

NUMERICAL SOLUTION FOR CLASS OF ONE DIMENSIONAL PARABOLIC PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS VIA CHEBYSHEV SPECTRAL-COLLOCATION METHOD

ABSTRACT. In this work, we apply the Chebyshev spectral-collocation method to obtain approximate solutions for some types of parabolic partial integro-differential equations (PPIDEs). In the first approach, we convert our equation to two coupled equivalent Volterra equations of the second kind. In the second approach, we approximate the integration by replacing the integrand by its interpolating polynomials with Lagrange basis functions in terms of the Chebyshev polynomials instead of using Gauss quadrature approximation to obtain a linear algebraic system. Finally, some numerical examples are included to illustrate the validity and applicability of the proposed technique.

1. Introduction

In recent years, the scientists have paid much attention to spectral method due to its high accuracy (see for instance, [1–5, 37] and the references therein). Spectral methods are powerful approaches for the numerical solution for ordinary or partial differential equations to high accuracy on a simple domain and if the data defining the problem are smooth [6, 7]. Many problems in science and engineering arise in the system mixed of a partial differential equation and the integral terms involving the unknown function.

We are concerned with an efficient numerical approximation scheme of the mathematical model of a physical phenomenon involving contaminant transport process with memory term. Mathematically, the simplest example of such a problem would be represented by the one dimensional parabolic partial integro-differential equation of the form:

$$(1.1) \quad \partial_t u(x, t) - u_{xx}(x, t) + p(x)u_x(x, t) = \int_0^t k(x, t-s)u_{xx}(x, s)ds + f(x, t),$$

associated with the initial condition and Dirichlet boundary conditions:

$$(1.2) \quad u(x, 0) = u_0(x), \quad x \in \Omega = [-1, 1],$$

$$(1.3) \quad u(0, t) = 0, \quad u(1, t) = 0, \quad t \in I = [0, T].$$

Here $(x, t) \in Q \equiv \Omega \times I$, where $I \equiv [0, T]$, $T > 0$ is a time interval and $\Omega \subseteq R \equiv [-1, 1]$ is a bounded domain with boundary $\partial\Omega \equiv \{-1, 1\}$. For implementation of high-order

2000 *Mathematics Subject Classification.* Primary 35R09, 45K05, 97N50, 45D05.

Key words and phrases. Spectral collocation method, Chebyshev polynomials, Chebyshev Differentiation matrix, Lagrange basis function, Parabolic partial integro-differential equations of Volterra type.

methods such as spectral methods, we assume that the functions $p(x)$, $f(x, t)$, $k(t - s)$ and $u_0(x)$ are smooth enough on Q .

Problems of type (1.1)–(1.3) represent the mathematical model of several physical phenomena involving process with memory term. For example, they describe compression of poro-viscoelastic media [8, 9], heat conduction through materials with memory term [9, 10], the nonlocal reactive flows in porous media [11, 12], and non-Fickian flow of fluid in porous media [13]. Many authors have considered the numerical solution of parabolic partial integro-differential equations by various methods. The method of lines used in [14] and the discontinuous Galerkin method in [15]. In [18], they use Galerkin mixed finite element methods and ADI orthogonal spline collocation [19, 20] and the references therein. Also in [16, 17] they introduce some solutions to a class of (PPIDEs). In [22, 23] the method of radial basis functions was used to nonlinear and linear equations respectively. Finally, F. Fakhari-Izadi and M. Dehghan proposed spectral method for parabolic Volterra integro-differential equations based on Legendre collocation scheme [21].

Also Volterra integral and ordinary Volterra integro-differential equations have a wide interest by using many methods of spectral type. In [24] H. Brunner used Collocation methods for second-order Volterra integro-differential equations. Multistep collocation method is also used for Volterra integral equations in [25]. Chebyshev spectral collocation method for the solution of Volterra integral and ordinary Volterra integro-differential equations are discussed in [26]. In [27] Tang introduces Legendre-spectral method with its error analysis for ordinary Volterra integro-differential equation of the second kind. Another spectral method using Legendre spectral Galerkin method was introduced for second-kind Volterra integral equations [28]. Also some authors [29–31] introduce Chebyshev spectral-collocation method for Volterra integral equations. Most papers these mentioned before were devoted to VIEs and ordinary VIDEs, but in this article we considered the solution to parabolic partial integro-differential ones (PPIDEs). Our goal in this article is to apply a Chebyshev spectral-collocation method for both space and time variables that are an extension of the method presented in [30, 31].

The organization of this work is as follows. In Section 2, we present the Chebyshev spectral-collocation method for discretizing the introduced problem. As a result a set of algebraic linear equations are formed and a solution of the considered problem is discussed. In Section 3, we present some numerical examples to demonstrate the effectiveness of the proposed method. Finally, in Section 4 we review some concluding remarks.

2. Chebyshev Spectral-Collocation Method

In this section, we apply the Chebyshev spectral-collocation method to problem (1.1)–(1.3). Before making this, we will introduce some basic properties of the most commonly used set of orthogonal polynomials [32, 33]; Chebyshev polynomials.

2.1. Chebyshev polynomials

The Chebyshev polynomials $T_n(x)$, $n = 0, 1, \dots$, are the Eigen-functions of the singular Sturm-Liouville problem

$$\frac{d}{dx} \left(\sqrt{1-x^2} \frac{dT_n(x)}{dx} \right) + \frac{n^2}{\sqrt{1-x^2}} T_n(x) = 0.$$

They are mutually orthogonal with respect to L_2 inner product on the interval $(-1, 1)$ with the weight function $\omega(x) = 1/\sqrt{1-x^2}$. This imply

$$\int_{-1}^1 T_n(x) T_m(x) \omega(x) dx = \frac{c_n \pi}{2} \delta_{nm},$$

where δ_{nm} is the Kronecker delta, $c_0 = 2$ and $c_n = 1 \forall n \geq 1$. The Chebyshev polynomials satisfy the following three-term recurrence relation

$$\begin{aligned} T_0(x) &= 1, & T_1(x) &= x, \\ (2.1) \quad T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), & n &\geq 1, \end{aligned}$$

and

$$\begin{aligned} T_0(x) &= T_1'(x), & 2T_1(x) &= 0.5T_2'(x), \\ (2.2) \quad T_n(x) &= \frac{1}{2(n+1)} T_{n+1}'(x) - \frac{1}{2(n-1)} T_{n-1}'(x), & n &\geq 1. \end{aligned}$$

A unique feature of the Chebyshev polynomials is their explicit relationship with a trigonometric function:

$$(2.3) \quad T_n(x) = \cos(n \arccos(x)).$$

In this work, we will use the Chebyshev–Gauss (CG) points and the corresponding quadrature weights as

$$(2.4) \quad x_n = -\cos \frac{(2n+1)\pi}{2N+2}, \quad \omega_n = \frac{\pi}{N+1}, \quad 0 \leq n \leq N,$$

beside, the Chebyshev–Gauss–Lobatto (CGL) points and the corresponding quadrature weights defined as

$$(2.5) \quad x_n = -\cos \left(\frac{n\pi}{N} \right), \quad \omega_n = \frac{\pi}{c_n N}, \quad c_n = \begin{cases} 2, & n = 0, N, \\ 1, & 1 \leq n \leq N-1. \end{cases}$$

2.2. Chebyshev Spectral–Collocation Scheme

Now, consider problem (1.1)–(1.3) on bounded domain Q . Because of the orthogonally property of the Chebyshev polynomials on the interval $[-1, 1]$, we transfer equation (1.1) from $[0, T]$ to an equivalent problem defined in the interval $[-1, 1]$, by using the substitution

$$t = \frac{T}{2}(\tau + 1), \quad \tau \in [-1, 1],$$

then equation (1.1) becomes, for all $x, \tau \in [-1, 1]$

$$(2.6) \quad \frac{2}{T} \frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial x^2} + p(x)U_x = \int_0^{\frac{T}{2}(\tau+1)} k\left(x, \frac{T}{2}(\tau+1) - s\right) u_{xx}(x, s) ds + g(x, \tau),$$

in which

$$U(x, \tau) := u\left(x, \frac{T}{2}(\tau+1)\right), \quad g(x, \tau) := f\left(x, \frac{T}{2}(\tau+1)\right).$$

By using the following linear transformation in equation (2.6)

$$s = \frac{T}{2}(\rho+1), \quad \rho \in [-1, 1],$$

by converting the integration interval from $[0, T(\tau+1)/2]$ to the interval $[-1, \tau]$, so that equation (2.6) becomes

$$(2.7) \quad \frac{2}{T} \frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial x^2} + p(x)U_x = \frac{T}{2} \int_{-1}^{\tau} K(x, \tau - \rho) U_{xx}(x, \rho) d\rho + g(x, \tau),$$

where $K(x, \tau - \rho) = k(x, T(\tau - \rho)/2)$. Define an auxiliary function

$$(2.8) \quad \Phi(x, \tau) = g(x, \tau) + \frac{T}{2} \int_{-1}^{\tau} K(x, \tau - \rho) U_{xx}(x, \rho) d\rho,$$

which we will use it in the next approximation. In order to approximate problem (1.1)–(1.3) by spectral methods, we restart equation (2.7) as two equivalent Volterra integro-differential equations by using equation (2.8) as the following

$$(2.9) \quad \begin{cases} \Phi(x, \tau) = g(x, \tau) + \frac{T}{2} \int_{-1}^{\tau} K(x, \tau - \rho) U_{xx}(x, \rho) d\rho, \\ \Phi(x, \tau) = \frac{2}{T} \frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial x^2} + p(x)U_x, \end{cases}$$

by integrating both sides in the second part of equation (2.9) over the interval $[-1, \tau]$ with respect to τ , we get the required two linear equations of (VDEs) as the following

$$(2.10) \quad \begin{cases} \Phi(x, \tau) = g(x, \tau) + \frac{T}{2} \int_{-1}^{\tau} K(x, \tau - \rho) U_{xx}(x, \rho) d\rho, \\ U(x, \tau) = u_0(x) + \frac{T}{2} \int_{-1}^{\tau} [\Phi(x, \xi) + U_{xx}(x, \xi) - p(x)U_x(x, \xi)] d\xi. \end{cases}$$

Let the collocation points $\{(x_i, \tau_j)\}_{i,j}$ be the set of $(N+1)(M+1)$ points, in which $\{x_i\}_{i=0}^N$ are the Chebyshev–Gauss–Lobatto points (CGL nodes), and $\{\tau_j\}_{j=0}^M$ are the Chebyshev–Gauss points (CG nodes). Equation (2.10) holds at the collocation points such that, for all $0 \leq j \leq M$ yields

$$(2.11) \quad \begin{cases} \Phi(x_i, \tau_j) = g(x_i, \tau_j) + \frac{T}{2} \int_{-1}^{\tau_j} K(x_i, \tau_j - \rho) U_{xx}(x_i, \rho) d\rho, & 0 \leq i \leq N, \\ U(x_i, \tau_j) = u_0(x_i) + \frac{T}{2} \int_{-1}^{\tau_j} [\Phi(x_i, \xi) + U_{xx}(x_i, \xi) - p(x_i)U_x(x_i, \xi)] d\xi, & 1 \leq i \leq N-1. \end{cases}$$

For approximating the integral terms in equation (2.11), first the integral interval will transfer from $[-1, \tau]$ to a fixed one $[-1, 1]$ by using the following change of variables

$$\xi(\tau_j, \theta) = \rho(\tau_j, \theta) = \frac{\tau_j + 1}{2}\theta + \frac{\tau_j - 1}{2}, \quad \theta \in [-1, 1],$$

where $\{\theta\}_{k=0}^p$ are roots of the $(p+1)$ -th Chebyshev polynomials. Then (2.11) becomes:

$$(2.12) \quad \begin{cases} \Phi(x_i, \tau_j) = g(x_i, \tau_j) + \frac{T(\tau_j + 1)}{4} \\ \quad \cdot \int_{-1}^1 K(x_i, \tau_j - \rho(\tau_j, \theta)) U_{xx}(x_i, \tau_j - \rho(\tau_j, \theta)) d\theta, \\ U(x_i, \tau_j) = u_0(x_i) + \frac{T(\tau_j + 1)}{4} \int_{-1}^1 \Phi(x_i, \xi(\tau_j, \theta)) d\theta + \frac{T(\tau_j + 1)}{4} \\ \quad \cdot \int_{-1}^1 [U_{xx}(x_i, \xi(\tau_j, \theta)) - p(x_i) U_x(x_i, \xi(\tau_j, \theta))] d\theta. \end{cases}$$

Now, we use

$$(2.13) \quad \begin{cases} \Phi_N^M(x, \sigma) \approx \sum_{n=0}^N \sum_{m=0}^M l_n(x) F_m(\sigma) \Phi(x_n, \tau_m), \\ U_N^M(x, \sigma) \approx \sum_{n=0}^N \sum_{m=0}^M l_n(x) F_m(\sigma) U(x_n, \tau_m), \end{cases}$$

to approximate the functions Φ and U , where $l_n(x)$ and $F_m(\sigma)$ are the n -th and m -th Lagrange basis functions corresponding to non-uniform meshes of $\{x_i\}$ and $\{\tau_j\}$ respectively. After enforcing the homogeneous boundary conditions at $x_0 = -1$ and $x_N = 1$ the first and the last terms in the interpolation polynomial of U are omitted. Therefore, we have

$$U_N^M(x, \sigma) \approx \sum_{n=1}^{N-1} \sum_{m=0}^M l_n(x) F_m(\sigma) U(x_n, \tau_m).$$

Now we can approximate U_x and U_{xx} in equation (2.12) by using the interpolation polynomial of U from the previous equation as the following

$$(2.14) \quad \begin{cases} (U_N^M)_x(x, \sigma) \approx \sum_{n=1}^{N-1} \sum_{m=0}^M l'_n(x) F_m(\sigma) U(x_n, \tau_m), \\ (U_N^M)_{xx}(x, \sigma) \approx \sum_{n=1}^{N-1} \sum_{m=0}^M l''_n(x) F_m(\sigma) U(x_n, \tau_m), \end{cases}$$

where $l'_n(x)$ and $l''_n(x)$ are the first and the second derivative of Lagrange interpolation basis polynomials $l_n(x)$ respectively, such that $l'_n(x) \in \mathbf{P}_{N-1}$ and $l''_n(x) \in \mathbf{P}_{N-2}$.

Now, we will define the differentiation matrices, they are an essential building block for collocation methods used in spectral schemes. The differentiation matrices have a connection with the first and the second derivative of Lagrange interpolation basis polynomials. Denoting $d_{i,n}^2 := l''_n(x_i)$, we introduce the second order differentiation

matrix as the following $D^2 = (d_{i,k}^2)_{0 \leq i,k \leq N}$ which is the second order to differentiation matrix D . By writing the entries of $D^2 = D_{i,k}^2$ for $\{x_i\}$ as

$$D_{i,k}^2 = \begin{cases} 2D_{i,k} \left(D_{i,i} - \frac{1}{x_i - x_k} \right), & i \neq k \\ - \sum_{l=0, l \neq i}^N D_{i,l}^2, & i = k, \end{cases}$$

where $D_{i,k}$ are the entries of the so-called differentiation matrix of dimension $(N+1)$. The entries of the differentiation matrix can be defined in [37] for (CGL) points as the following

$$D_{i,k} = \begin{cases} -\frac{2N^2+1}{6}, & i = k = 0, \\ \frac{c_i}{c_k} \frac{(-1)^{i+k}}{x_i - x_k}, & i \neq k, \\ -\frac{2(1-x_i^2)}{6}, & 1 \leq i = k \leq N-1, \\ \frac{2N^2+1}{6}, & i = k = N, \end{cases}$$

with $c_i = 2$ for $i = 0, N$, where $c_i = 1$ otherwise.

Now we will introduce our work in this article, to approximate the integration in equation (2.12) instead of using Gauss quadrature we replace the integral function by its interpolation polynomial approximation of Φ , U , and approximation to U_x and U_{xx} from equations (2.13) and (2.14), respectively in equation (2.12) and write

$$\begin{aligned} \Phi(x_i, \tau_j) &= \Phi_{i,j}, \quad U(x_i, \tau_j) = U_{i,j}, \quad g(x_i, \tau_j) = g_{i,j}, \\ \rho(\tau_j, \theta_k) &= \rho_{i,k}, \quad \xi(\tau_j, \theta_k) = \xi_{i,k}. \end{aligned}$$

Our main goal is to find the approximation $U_{i,j}$ so after replacing in equation (2.12) we get

$$(2.15) \quad \begin{cases} \Phi_{i,j} \approx g_{i,j} + \frac{T(\tau_j+1)}{4} \\ \quad \cdot \int_{-1}^1 I_N^M \left[K(x_i, \tau_j - \rho(\tau_j, \theta)) (U_N^M)_{xx}(x_i, \tau_j - \rho(\tau_j, \theta)) \right] d\theta, \\ U_{i,j} \approx u_0(x_i) + \frac{T(\tau_j+1)}{4} \int_{-1}^1 I_N^M \left[\Phi_N^M(x_i, \xi(\tau_j, \theta)) \right] d\theta + \frac{T(\tau_j+1)}{4} \\ \quad \cdot \int_{-1}^1 I_N^M \left[(U_N^M)_{xx}(x_i, \xi(\tau_j, \theta)) - p(x_i) (U_N^M)_x(x_i, \xi(\tau_j, \theta)) \right] d\theta, \end{cases}$$

where I_N^M is the interpolation operator associated with the Chebyshev mesh points $\{x_i, \tau_j\}_{i,j}$ defined as the following

$$I_N^M Q(x, \tau) \approx \sum_{n=0}^N \sum_{m=0}^M l_n(x) F_m(\sigma) Q(x_n, \tau_m).$$

Now each equation in (2.15) can be reformulated respectively as

$$(2.16) \quad \begin{aligned} \Phi_{i,j} &\approx g_{i,j} + \frac{T(\tau_j + 1)}{4} \sum_{n=1}^{N-1} \sum_{m=0}^M l_n''(x_i) U(x_n, \tau_m) \\ &\quad \cdot \sum_{z=1}^{p-1} l_z(x_i) \sum_{k=0}^p K(x_i, \tau_j - \rho_{j,k}) F_m(\rho_{j,k}) \int_{-1}^1 F_k(\theta) d\theta, \end{aligned}$$

$$(2.17) \quad \begin{aligned} U_{i,j} &\approx u_0(x_i) + \frac{T(\tau_j + 1)}{4} \left\{ \sum_{n=0}^N \sum_{m=0}^M l_n(x_i) \Phi(x_n, \tau_m) + \sum_{n=1}^{N-1} \sum_{m=0}^M l_n''(x_i) U(x_n, \tau_m) \right. \\ &\quad \left. - p(x_i) \sum_{n=1}^{N-1} \sum_{m=0}^M l_n'(x_i) U(x_n, \tau_m) \right\} \sum_{z=1}^{p-1} l_z(x_i) \sum_{k=0}^p F_m(\xi_{j,k}) \int_{-1}^1 F_k(\theta) d\theta. \end{aligned}$$

Now we discuss an efficient way to find $d_k = \int_{-1}^1 F_k(\theta) d\theta$ and express it in an explicit form. First we express $F_j(s)$ in terms of Chebyshev functions as in [27, 30]:

$$(2.18) \quad F_j(s) = \omega_j^C \sum_{p=0}^M (T_p(x_j)/\gamma_p) T_p(s),$$

where ω_j^C is the Chebyshev weight corresponding to Chebyshev points $\{x_j\}_{j=0}^M$ and let

$$(2.19) \quad \gamma_p = \sum_{i=0}^M T_p^2(x_i) \omega_i^C = \begin{cases} \pi, & p = 0, \\ \pi/2, & 1 \leq p \leq M, \end{cases}$$

and $\gamma_M = \pi/2$ if $\{x_i\}$ are the Chebyshev–Gauss or the Chebyshev–Gauss–Radau points, but if we use $\{x_i\}$ as the Chebyshev–Gauss–Lobatto then $\gamma_M = \pi$. Now from equations (2.18) and (2.19), we can calculate d_k as the following

$$(2.20) \quad d_k = \omega_k^C \sum_{p=0}^M (T_p(x_k)/\gamma_p) \int_{-1}^1 T_p(\theta) d\theta.$$

To compute $\int_{-1}^1 T_p(\theta) d\theta$ we use the recurrence relation in equation (2.2), yields

$$(2.21) \quad \int_{-1}^1 T_p(\theta) d\theta = \begin{cases} 2/(1-p^2), & p \text{ is +ve even number,} \\ 0, & p \text{ is +ve odd number.} \end{cases}$$

Now rewrite equations (2.16) and (2.17) in the matrix form as

$$(2.22) \quad \begin{cases} \Phi_{(N+1)(M+1)} = \mathbf{G}_{(N+1)(M+1)} + \mathbb{L} \mathbf{U}_{(N-1)(M+1)}, \\ \mathbf{U}_{(N-1)(M+1)} = \mathbf{U}_{-1} + \mathbb{A} \Phi_{(N+1)(M+1)} + (\mathbb{B} + \mathbb{C}) \mathbf{U}_{(N-1)(M+1)}, \end{cases}$$

where $\Phi, \mathbf{G}, \mathbf{U}$ and $\mathbf{U}_{-1}$ represent vectors defined as in the following:

$$\begin{aligned} \Phi_{(N+1)(M+1)} &= \text{vec}[\Phi_{i,j}], & \mathbf{G}_{(N+1)(M+1)} &= \text{vec}[g_{i,j}], & 0 \leq i \leq N, \\ \mathbf{U}_{(N-1)(M+1)} &= \text{vec}[U_{i,j}], & & & 1 \leq i \leq N, \quad 0 \leq j \leq M, \end{aligned}$$

$$\mathbf{U}_{-1} = \text{vec} \begin{bmatrix} u_0(x_2) & u_0(x_3) & \cdots & u_0(x_{N-1}) \\ \vdots & \vdots & \vdots & \vdots \\ u_0(x_2) & u_0(x_3) & \cdots & u_0(x_{N-1}) \end{bmatrix},$$

in which the (vec) operator reshapes any matrix into a vector by placing columns of the matrix below each other from the first to the last. For the other matrices each one can be described as block ones as the following:

- $\mathbb{A} = (A_j^i)$, $\mathbb{A}$ is a matrix with a dimension $(N+1)(N-1) \times (N+1)^2$ in which the first and last $(N+1)$ columns are zeros, and each block matrix (A_j^i) has a dimension of $(N+1) \times (N+1)$. For each $0 \leq m \leq M$, $0 \leq j \leq M$, $1 \leq i \leq N-1$, $1 \leq n \leq N-1$, the other entries of (A_j^i) can be given by the following formula

$$(A_j^i)_{i,n+1} = \frac{T(\tau_j+1)}{4} \sum_{k=0}^p d_k F_m(\xi_{j,k}),$$

the shape of the global matrix $\mathbb{A}$ will be

$$\mathbb{A} = \begin{bmatrix} \mathbf{0} & A_j^1 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & A_j^{N-1} & \mathbf{0} \end{bmatrix}.$$

- $\mathbb{L} = (L_j^i)$, $\mathbb{L}$ is a matrix with a dimension $(N+1)^2 \times (N+1)(N-1)$ in which the first and last $(N+1)$ rows are zeros, and each block matrix (L_j^i) has a dimension of $(N+1) \times (N+1)$. For each m, j, i, n , the other entries of (L_j^i) can be given by the following formula

$$(L_j^i)_{i+1,n} = \frac{T(\tau_j+1)}{4} \sum_{k=0}^p d_k K(x_i, \tau_j - \rho_{j,k}) F_m(\rho_{j,k}) D_{i,n}^2,$$

- $\mathbb{B} = (B_j^i)$, $\mathbb{B}$ is a matrix with a dimension $(N+1)(N-1) \times (N+1)(N-1)$ in which each block matrix (B_j^i) has a dimension of $(N+1) \times (N+1)$. For each m, j, i, n , the entries of (B_j^i) can be found from the following relation

$$(B_j^i)_{i,n} = \frac{T(\tau_j+1)}{4} \sum_{k=0}^p d_k F_m(\xi_{j,k}) D_{i,n}^2,$$

- Finally, $\mathbb{C} = (C_j^i)$ which is similar to the matrix $\mathbb{B}$. For each m, j, i, n , the entries of (C_j^i) can be found from the following relation

$$(C_j^i)_{i,n} = -\frac{T(\tau_j+1)}{4} p(x_i) \sum_{k=0}^p d_k F_m(\xi_{j,k}) D_{i,n}.$$

To solve the coupled equations system in equation (2.22), we convert them to a linear algebraic system as follows

$$(I - (\mathbb{B} + \mathbb{C}) - \mathbb{A}\mathbb{L}) \mathbf{U}_{(N-1)(M+1)} = \mathbf{U}_{-1} + \mathbb{A}\mathbf{G}_{(N+1)(M+1)}.$$

By solving the previous system, we obtain an approximation to $\mathbf{U}_{(N-1)(M+1)}$ then the approximation to the original problem for all $x \in (-1, 1)$ and $t \in [0, T]$ can obtain by

$$u(x, t) \approx \sum_{n=1}^{N-1} \sum_{m=0}^M l_n(x) F_m\left(\frac{2}{T}t - 1\right) U(x_n, \tau_m).$$

3. Numerical examples

In order to test the utility of the proposed new method, we apply the new scheme to the following two examples whose exact solutions are provided in each case. For both examples, we take $T = 1$, $p = N = M$, and consider $\{\theta_k\}_{k=0}^N$ as the Chebyshev-Gauss points, and proceed as in Section 2. To show the efficiency of the present method for our problem in comparison with the exact solution, we calculate for different values of N the maximum error defined by

$$\|E\|_{\infty} = \max_{\substack{1 \leq i \leq N-1 \\ 0 \leq j \leq M}} \left| U_{i,j} - u\left(x_i, \frac{T(\tau_j + 1)}{2}\right) \right|.$$

All the computations are carried out in double precision arithmetic using Matlab 7.9.0 (R2009b). To obtain sufficient accurate calculations, variable arithmetic precision (vpa) is employed with digit being assigned to be 32. The code was executed on a second generation Intel Core i5-2410M, 2.3 Ghz Laptop. Finally, the CPU time indicates the time for all calculations for the solution of the entire problem is presented.

Example 3.1. Consider the linear PVIDE (1.1)–(1.3) with the kernel $k(x, t) = xt$ and the forcing function

$$\begin{aligned} f(x, t) = & (9x^4 - 8x^3 - 1 - p(x)(1 - 4x^4)) \sin(x + t) - 8x^4(t \cos(x) + \sin(x)) \\ & + (x^5 - x^4 - 12x^3 + 12x^2 - x + 1 - 4x^3 p(x)) \cos(x + t) \\ & - (x^5 - 12x^3 - x)(t \sin(x) - \cos(x)), \end{aligned}$$

so that the exact solution will be $u(x, t) = (1 - x^4) \cos(x + t)$.

TABLE 1. Maximum L_{∞} errors for Example 3.1. with different values of N .

$p_1(x)$			$p_2(x)$	
N	Maximum error	CPU(s)	Maximum error	CPU(s)
4	9.54993E-02	1.694646	9.77251E-02	1.699577
6	1.91331E-03	3.866402	1.77596E-03	4.940960
8	1.52747E-05	8.436051	1.44350E-05	10.440083
10	7.02970E-08	17.13380	6.98292E-08	19.006552
12	2.01326E-10	24.70313	2.00289E-10	28.995915
14	3.88335E-13	55.79182	3.89168E-13	61.264216
16	1.34615E-14	114.94758	1.92346E-14	116.30818

In “Table 1”, we take $p_1(x) = x^3 \sin(x)e^x/(1+x^2)$ and $p_2(x) = x^3 \cos(x)e^x/(1+x^2)$. In “Figure 1”, the exact solution, numerical solution are plotted with $p(x) =$

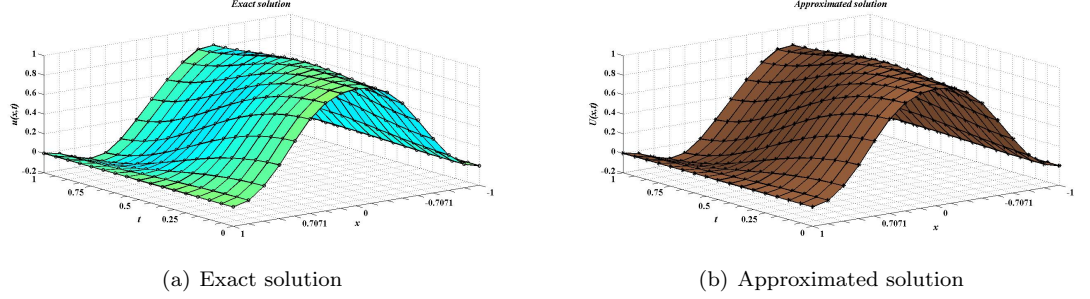


FIGURE 1. Exact and Approximated solutions for $p_1(x)$ with $x \in [-1, 1]$ and $t \in [0, T]$ at $N = 16$

$x^3 \sin(x)e^x/(1+x^2)$ and the nonuniform set of collocation points (x_i, t_j) at $N = 16$.

Example 3.2. Consider the linear PVIDE (1.1)-(1.3) with the kernel $k(x, t) = \sin(xt)$ and the forcing function

$$f(x, t) = (8x^3 + 12x^2 + p(x)(1 - 4x^3 - x^4))e^{(x+t)} + \left(\frac{x^4 + 12x^2 + 8x^3 - 1}{1 + x^2} \right)(xe^t - \sin(xt) - x \cos(xt))e^x,$$

so that the exact solution will be $u(x, t) = (1 - x^4)e^{(x+t)}$.

TABLE 2. Maximum L_∞ errors for Example 3.2. with different values of N .

N	$p_1(x)$	CPU(s)	$p_2(x)$	CPU(s)
	Maximum error		Maximum error	
4	3.01086E-01	2.454881	4.06238E-01	1.989695
6	7.41424E-03	5.251765	6.11058E-03	5.149693
8	5.67288E-05	12.90018	5.16640E-05	9.960659
10	2.59662E-07	22.52414	2.38373E-07	19.88537
12	7.21778E-10	31.28570	7.0061E-10	37.18235
14	1.36713E-12	57.69717	1.38201E-12	53.88282
16	7.10543E-14	114.5657	7.26086E-14	109.4029

In “Table 2”, we take $p_1(x) = x^3 \cos(x)e^x/(1+x^2)$ and $p_2(x) = e^x$. In “Figure 2”, the exact solution, numerical solution are plotted with $p(x) = x^3 \cos(x)e^x/(1+x^2)$ and the nonuniform set of collocation points (x_i, t_j) at $N = 16$.

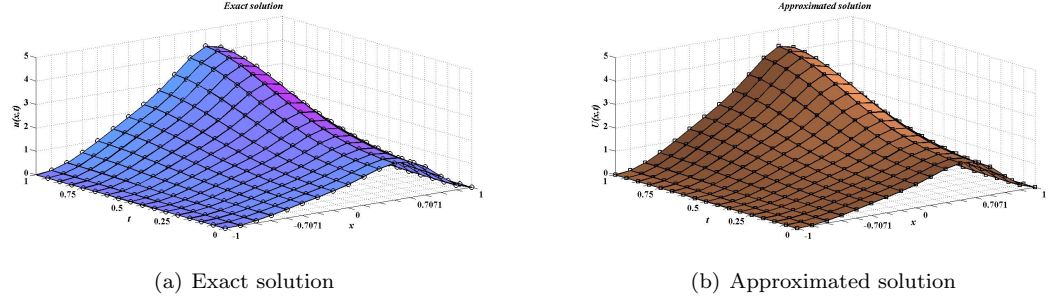


FIGURE 2. Exact and Approximated solutions for $p_1(x)$ with $x \in [-1, 1]$ and $t \in [0, T]$ at $N = 18$. See Example 3.2.

4. Conclusion

In this work, we used a new competitive numerical scheme based on developing Chebyshev spectral-collocation method to find the approximate solution for parabolic partial integro-differential equations. The main advantage of using Chebyshev scheme instead of using Legendre one is that its quadrature points have explicit and simple expressions as well as the corresponding weights. This enables us to avoid the complex computation of the Legendre quadrature points and the corresponding weights. Moreover, we made a minor modification on approximating the integration by replacing the integrand by its interpolating polynomials instead of using Gauss quadrature approximation and this increases the accuracy of the suggested method. The numerical examples given in this work have demonstrated the potential of the newly proposed numerical scheme in solving parabolic partial integro-differential and similar equations even with using a small number of collocation points.

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