\frac{1}{3}}, Q^-_0\right] = Q^+_{-\frac{1}{3}} \qquad \left[Q^-_0, Q^-_{-\frac{2}{3}}\right] = Q^+_{-\frac{2}{3}}$$
$$\left[Q^-_{-\frac{2}{3}}, Q^-_{-\frac{2}{3}}\right] = 0 \qquad \left[Q^-_{-\frac{1}{3}}, Q^-_{-\frac{2}{3}}\right] = -Q^+_0$$
$$\left[Q^+_{-\frac{2}{3}}, Q^+_{-\frac{1}{3}}\right] = Q^-_0 \qquad \left[Q^+_{-\frac{1}{3}}, Q^+_{-\frac{1}{3}}\right] = 0 \qquad \left[Q^+_0, Q^+_{-\frac{1}{3}}\right] = -Q^-_{-\frac{1}{3}}$$
$$\left[Q^+_0, Q^+_0\right] = 0$$
$$\left[Q^+_{-\frac{2}{3}}, Q^+_0\right] = -Q^-_{-\frac{2}{3}} \qquad \left[Q^+_{-\frac{1}{3}}, Q^+_0\right] = Q^-_{-\frac{1}{3}} \qquad \left[Q^+_0, Q^+_{-\frac{2}{3}}\right] = Q^-_{-\frac{2}{3}} \tag{33}$$
$$\left[Q^+_{-\frac{1}{3}}, Q^+_0\right] = Q^-_{-\frac{1}{3}} \qquad \left[Q^+_{-\frac{1}{3}}, Q^+_{-\frac{2}{3}}\right] = -Q^-_0$$

$$\left[Q^-_{-\frac{2}{3}}, Q^+_{-\frac{2}{3}}\right] = 0 \qquad \left[Q^-_{-\frac{1}{3}}, Q^+_{-\frac{2}{3}}\right] = Q^{-\frac{2}{3}}_{-\frac{1}{3}} = c_1c_1^\dagger - c_1^\dagger c_1 \qquad \left[Q^-_0, Q^+_{-\frac{2}{3}}\right] = 0$$
$$\left[Q^-_{-\frac{2}{3}}, Q^+_{-\frac{1}{3}}\right] = Q^{-\frac{1}{3}}_{-\frac{2}{3}} = c_2^\dagger c_2 - c_2c_2^\dagger \qquad \left[Q^-_{-\frac{1}{3}}, Q^+_{-\frac{1}{3}}\right] = 0 \qquad \left[Q^-_0, Q^+_{-\frac{1}{3}}\right] = 0$$
$$\left[Q^-_{-\frac{2}{3}}, Q^+_0\right] = 0 \qquad \left[Q^-_{-\frac{1}{3}}, Q^+_0\right] = 0 \qquad \left[Q^-_0, Q^+_0\right] = Q^0_0 = c_1^\dagger c_1 - c_2^\dagger c_2$$

which implies

$$\left[Q^-_{-\frac{2}{3}}, Q^+_{-\frac{1}{3}}\right] + \left[Q^-_{-\frac{1}{3}}, Q^+_{-\frac{2}{3}}\right] + \left[Q^-_0, Q^+_0\right] = \left[Q^-, Q^+\right] = c_1 c_1^\dagger - c_2 c_2^\dagger = \Pi - \Pi = 0$$

Denoting that the Q_0^0, $Q^{-\frac{1}{3}}_{-\frac{2}{3}}$ and $Q^{-\frac{2}{3}}_{-\frac{1}{3}}$ are acting on the $Q^{\pm}_{-x}$, $x = 0, \frac{1}{3}, \frac{2}{3}$ charges as:

$$\begin{array}{lll}
\left[Q^{-\frac{1}{3}}_{-\frac{2}{3}}, Q^-_{-\frac{2}{3}}\right] = +\mathbf{2}Q^-_{-\frac{2}{3}} & \left[Q^{-\frac{2}{3}}_{-\frac{1}{3}}, Q^-_{-\frac{1}{3}}\right] = +\mathbf{2}Q^-_{-\frac{1}{3}} & \left[Q^0_0, Q^-_0\right] = +\mathbf{2}Q^-_0 \\
 & & \left[Q^0_0, Q^+_0\right] = -\mathbf{2}Q^+_0 \\
\left[Q^{-\frac{1}{3}}_{-\frac{2}{3}}, Q^+_{-\frac{1}{3}}\right] = -\mathbf{2}Q^+_{-\frac{1}{3}} & \left[Q^{-\frac{2}{3}}_{-\frac{1}{3}}, Q^+_{-\frac{2}{3}}\right] = -\mathbf{2}Q^+_{-\frac{2}{3}} & \left[Q^0_0, Q^-_{-\frac{2}{3}}\right] = -Q^-_{-\frac{2}{3}} \\
\left[Q^{-\frac{1}{3}}_{-\frac{2}{3}}, Q^-_{-\frac{1}{3}}\right] = -Q^-_{-\frac{1}{3}} & \left[Q^{-\frac{2}{3}}_{-\frac{1}{3}}, Q^-_{-\frac{2}{3}}\right] = -Q^-_{-\frac{2}{3}} & \left[Q^0_0, Q^+_{-\frac{1}{3}}\right] = +Q^+_{-\frac{1}{3}} \\
\left[Q^{-\frac{1}{3}}_{-\frac{2}{3}}, Q^+_{-\frac{2}{3}}\right] = +Q^+_{-\frac{2}{3}} & \left[Q^{-\frac{2}{3}}_{-\frac{1}{3}}, Q^+_{-\frac{1}{3}}\right] = +Q^+_{-\frac{1}{3}} & \left[Q^0_0, Q^-_{-\frac{1}{3}}\right] = -Q^-_{-\frac{1}{3}} \\
\left[Q^{-\frac{1}{3}}_{-\frac{2}{3}}, Q^-_0\right] = -Q^-_0 & \left[Q^{-\frac{2}{3}}_{-\frac{1}{3}}, Q^-_0\right] = -Q^-_0 & \left[Q^0_0, Q^+_{-\frac{2}{3}}\right] = +Q^+_{-\frac{2}{3}} \\
\left[Q^{-\frac{1}{3}}_{-\frac{2}{3}}, Q^+_0\right] = +Q^+_0 & \left[Q^{-\frac{2}{3}}_{-\frac{1}{3}}, Q^+_0\right] = +Q^+_0 &
\end{array} \qquad (34)$$

7. The charge operators in this representation satisfy the equalities (4)

$$\begin{aligned}
Q^{\mp}_{-\frac{1}{3}} &= Q^{\pm}_{-\frac{1}{3}} Q^{\pm}_0 \\
Q^{\mp}_{-\frac{2}{3}} &= Q^{\pm}_0 Q^{\pm}_{-\frac{2}{3}} \qquad (35) \\
Q^{\mp}_{-\frac{1}{3}} &= Q^{\pm}_{-\frac{1}{3}} Q^{\pm}_{-\frac{1}{3}}
\end{aligned}$$

8. From equation (35), we deduce that:

$$\left(Q^+\right)^2 = Q'^- \quad \text{and} \quad \left(Q^-\right)^2 = Q'^+ \qquad (35)$$

3.3. The Hamiltonian of the system

Like in [5], the expression of the Hamiltonian in this representation is:

$$2H = P^-_{-1} + P^+_{-1} = \left(Q^-\right)^3 + \left(Q^+\right)^3 \qquad (37)$$

$$= Q^-_{-\frac{1}{3}} Q^-_0 Q^-_{-\frac{2}{3}} + Q^-_0 Q^-_{-\frac{2}{3}} Q^-_{-\frac{1}{3}} + Q^-_{-\frac{2}{3}} Q^-_{-\frac{1}{3}} Q^-_0 \qquad (38)$$

$$+ Q^+_{-\frac{1}{3}} Q^+_0 Q^+_{-\frac{2}{3}} + Q^+_0 Q^+_{-\frac{2}{3}} Q^+_{-\frac{1}{3}} + Q^+_{-\frac{2}{3}} Q^+_{-\frac{1}{3}} Q^+_0$$

Using (35) and (36), the new expressions of the Hamiltonian will be:

$$2H = \left(Q^-_{-\frac{1}{3}} Q^+_{-\frac{2}{3}} + Q^+_{-\frac{1}{3}} Q^-_{-\frac{2}{3}}\right) + \left(Q^-_{-\frac{2}{3}} Q^+_{-\frac{1}{3}} + Q^+_{-\frac{2}{3}} Q^-_{-\frac{1}{3}}\right)$$

$$+ \left(Q^-_0 Q^+_{-1} + Q^+_0 Q^-_{-1}\right) \qquad (39)$$

$$= Q^- Q'^+ + Q'^+ Q^- = Q^+ Q'^- + Q'^- Q^+ \qquad (40)$$

The Hamiltonian H is hermitian and verify the following relations of commutation with the charge operators Q^- and Q^+ :

$$\left[Q^{\pm}, H\right] = \left[Q'^{\pm}, H\right] = 0 \tag{41}$$

Furthermore, the Hamiltonian H can be decomposed on three hermitian terms

$$2H = 2H_0 + 2H_1 + 2H_2 \tag{42}$$

where

$$\begin{aligned} 2H_0 &= Q^-_{-\frac{1}{3}} Q^+_{-\frac{2}{3}} + Q^+_{-\frac{1}{3}} Q^-_{-\frac{2}{3}} \\ 2H_1 &= Q^-_{-1} Q^+_0 + Q^+_{-\frac{2}{3}} Q^-_{-\frac{1}{3}} \\ 2H_2 &= Q^-_0 Q^+_{-1} + Q^-_{-\frac{2}{3}} Q^+_{-\frac{1}{3}} \end{aligned} \tag{43}$$

these hermetian terms verifies

$$\begin{aligned} &\left[H_0, Q^{\pm}_{-\frac{1}{3}}\right] = \left[H_0, Q^{\pm}_{-\frac{2}{3}}\right] = 0 \\ &\left[H_1, Q^-_{-\frac{1}{3}}\right] = \left[H_1, Q^+_{-\frac{2}{3}}\right] = 0 \\ &\left[H_1, Q^-_{-1}\right] = \left[H_1, Q^+_0\right] = 0 \\ &\left[H_2, Q^-_{-\frac{2}{3}}\right] = \left[H_2, Q^+_{-\frac{1}{3}}\right] = 0 \\ &\left[H_2, Q^-_0\right] = \left[H_2, Q^+_{-1}\right] = 0 \end{aligned} \tag{44}$$

3.4. The automorphism groups of the D = 2(1/3, 0) Model

This FSA is invariant under two kinds of discrete symmetries:

First the Z_3 symmetry operating as:

$$\begin{aligned} &Q^+ \to qQ^+ \quad ; \quad P^+_{-1} \to q^3 P^+_{-1} = P^+_{-1} \\ &Q^- \to \bar{q}Q^- \quad ; \quad P^-_{-1} \to \bar{q}^3 P^-_{-1} = P^-_{-1} \end{aligned} \tag{45}$$

where $q^3 = \bar{q}^3 = 1$.

Second, the Z_2 symmetry, generated by the charge conjugation operators C_0, C_1 and C_2 acting on the $Q^{\pm}$ components and $P^{\pm}$ as:

$$\begin{aligned} C_0 Q^-_{-\frac{1}{3}} &= Q^+_{-\frac{2}{3}} C_0 \\ C_1 Q^-_0 &= Q^+_0 C_1 \\ C_2 Q^-_{-\frac{2}{3}} &= Q^+_{-\frac{1}{3}} C_2 \end{aligned} \tag{46}$$

and

$$\begin{aligned} &C_0 P^-_{-1} = P^-_{-1} C_0 \quad ; \quad C_0 P^+_{-1} = P^+_{-1} C_0 \\ &C_1 P^-_{-1} = P^-_{-1} C_1 \quad ; \quad C_1 P^+_{-1} = P^+_{-1} C_1 \\ &C_2 P^-_{-1} = P^-_{-1} C_2 \quad ; \quad C_2 P^+_{-1} = P^+_{-1} C_2 \end{aligned} \tag{47}$$

where

$$
\begin{aligned}
C_0 &= Q^-_{-\frac{1}{3}} + Q^+_{-\frac{2}{3}} \\
C_1 &= Q^-_0 + Q^+_0 \\
C_2 &= Q^-_{-\frac{2}{3}} + Q^+_{-\frac{1}{3}}
\end{aligned}
\tag{48}
$$

To enclose this paragraph, the knowledge of these charge conjugation operators allows us to give the new expression of the hamiltonien:

$$2H = \sum_{i=j=k=1; i\neq j\neq k}^{3} C_i C_j C_k \tag{49}$$

4. The Fractional supersymmetric Algebra (FSA) (k ≥ 3) :

4.1. Representation of FSA components

Before starting this section, two schemes will be introduced. The first one will be for odd number *k* (Scheme 2) and the second one will be for even *k* (Scheme 3)

$$|0\rangle \xrightarrow{Q^+_{-\frac{k-1}{k}}} |1\rangle \xrightarrow{Q^+_{-\frac{k-3}{k}}} |2\rangle \xrightarrow{Q^+_{-\frac{k-5}{k}}} |3\rangle \ldots \xrightarrow{Q^+_{-\frac{2}{k}}} \left|\frac{k-1}{2}\right\rangle \xrightarrow{Q^+_0} \left|\frac{k+1}{2}\right\rangle \xrightarrow{Q^+_{\frac{2}{k}}} \ldots \xrightarrow{Q^+_{\frac{k-3}{k}}} \left|\frac{2k-2}{2}\right\rangle \xrightarrow{Q^+_{-\frac{1}{k}}} |0\rangle$$

$$|0\rangle \xleftarrow{Q^-_{-\frac{1}{k}}} |1\rangle \xleftarrow{Q^-_{\frac{k-3}{k}}} |2\rangle \xleftarrow{Q^-_{\frac{k-5}{k}}} |3\rangle \ldots \xleftarrow{Q^-_{\frac{2}{k}}} \left|\frac{k-1}{2}\right\rangle \xleftarrow{Q^-_0} \left|\frac{k+1}{2}\right\rangle \xleftarrow{Q^-_{-\frac{2}{k}}} \ldots \xleftarrow{Q^-_{-\frac{k-3}{k}}} \left|\frac{2k-2}{2}\right\rangle \xleftarrow{Q^-_{-\frac{k-1}{k}}} |0\rangle$$

Scheme 2

$$|0\rangle \xrightarrow{Q^+_{-\frac{k-1}{k}}} |1\rangle \xrightarrow{Q^+_{-\frac{k-3}{k}}} |2\rangle \xrightarrow{Q^+_{-\frac{k-5}{k}}} |3\rangle \ldots \left|\frac{k-1}{2}\right\rangle \xrightarrow{Q^+_{-\frac{1}{k}}} \left|\frac{k}{2}\right\rangle \xrightarrow{Q^+_{\frac{2}{k}}} \left|\frac{k+2}{2}\right\rangle \ldots \xrightarrow{Q^+_{\frac{k-2}{k}}} \left|\frac{2k-2}{2}\right\rangle \xrightarrow{Q^+_{-\frac{1}{k}}} |0\rangle$$

$$|0\rangle \xleftarrow{Q^-_{-\frac{1}{k}}} |1\rangle \xleftarrow{Q^-_{\frac{k-3}{k}}} |2\rangle \xleftarrow{Q^-_{\frac{k-5}{k}}} |3\rangle \ldots \left|\frac{k-1}{2}\right\rangle \xleftarrow{Q^-_{\frac{1}{k}}} \left|\frac{k+1}{2}\right\rangle \xleftarrow{Q^-_{-\frac{2}{k}}} \ldots \xleftarrow{Q^-_{-\frac{k-3}{k}}} \left|\frac{2k-2}{2}\right\rangle \xleftarrow{Q^-_{-\frac{k-1}{k}}} |0\rangle$$

Scheme 3

where $Q^{\pm}$ are the charge operators components. From scheme (2) and (3), the representation of the components of operators charge Q^+ and Q^- of the FSA in orthofermion algebra are

$$
\begin{aligned}
&Q^-_{\frac{1-2n+k}{k}} = c^\dagger_{n-1} c_n \quad ; \quad 2 \leq n \leq k-1 \\
&Q^+_{\frac{1+2n-k}{k}} = c^\dagger_{n+1} c_n \quad ; \quad 1 \leq n \leq k-2 \\
&Q^-_{-\frac{1}{k}} = c_1 \quad ; \quad Q^-_{-\frac{k-1}{k}} = c^\dagger_{k-1} \\
&Q^+_{-\frac{1}{k}} = c_{k-1} \quad ; \quad Q^+_{-\frac{k-1}{k}} = c^\dagger_1
\end{aligned}
\tag{50}
$$

so, we can define the charge operators

$$
\begin{aligned}
Q^- &= \sum_{n=1}^{k-2} Q^-_{\frac{1-2n+k}{k}} + Q^-_{-\frac{1}{k}} + Q^-_{-\frac{k-1}{k}} \\
Q^+ &= \sum_{n=2}^{k-1} Q^+_{\frac{1+2n-k}{k}} + Q^+_{-\frac{1}{k}} + Q^+_{-\frac{k-1}{k}}
\end{aligned}
\tag{51}
$$

From these representations, we can see that every component may act only on one state. This particularity was in accordance with the sectors particularity in parafermionic algebra.

4.2. Remarks and Properties:

Like in previous section, the representation of the FSA in orthofermion algebra will allow us to deduce some interesting properties.

1. $\forall Q^-_{\frac{1-2n+k}{k}}$ and $\forall Q^+_{\frac{1+2n-k}{k}}$; $\left(Q^-_{\frac{1-2n+k}{k}}\right)^2 = \left(Q^+_{\frac{1+2n-k}{k}}\right)^2 = 0$

2. Every operator charge has his adjoint operator

$$\left(Q^{-}_{\frac{1-2n+k}{k}}\right)^{\dagger}=Q^{+}_{-1-\frac{1-2n+k}{k}}=P_{-1}Q^{+}_{-\frac{1-2n+k}{k}} \qquad ; \qquad n\in\{2,....,k-1\}$$

$$\left(Q^{+}_{\frac{1+2n-k}{k}}\right)^{\dagger}=Q^{-}_{-1-\frac{1+2n-k}{k}}=P_{-1}Q^{+}_{-\frac{1+2n-k}{k}} \qquad ; \qquad n\in\{1,....,k-2\} \tag{52}$$

$$\left(Q^{\pm}_{-\frac{1}{k}}\right)^{\dagger}=Q^{\mp}_{-\frac{k-1}{k}}$$

3. Knowing that $Q^{-}_{\frac{1-2n+k}{k}}=Q^{-}_{\frac{1-2(n\pm k)+k}{k}}$ and $Q^{+}_{\frac{1+2n-k}{k}}=Q^{-}_{\frac{1+2(n\pm k)-k}{k}}$, only these combinations are no null:

$$Q^{+}_{\frac{1+2(n-1)-k}{k}}\dots Q^{+}_{-\frac{k-1}{k}}Q^{+}_{-\frac{1}{k}}\dots Q^{+}_{\frac{1+2(n+1)-k}{k}}Q^{+}_{\frac{1+2n-k}{k}}=c_n^{\dagger}c_n \qquad ; \qquad \text{if } 1\le n\le k-2$$

$$Q^{+}_{-\frac{k-3}{k}}\dots Q^{+}_{-\frac{k-1}{k}}Q^{+}_{-\frac{1}{k}}.=c_{k-1}^{\dagger}c_{k-1} \tag{53}$$

$$Q^{+}_{-\frac{1}{k}}Q^{+}_{-\frac{k-3}{k}}\dots Q^{+}_{-\frac{k-1}{k}}.=\Pi$$

$$Q^{-}_{\frac{1-2(n+1)+k}{k}}\dots Q^{-}_{-\frac{k-1}{k}}Q^{-}_{-\frac{1}{k}}\dots Q^{-}_{\frac{1-2(n+1)+k}{k}}Q^{-}_{\frac{1-2n+k}{k}}=c_n^{\dagger}c_n \qquad ; \qquad \text{if } 2\le n\le k-1$$

$$Q^{-}_{\frac{k-3}{k}}\dots Q^{-}_{-\frac{k-1}{k}}Q^{-}_{-\frac{1}{k}}.=c_1^{\dagger}c_1 \tag{54}$$

$$Q^{-}_{-\frac{1}{k}}Q^{-}_{-\frac{k-3}{k}}\dots Q^{+}_{-\frac{k-1}{k}}.=\Pi$$

4. Relations of commutations

$$\left[Q^{-}_{\frac{1-2n+k}{k}},Q^{+}_{\frac{1+2m-k}{k}}\right]=Q^{\frac{1+2m-k}{k}}_{\frac{1-2n+k}{k}}=\begin{cases}c_{n-1}^{\dagger}c_{n-1}-c_n^{\dagger}c_n & \text{if } m=n-1 \text{ and } 2\le n\le k-1\\ 0 & \text{if } m\ne n-1 \text{ and } 2\le n\le k-1\end{cases}$$

$$\left[Q^{-}_{-\frac{1}{k}},Q^{+}_{-\frac{k-1}{k}}\right]=Q^{-\frac{k-1}{k}}_{-\frac{1}{k}}=c_1^{\dagger}c_1-c_1c_1^{\dagger} \tag{55}$$

$$\left[Q^{-}_{-\frac{k-1}{k}},Q^{+}_{-\frac{1}{k}}\right]=Q^{-\frac{1}{k}}_{-\frac{k-1}{k}}=c_{k-1}^{\dagger}c_{k-1}-c_{k-1}c_{k-1}^{\dagger}$$

which implies that

$$\sum_{n=2}^{k-1}\left[Q^{-}_{\frac{1-2n+k}{k}},Q^{+}_{\frac{1+2(n-1)-k}{k}}\right]+\left[Q^{-}_{-\frac{1}{k}},Q^{+}_{-\frac{k-1}{k}}\right]+\left[Q^{-}_{-\frac{k-1}{k}},Q^{+}_{-\frac{1}{k}}\right]=\left[Q^{-},Q^{+}\right]$$

de plus

$$\left[Q^{\frac{1+2(n-1)-k}{k}}_{\frac{1-2n+k}{k}},Q^{-}_{\frac{1-2m+k}{k}}\right]=\begin{cases}2Q^{-}_{\frac{1-2n+k}{k}} & \text{if } m=n \text{ and } 2<n<k-1\\ -Q^{-}_{\frac{1-2(n-1)+k}{k}} & \text{if } m=n-1 \text{ and } 2<n\le k-1\\ -Q^{-}_{\frac{1-2(n+1)+k}{k}} & \text{if } m=n+1 \text{ and } 2<n\le k-1\\ 0 & \text{if } m\ne\{n-1,n,n+1\}\end{cases}$$

$$\left[Q^{\frac{1+2(n-1)-k}{k}}_{\frac{1-2n+k}{k}},Q^{+}_{\frac{1+2m-k}{k}}\right]=\begin{cases}Q^{+}_{\frac{1+2n-k}{k}} & \text{if } m=n \text{ and } 2<n<k-1\\ -2Q^{+}_{\frac{1+2(n-1)-k}{k}} & \text{if } m=n-1 \text{ and } 2<n\le k-1\\ Q^{+}_{\frac{1-2(n-2)-k}{k}} & \text{if } m=n-2 \text{ and } 2<n\le k-1\\ 0 & \text{if } m\ne\{n-2,n-1,n\}\end{cases}$$

$$\left[Q^{-\frac{k-1}{k}}_{-\frac{1}{k}}, Q_x^-\right] = \begin{cases} -2Q^-_{-\frac{1}{k}} \text{ if } x = -\frac{1}{k} \\ Q^-_{-\frac{k-3}{k}} \text{ if } x = -\frac{k-3}{k} \\ Q^-_{-\frac{k-1}{k}} \text{ if } x = -\frac{k-1}{k} \\ 0 \text{ if } m \neq \left\{-\frac{1}{k}, -\frac{k-1}{k}, -\frac{k-3}{k}\right\} \end{cases}$$

$$\left[Q^{-\frac{k-1}{k}}_{-\frac{1}{k}}, Q_x^+\right] = \begin{cases} -Q^+_{-\frac{1}{k}} \text{ if } x = -\frac{1}{k} \\ -Q^+_{-\frac{k-3}{k}} \text{ if } x = -\frac{k-3}{k} \\ 2Q^+_{-\frac{k-1}{k}} \text{ if } x = -\frac{k-1}{k} \\ 0 \text{ if } m \neq \left\{-\frac{1}{k}, -\frac{k-1}{k}, -\frac{k-3}{k}\right\} \end{cases}$$

$$\left[Q^{-\frac{1}{k}}_{-\frac{k-1}{k}}, Q_x^-\right] = \begin{cases} -Q^-_{-\frac{1}{k}} \text{ if } x = -\frac{1}{k} \\ -Q^-_{-\frac{k-3}{k}} \text{ if } x = -\frac{k-3}{k} \\ 2Q^-_{-\frac{k-1}{k}} \text{ if } x = -\frac{k-1}{k} \\ 0 \text{ if } m \neq \left\{-\frac{1}{k}, -\frac{k-1}{k}, -\frac{k-3}{k}\right\} \end{cases}$$

$$\left[Q^{-\frac{1}{k}}_{-\frac{k-1}{k}}, Q_x^+\right] = \begin{cases} -2Q^+_{-\frac{1}{k}} \text{ if } x = -\frac{1}{k} \\ Q^+_{-\frac{k-3}{k}} \text{ if } x = -\frac{k-3}{k} \\ Q^+_{-\frac{k-1}{k}} \text{ if } x = -\frac{k-1}{k} \\ 0 \text{ if } m \neq \left\{-\frac{1}{k}, -\frac{k-1}{k}, -\frac{k-3}{k}\right\} \end{cases}$$

$$\left[Q^{\frac{k-3}{k}}_{-\frac{k-3}{k}}, Q_x^-\right] = \begin{cases} -Q^-_{-\frac{1}{k}} \text{ if } x = -\frac{1}{k} \\ -2Q^-_{\frac{k-3}{k}} \text{ if } x = \frac{k-3}{k} \\ 2Q^-_{\frac{k-5}{k}} \text{ if } x = \frac{k-5}{k} \\ 0 \text{ if } m \neq \left\{-\frac{1}{k}, \frac{k-3}{k}, \frac{k-5}{k}\right\} \end{cases}$$

$$\left[Q^{\frac{k-3}{k}}_{-\frac{k-3}{k}}, Q_x^+\right] = \begin{cases} -Q^+_{-\frac{k-1}{k}} \text{ if } x = -\frac{k-1}{k} \\ 2Q^+_{-\frac{k-3}{k}} \text{ if } x = -\frac{k-3}{k} \\ -Q^+_{-\frac{k-5}{k}} \text{ if } x = -\frac{k-5}{k} \\ 0 \text{ if } m \neq \left\{-\frac{k-1}{k}, -\frac{k-3}{k}, -\frac{k-5}{k}\right\} \end{cases}$$

$$\left[Q^{-\frac{k-3}{k}}_{\frac{k-3}{k}}, Q_x^-\right] = \begin{cases} -Q^-_{-\frac{1}{k}} \text{ if } x = -\frac{1}{k} \\ -2Q^-_{\frac{k-3}{k}} \text{ if } x = \frac{k-3}{k} \\ 2Q^-_{\frac{k-5}{k}} \text{ if } x = \frac{k-5}{k} \\ 0 \text{ if } m \neq \left\{-\frac{1}{k}, \frac{k-3}{k}, \frac{k-5}{k}\right\} \end{cases}$$

$$\left[Q^{-\frac{k-3}{k}}_{\frac{k-3}{k}}, Q_x^+\right] = \begin{cases} -Q^+_{-\frac{k-1}{k}} \text{ if } x = -\frac{1}{k} \\ 2Q^+_{\frac{k-3}{k}} \text{ if } x = \frac{k-3}{k} \\ -Q^+_{\frac{k-5}{k}} \text{ if } x = \frac{k-5}{k} \\ 0 \text{ if } m \neq \left\{-\frac{k-1}{k}, \frac{k-3}{k}, -\frac{k-5}{k}\right\} \end{cases}$$

5. The expressions of P_{-1}^+ and P_{-1}^- are:

$$\begin{aligned} P_{-1}^- = \left(Q^-\right)^k &= \sum_{n=2}^{k-1} Q^-_{\frac{1-2(n+1)+k}{k}} \cdots Q^-_{-\frac{k-1}{k}} Q^-_{-\frac{1}{k}} \ldots Q^-_{\frac{1-2(n-1)+k}{k}} Q^-_{\frac{1-2n+k}{k}} \\ &+ Q^-_{\frac{k-3}{k}} \ldots Q^-_{-\frac{k-1}{k}} Q^-_{-\frac{1}{k}} + Q^-_{-\frac{1}{k}} Q^-_{\frac{k-3}{k}} \ldots Q^-_{-\frac{k-1}{k}} \\ &= \sum_{\text{n}=1}^{\text{k-1}} c_n^\dagger c_n + \Pi \end{aligned} \tag{56}$$

and

$$\begin{aligned} P_{-1}^+ = \left(Q^+\right)^k &= \sum_{n=2}^{k-1} Q^+_{\frac{1-2(n-1)+k}{k}} \cdots Q^+_{-\frac{k-1}{k}} Q^+_{-\frac{1}{k}} \ldots Q^+_{\frac{1-2(n+1)+k}{k}} Q^+_{\frac{1-2n+k}{k}} \\ &+ Q^+_{\frac{k-3}{k}} \ldots Q^+_{-\frac{k-1}{k}} Q^+_{-\frac{1}{k}} + Q^+_{-\frac{1}{k}} Q^+_{\frac{k-3}{k}} \ldots Q^+_{-\frac{k-1}{k}} \\ &= \sum_{\text{n}=1}^{\text{k-1}} c_n^\dagger c_n + \Pi \end{aligned} \tag{57}$$

which implies that $P_{-1}^+ = P_{-1}^- = P_{-1}$

6. The generalisation of the equations (4) and (35) for $k \geq 3$ will be:

$$Q'^{+}_{-1-\frac{1-2(n+1)+k}{k}} = \left(Q^{-}_{\frac{1-2(n+1)+k}{k}} \right)^{\dagger} = Q^{-}_{\frac{1-2(n+2)+k}{k}} \ldots Q^{-}_{-\frac{k-1}{k}} Q^{-}_{-\frac{1}{k}} \ldots Q^{-}_{\frac{1-2(n-1)+k}{k}} Q^{-}_{\frac{1-2n+k}{k}} \tag{58}$$

$$Q'^{-}_{-1-\frac{1+2(m-1)-k}{k}} = \left(Q^{+}_{\frac{1+2(m-1)-k}{k}} \right)^{\dagger} = Q^{+}_{\frac{1+2(m-2)-k}{k}} \ldots Q^{+}_{-\frac{k-1}{k}} Q^{+}_{-\frac{1}{k}} \ldots Q^{-}_{\frac{1+2(m+1)-k}{k}} Q^{-}_{\frac{1+2m-k}{k}} \tag{59}$$

$$Q'^{\pm}_{-\frac{1}{k}} = Q^{\mp}_{-\frac{1}{k}} Q^{\mp}_{\frac{k-3}{k}} Q^{\mp}_{\frac{k-5}{k}} \ldots Q^{\mp}_{-\frac{k-5}{k}} Q^{\mp}_{-\frac{k-3}{k}} \tag{60}$$

$$Q'^{\pm}_{-\frac{k-1}{k}} = Q^{\mp}_{\frac{k-3}{k}} Q^{\mp}_{\frac{k-5}{k}} \ldots Q^{\mp}_{-\frac{k-5}{k}} Q^{\mp}_{-\frac{k-3}{k}} Q^{\mp}_{-\frac{k-1}{k}} \tag{61}$$

where $1 \leq n \leq k-2$ and $2 \leq m \leq k-1$. From (59 – 62), we can deduce that:

$$\left(Q^{+}\right)^{k-1} = Q'^{-} \qquad \text{and} \qquad \left(Q^{+}\right)^{k-1} = Q'^{-} \tag{62}$$

where

$$\begin{aligned} Q'^{+} &= \sum_{n=1}^{k-2} Q^{-}_{-1-\frac{1-2n+k}{k}} + Q^{-}_{-\frac{1}{k}} + Q^{-}_{-\frac{k-1}{k}} \\ Q'^{-} &= \sum_{n=2}^{k-1} Q^{+}_{-1-\frac{1+2n-k}{k}} + Q^{+}_{-\frac{1}{k}} + Q^{+}_{-\frac{k-1}{k}} \end{aligned} \tag{63}$$

4.3. The Hamiltonian of the system

The Hamiltonian expression is:

$$2H = P^{-}_{-1} + P^{+}_{-1} = \left(Q^{-}\right)^{k} + \left(Q^{+}\right)^{k} \tag{64}$$

$$= Q^{-} Q'^{+} + Q'^{+} Q^{-} = Q^{+} Q'^{-} + Q'^{-} Q^{+} \tag{65}$$

$$= 2\left(\sum_{\text{n=1}}^{\text{k-1}} c_n^{\dagger} c_n + \Pi \right) \tag{66}$$

The Hamitonian H is hermitian and verify the following relations of commutation with the charge operators Q^{+} and Q^{-}:

$$\left[Q^{\pm}, H\right] = \left[Q'^{\pm}, H\right] = 0 \tag{67}$$

Furthermore, the Hamitonian H can be decomposed on *k* hermitian terms

$$2H = \sum_{n=0}^{k-1} 2H_n \tag{68}$$

where

$$\begin{aligned} 2H_0 &= Q^{-}_{-\frac{1}{k}} Q^{+}_{-\frac{k-1}{k}} + Q^{+}_{-\frac{1}{k}} Q^{-}_{-\frac{k-1}{k}} \\ 2H_1 &= Q^{+}_{-\frac{k-1}{k}} Q^{-}_{-\frac{1}{k}} + Q^{-}_{-1+\frac{k-3}{k}} Q^{+}_{-\frac{k-3}{k}} \\ &\vdots \\ 2H_n &= Q^{-}_{-1-\frac{1+2n-k}{k}} Q^{+}_{\frac{1+2n-k}{k}} + Q^{+}_{-1-\frac{1-2n+k}{k}} Q^{-}_{\frac{1-2n+k}{k}} \\ &\vdots \\ 2H_{k-1} &= Q^{+}_{-1+\frac{k-3}{k}} Q^{-}_{-\frac{k-3}{k}} + Q^{-}_{-\frac{k-1}{k}} Q^{+}_{-\frac{1}{k}} \end{aligned} \tag{69}$$

this hermitian terms verifies

$$
\begin{aligned}
&\left[H_0, Q^-_{-\frac{1}{k}}\right] = \left[H_0, Q^+_{-\frac{k-1}{k}}\right] = 0 \\
&\left[H_1, Q^-_{-\frac{1}{k}}\right] = \left[H_1, Q^+_{-\frac{k-1}{k}}\right] = 0 \\
&\left[H_1, Q^-_{-1+\frac{k-3}{k}}\right] = \left[H_1, Q^+_{-\frac{k-3}{k}}\right] = 0 \\
&\vdots \\
&\left[H_n, Q^-_{-1-\frac{1+2n-k}{k}}\right] = \left[H_n, Q^+_{\frac{1+2n-k}{k}}\right] = 0 \\
&\vdots \\
&\left[H_{k-1}, Q^-_{-\frac{k-3}{k}}\right] = \left[H_{k-1}, Q^+_{-1+\frac{k-3}{k}}\right] = 0 \\
&\left[H_{k-1}, Q^-_{-\frac{k-1}{k}}\right] = \left[H_{k-1}, Q^+_{-\frac{1}{k}}\right] = 0
\end{aligned}
\tag{70}
$$

4.4. The automorphism groups of the *D=2(1/k, 0)* Model

Like in subsection (3.4), this is also invariant under two kinds of discrete symmetries:
First the Z_k symmetry operating as:

$$
\begin{aligned}
Q^+ &\to qQ^+ \quad ; \quad P^+_{-1} \to q^k P^+_{-1} = P^+_{-1} \\
Q^- &\to \bar{q}Q^- \quad ; \quad P^-_{-1} \to \bar{q}^k P^-_{-1} = P^-_{-1}
\end{aligned}
\tag{71}
$$

where $q^k = \bar{q}^k = 1$.

Second, the Z_2 symmetry, generated by the charge conjugation operators C_n ($n=0, 1, 2,...,k-1$) acting on the $Q^{\pm}$ components and $P^{\pm}$ as:

$$
C_n Q^-_{\frac{1-2n+k}{k}} = Q^+_{\frac{1+2(n-1)-k}{k}} C_n \tag{72}
$$

and

$$
C_n P^+ = P^+ C_n \quad ; \quad C_n P^- = P^- C_n \tag{73}
$$

where

$$
C_n = Q^-_{\frac{1-2n+k}{k}} + Q^+_{\frac{1+2(n-1)-k}{k}} \tag{74}
$$

To enclose this paragraph, the knowledge of these charge conjugation operators allows us to give the new expression of the hamiltonien:

$$
2H = \sum_{i_0=i_1=...=i_{k-1}=1; i\neq j\neq k}^{k-1} C_{i_0} C_{i_1} C_{i_2} ... C_{i_k-1} \tag{75}
$$

5. Conclusion

In this paper, we have proved that the representation of the FSA with the Orthofermion algebra is more appropriate than the representation in W_k algebra [5]. Moreover, we demonstrated, in eq. (69), that the Hamiltonian H of the fractional supersymmetry of order k can be expressed as a sum of the k Hamiltonians of ordinary supersymmetry H_n where $0 \leq n \leq k-1$. Moreover, we can prove in Part II of this article that the

Fractional supersymmetric Algebra (FSA) (k = 3) component constitute by these 9 generators is a simple Lie algebra.

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