

ECCENTRIC CONNECTIVITY INDEX AND POLYNOMIAL OF SOME GRAPHS

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ABSTRACT. Let G be a simple and connected graph with n vertices and m edges. The Eccentric connectivity index of G is defined as the summation of the product of degree and eccentricity of the vertices [15]. Eccentric connectivity polynomial is a topological polynomial of G which is related to its Eccentric connectivity index [8]. In this article, exact expressions of these indices for the double graph of a given graph is presented. In addition to it, a lower bound of these invariants for the subdivision graph of the double graph of a given graph and a lower bound of these invariants for the extended double cover graph of a given graph is also proposed.

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1. INTRODUCTION AND TERMINOLOGIES

The eccentric connectivity index is one of the important graph invariant which has been extensively used in molecular chemistry for studies related to structure activity/property relationships. It is defined as follows. Consider a finite, simple and connected graph G with n vertices and m edges. Let the vertex set of G is denoted by $V(G)$ and the edge set is denoted by $E(G)$. The distance $d(u, v)$ between the vertices u and v is the length of the shortest path between u and v . The eccentricity $\varepsilon(u)$ for a given vertex u of $V(G)$ is the maximum distance between u and any other vertex v of G . The **Eccentric connectivity index** $\xi^c(G)$ of a graph G , proposed by Sharma, Goswami and Madan [15] is defined as

$$\xi^c(G) = \sum d(u)\varepsilon(u)$$

, where $d(u)$ denotes the degree of vertex u in G .

Recently, scientists are interested in exploring the mathematical properties of this index. Characterization of trees with maximum and minimum vertex degree has been reported by I. Gutman in his work. In [10], the author identified trees and unicyclic graphs with highest and lowest value for eccentric connectivity index, under certain conditions. In the same work, he presented values of Sharp lower bounds and asymptotic upper bounds for graphs and established

expressions for connecting this index with other important topological indices. In addition, he calculated the values of eccentric connectivity index for families of composite graphs and developed a linear algorithm for calculating the eccentric connectivity index value for trees. In [5], the authors presented a comparison between eccentric connectivity index and the Zagreb indices for chemical trees. In [14], authors studied about upper and lower bounds of graphs and trees in terms of order and diameter respectively. Eccentric connectivity index of various families of graphs are studied by Došlić in [7] and presented explicit formulae. In [17], the authors characterized unicyclic graphs with minimal and second minimal eccentric distance sum among n -vertex unicyclic graphs with given girth. In addition, the extremal trees with given diameter and minimal eccentric distance sum is also characterized. In [12], the authors studied about extremal trees and graphs with maximal eccentric distance sum and derived expressions for lower and upper bounds for the eccentric distance sum in terms of other graph invariants such as Wiener index, degree distance, eccentric connectivity index, independence number, connectivity, matching number, chromatic number and clique number. In addition, the eccentric distance sum for the Cartesian product of nanotubes and nanotori are also presented. In [16], the authors studied about edge subdivisions and derived expressions for their eccentric connectivity index in terms of the eccentric connectivity index of the parent graph and some other topological parameters. They studied about four classes of graphs formed from a given graph by performing different types subdivision operations on edges.

The eccentric connectivity polynomial is a polynomial version of eccentric connectivity index. The **Eccentric connectivity polynomial** of a graph G is defined as [8]

$$\xi^c(G, x) = \sum d(u)x^{\varepsilon(u)}$$

, where $d(u)$ and $\varepsilon(u)$ are the degree of vertex u and eccentricity of u respectively, where value of x is greater than 1. The connection between the eccentric connectivity polynomial and the eccentric connectivity index is given by

$$\xi^c(G) = \xi^c(G, 1)$$

where $\xi^c(G, 1)$ is the first derivative of $\xi^c(G, x)$.

In [3], the authors studied about cartesian product, symmetric difference, disjunction and join of graphs and presented expressions for the eccentric connectivity polynomial for the above graphs. In [6], the authors studied about some special cases of thorn graphs and derived general expressions for the eccentric connectivity index and polynomial of the these graphs. Fullerenes are connected graphs with exactly 12 pentagonal faces. In [9], eccentric connectivity polynomial is computed for an infinite family of fullerenes. In [1], the authors studied about eccentric connectivity index and polynomial of $CNC_m[n]$ carbon nanocones for $n \geq 1$, $3 \leq m \leq 7$.

2. MAIN RESULTS

In this section, we start with definitions and then derive expressions for the *Eccentric connectivity index* and *Eccentric connectivity polynomial* of *Double graph* G' and *Extended Double cover graph* G'' . We present lower bounds for the *Subdivisiongraph* of *Double graph* as well.

Definition 2.1. For a graph G , the **double graph** G' is defined as follows. Let $V(G) = v_1, v_2, \dots, v_n$ be the set vertices of G . Corresponding to $V(G)$, there are two sets of vertices $A = a_1, a_2, \dots, a_n$ and $B = b_1, b_2, \dots, b_n$ in G' . For every edge $(v_i, v_j) \in E(G)$, two additional edges (a_i, b_j) and (a_j, b_i) will be added in G' [2, 11].

Definition 2.2. Let G be a simple connected graph with vertex set $V = v_1, v_2, \dots, v_n$. The **extended double cover of G** , introduced by Alon [2] in 1986, is as follows. For a simple connected graph G , its extended double cover graph G'' is a bipartite graph with bipartition $a = a_1, a_2, \dots, a_n$ and $B = b_1, b_2, \dots, b_n$. For any edge $(v_i, v_j) \in E(G)$ or if $i = j$, one edge (a_i, b_j) will be added in G'' .

2.1. Eccentric connectivity index and polynomial of double graph and its subdivision graph.

Theorem 2.1. For a Double graph G' , the Eccentric connectivity index and Eccentric connectivity polynomial are given by

$$\xi^c(G') = 4[\xi^c(G) + (n-1)\|n-1\|]$$

and

$$\xi^c(G', x) = 4[\xi^c(G, x) + x(n-1)\|n-1\|]$$

respectively. Where $\|n-1\|$ is the number of vertices having eccentricity 1.

Proof. From the definition of *Double graph*, it is clear that $d_{G'}(a_i) = d_{G'}(b_i) = 2d_G(v_i)$ and $\varepsilon_{G'}(a_i) = \varepsilon_{G'}(b_i) = \varepsilon_G(v_i)$, when $\varepsilon_G(v_i) \geq 2$ and $\varepsilon_{G'}(a_i) = \varepsilon_{G'}(b_i) = \varepsilon_G(v_i) + 1$, when $\varepsilon_G(v_i) = 1$.

Hence the *Eccentric connectivity index* of the *Double graph* G' is,

$$\begin{aligned} \xi^c(G') &= \sum_{i=1,2,\dots,n} d(a_i)\varepsilon(a_i) + \sum_{i=1,2,\dots,n} d(b_i)\varepsilon(b_i) \\ &= 2\left[\sum_{\varepsilon(v_i) \geq 2} 2d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} 2d(v_i)(\varepsilon(v_i) + 1) \right] \\ &= 4\left[\sum_{\varepsilon(v_i) \geq 2} d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} d(v_i) \right] \\ &= 4[\xi^c(G) + (n-1)\|n-1\|] \end{aligned}$$

Where $\|n-1\|$ is the number of vertices having eccentricity 1.

The *Eccentric connectivity polynomial* of the double graph G' is given by,

$$\begin{aligned}
\xi^c(G', x) &= \sum_{i=1,2,\dots,n} d(a_i)x^{\varepsilon(a_i)} + \sum_{i=1,2,\dots,n} d(b_i)x^{\varepsilon(b_i)} \\
&= 2\left[\sum_{\varepsilon(v_i) \geq 2} 2d(v_i)x^{\varepsilon(v_i)} + \sum_{\varepsilon(v_i)=1} 2d(v_i)x^{\varepsilon(v_i)+1} \right] \\
&= 4\left[\sum_{\varepsilon(v_i) \geq 2} d(v_i)x^{\varepsilon(v_i)} + \sum_{\varepsilon(v_i)=1} d(v_i)x^{\varepsilon(v_i)} + \sum_{\varepsilon(v_i)=1} d(v_i)x^1 \right] \\
&= 4[\xi^c(G, x) + x(n-1)\|n-1\|]
\end{aligned}$$

□

Theorem 2.2. *For the subdivision graph of the Double graph $S(G')$, the Eccentric connectivity index and Eccentric connectivity polynomial are bounded below as follows.*

$$\xi^c(S(G')) \geq 8[\xi^c(G) + (n-1)\|n-1\|]$$

and

$$\xi^c(S(G', x)) \geq 4[\xi^c(G, x^2) + x^2(n-1)\|n-1\|]$$

Where $\|n-1\|$ is the number of vertices having eccentricity 1.

Proof. In $S(G')$, $d_{S(G')}(a_i) = d_{S(G')}(b_i) = 2d_G(v_i)$ and $\varepsilon_{S(G')}(a_i) = \varepsilon_{S(G')}(b_i) = 2\varepsilon_G(v_i)$, when $\varepsilon_G(v_i) \geq 2$ and $\varepsilon_{S(G')}(a_i) = \varepsilon_{S(G')}(b_i) = 2[\varepsilon_G(v_i) + 1]$, when $\varepsilon_G(v_i) = 1$.

Let $z_1, z_2, \dots, z_m$ be the m subdivision vertices in $S(G')$. Then $d_{S(G')}(z_i) = 2$ and $\varepsilon_{S(G')}(z_i) \geq 1$

$$\begin{aligned}
\xi^c(S(G')) &= \sum_{i=1,2,\dots,n} d_{S(G')}(a_i)\varepsilon_{S(G')}(a_i) + \sum_{i=1,2,\dots,n} d_{S(G')}(b_i)\varepsilon_{S(G')}(b_i) + \sum_{i=1,2,\dots,m} d_{S(G')}(z_i)\varepsilon_{S(G')}(z_i) \\
&= 2\left[\sum_{\varepsilon_G(v_i) \geq 2} 2d_G(v_i)2\varepsilon_G(v_i) + \sum_{\varepsilon_G(v_i)=1} 2d_G(v_i)(2\varepsilon_G(v_i) + 1) \right] + 2 \sum_{i=1,2,\dots,m} \varepsilon_{S(G')}(z_i) \\
&= 8\left[\sum_{\varepsilon(v_i) \geq 2} d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} d(v_i) \right] + \sum_{i=1,2,\dots,m} \varepsilon_{S(G')}(z_i) \\
&\geq 8[\xi^c(G) + (n-1)\|n-1\|]
\end{aligned}$$

Similarly,

$$\begin{aligned}
\xi^c(S(G', x)) &= \sum_{i=1,2,\dots,n} d_{S(G')}(a_i) x^{\varepsilon_{S(G')}(a_i)} + \sum_{i=1,2,\dots,n} d_{S(G')}(b_i) x^{\varepsilon_{S(G')}(b_i)} + \sum_{i=1,2,\dots,m} d_{S(G')}(z_i) x^{\varepsilon_{S(G')}(z_i)} \\
&= 2 \left[\sum_{\varepsilon_G(v_i) \geq 2} 2d_G(v_i) x^{2\varepsilon_G(v_i)} + \sum_{\varepsilon_G(v_i)=1} 2d_G(v_i) x^{(2\varepsilon_G(v_i)+1)} \right] + 2 \sum_{i=1,2,\dots,m} x^{\varepsilon_{S(G')}(z_i)} \\
&= 4[\xi^c(G, x^2) + x^2(n-1)\|n-1\|] + 2 \sum_{i=1,2,\dots,m} x^{\varepsilon_{S(G')}(z_i)} \\
&\geq 4[\xi^c(G, x^2) + x^2(n-1)\|n-1\|]
\end{aligned}$$

□

2.2. Eccentric connectivity index and polynomial of Extended double cover of a graph and its subdivision graph.

Theorem 2.3. *The Eccentric connectivity index of the Extended double cover graph G'' satisfies the inequality*

$$\xi^c(G'') \geq 2[\xi^c(G) + n\|n-1\|]$$

and Eccentric connectivity polynomial of the Extended double cover graph G'' satisfies the inequality

$$\xi^c(G'', x) \geq 2x[\xi^c(G, x) + x\|n-1\|].$$

Where $\|n-1\|$ is the number of vertices having eccentricity 1.

Proof. Let there are n vertices and m edges in graph G . Then its extended double cover graph will consists of $2n$ vertices and $n+2m$ edges. In G'' , $d_{G''}(a_i) = d_{G''}(b_i) = d_G(v_i) + 1$ and $\varepsilon_{G''}(a_i) = \varepsilon_{G''}(b_i) = \varepsilon_G(v_i) + 1$, for $i = 1, 2, \dots, n$.

$$\begin{aligned}
\xi^c(G'') &= \sum_{i=1,2,\dots,n} d_{G''}(a_i) \varepsilon_{G''}(a_i) + \sum_{i=1,2,\dots,n} d_{G''}(b_i) \varepsilon_{G''}(b_i) \\
&= 2 \left[\sum_{\varepsilon(v_i) \geq 2} (d(v_i) + 1)(\varepsilon(v_i) + 1) + \sum_{\varepsilon(v_i)=1} 2(d(v_i) + 1)(\varepsilon(v_i) + 1) \right] \\
&\geq 2 \left[\sum_{\varepsilon(v_i) \geq 2} d(v_i) \varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} (d(v_i) + 1)(\varepsilon(v_i) + 1) \right] \\
&\geq 2 \left[\sum_{\varepsilon(v_i) \geq 1} d(v_i) \varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} d(v_i) \right] + \left[\sum_{\varepsilon(v_i)=1} \varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} 1 \right] \\
&\geq 2[\xi^c(G) + (n-1)\|n-1\|] + 2\|n-1\| \\
&\geq 2[\xi^c(G) + n\|n-1\|]
\end{aligned}$$

The *Eccentric connectivity polynomial* of the Extended double cover graph G'' is given by,

$$\begin{aligned}
\xi^c(G'', x) &= \sum_{i=1,2,\dots,n} d(a_i)x^{\varepsilon(a_i)} + \sum_{i=1,2,\dots,n} d(b_i)x^{\varepsilon(b_i)} \\
&= 2\left[\sum_{\varepsilon(v_i) \geq 2} (d(v_i) + 1)x^{(\varepsilon(v_i)+1)} + \sum_{\varepsilon(v_i)=1} (d(v_i) + 1)x^{(\varepsilon(v_i)+1)} \right] \\
&= 2x\left[\sum_{\varepsilon(v_i) \geq 2} d(v_i)x^{\varepsilon(v_i)} + \sum_{\varepsilon(v_i)=1} d(v_i)x^{\varepsilon(v_i)} \right] + 2x\left[\sum_{\varepsilon(v_i)=1} x^{\varepsilon(v_i)} + \sum_{\varepsilon(v_i) \geq 2} x^{\varepsilon(v_i)} \right] \\
&= 2x[\xi^c(G, x) + x\|n-1\| + \sum_{\varepsilon(v_i) \geq 2} x^{\varepsilon(v_i)}] \\
&\geq 2x[\xi^c(G, x) + x\|n-1\|].
\end{aligned}$$

□

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