

ECCENTRIC CONNECTIVITY INDEX AND POLYNOMIAL OF SOME GRAPHS

ABSTRACT. The eccentric connectivity index based on degree and eccentricity of the vertices of a graph is one of the ideal graph in-variant in mathematics. In this article, the explicit generalized expressions for the eccentric connectivity index and polynomial of the double graphs, double cover and corona of a graph are established.

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1. INTRODUCTION AND TERMINOLOGIES

The eccentric connectivity index is a novel topological index that offers a vast potential for structure activity/property relationships. It is defined as follows. Let G be a finite, simple, connected graph with n vertices and m edges. The vertex and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. For two vertices u and v of $V(G)$ their distance $d(u, v)$ is defined as the length of any shortest path connecting u and v in G . For a given vertex u of $V(G)$ its eccentricity $\varepsilon(u)$ is the largest distance between u and any other vertex v of G . Hence, $\varepsilon(u) = \max_{v \in V(G)} d(u, v)$.

The **Eccentric connectivity index** $\xi^c(G)$ of a graph G is defined as

$$\xi^c(G) = \sum d(u)\varepsilon(u),$$

where $d(u)$ denotes the degree of vertex u in G , the number of its neighbors in G .

Recently the mathematical properties of this invariant has been explored by some scientists in their work. I. Gutman in his work, characterized trees with maximum and minimum vertex degree and proposed a simple linear algorithm for calculating eccentric connectivity index of trees. In [8], the author presented the extremal trees and unicyclic graphs with maximum and minimum eccentric connectivity index to the certain graph constraints. Sharp lower and asymptotic upper binds for all graphs are given and various connections with other important graph invariants are established. In addition, author presented explicit formulae for the values of eccentric connectivity index for several families of composite graphs and designed a linear algorithm for calculating

the eccentric connectivity index of trees. A comparison of the eccentric connectivity index and the Zagreb indices for chemical trees are reported in [4]. The authors obtained an exact lower bound on the eccentric connectivity index in terms of order in [11]. Also, an asymptotically sharp upper bound is also derived. In addition, for trees of given order, when the diameter is prescribed, precise upper and lower bounds are provided. In [6], the authors presented explicit formulas for eccentric connectivity index of various families of graphs. Characterization of extremal unicyclic graphs among n -vertex unicyclic graphs with given girth having the minimal and second minimal eccentric distance sum is reported in [13]. In addition, the extremal trees with given diameter and minimal eccentric distance sum is characterized. In [10], the authors studied about extremal trees and graphs with maximal eccentric distance sum. Various lower and upper bounds for the eccentric distance sum in terms of other graph invariants including the Wiener index, the degree distance, eccentric connectivity index, independence number, connectivity, matching number, chromatic number and clique number are established. In addition, explicit formulae for the values of eccentric distance sum for the Cartesian product, applied to some graphs of chemical interest (like nanotubes and nanotori) are also presented. In [12], four classes of graphs arising from a given graph via different types of edge subdivisions and presented explicit formulas expressing their eccentric connectivity index in terms of the eccentric connectivity index of the original graph and some auxiliary invariants are taken care.

The eccentric connectivity polynomial is a polynomial version eccentric connectivity index. The **Eccentric connectivity polynomial** of a graph G is defined as

$$\xi^c(G, x) = \sum d(u)x^{\varepsilon(u)}$$

, where $d(u)$ and $\varepsilon(u)$ are the degree of vertex u and eccentricity of u respectively. It is defined for $x \geq 1$.

In [3], exact formulas for the eccentric connectivity polynomial of cartesian product, symmetric difference, disjunction and join of graphs are presented. In [5], the authors presented the explicit generalized expressions for the eccentric connectivity index and polynomial of the thorn graphs and considered some particular cases. Fullerenes are 3-connected graphs with exactly 12 pentagonal faces. In [7], eccentric connectivity polynomial is computed for an infinite family of fullerenes. In [1], eccentricity connectivity index and polynomial of $CNC_m[n]$ carbon nanocones for any arbitrary amount of n and m so that $n \geq 1$, $3 \leq m \leq 7$ is computed.

2. MAIN RESULTS

In this section, starts with definition and then derive expressions for the *Eccentric connectivity index* and *Eccentric connectivity polynomial* of *Double graph* G' and *Extended double cover graph* G'' . Also, lower bounds for the subdivision graph of *Double graph* are presented.

2.1. Eccentric connectivity index and polynomial of double graph and its subdivision graph. The double graph of a graph G is denoted by G' , which is constructed from two copies of

G in the following manner [2, 9]. Let the vertex set of G be $V(G) = v_1, v_2, \dots, v_n$, and the vertices of G are given by the two sets $X = x_1, x_2, \dots, x_n$ and $Y = y_1, y_2, \dots, y_n$. Thus for each vertex $v_i \in V(G)$, there are two vertices x_i and y_i in $V(G)$. The double graph G includes the initial edge set of each copies of G , and for any edge $(v_i, v_j) \in E(G)$, two more edges (x_i, y_j) and (x_j, y_i) are added.

Theorem 2.1. *The Eccentric connectivity index and Eccentric connectivity polynomial of Double graph G' are given by*

$$\xi^c(G') = 4[\xi^c(G) + (n-1)\|n-1\|]$$

and

$$\xi^c(G', x) = 4[\xi^c(G, x) + x(n-1)\|n-1\|]$$

Proof. From the definition of Double graph, it is clear that $d_{G'}(x_i) = d_{G'}(y_i) = 2d_G(v_i)$ and $\varepsilon_{G'}(x_i) = \varepsilon_{G'}(y_i) = \varepsilon_G(v_i)$, when $\varepsilon_G(v_i) \geq 2$ and $\varepsilon_{G'}(x_i) = \varepsilon_{G'}(y_i) = \varepsilon_G(v_i) + 1$, when $\varepsilon_G(v_i) = 1$.

Hence the Eccentric connectivity index of the Double graph, G' is,

$$\begin{aligned} \xi^c(G') &= \sum_{i=1,2,\dots,n} d(x_i)\varepsilon(x_i) + \sum_{i=1,2,\dots,n} d(y_i)\varepsilon(y_i) \\ &= 2\left[\sum_{\varepsilon(v_i) \geq 2} 2d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} 2d(v_i)(\varepsilon(v_i) + 1) \right] \\ &= 4\left[\sum_{\varepsilon(v_i) \geq 2} d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} d(v_i) \right] \\ &= 4[\xi^c(G) + (n-1)\|n-1\|] \end{aligned}$$

where $\|n-1\|$ is the number of vertices having eccentricity 1.

The Eccentric connectivity polynomial of the double graph G' is given by,

$$\begin{aligned} \xi^c(G', x) &= \sum_{i=1,2,\dots,n} d(x_i)x^{\varepsilon(x_i)} + \sum_{i=1,2,\dots,n} d(y_i)x^{\varepsilon(y_i)} \\ &= 2\left[\sum_{\varepsilon(v_i) \geq 2} 2d(v_i)x^{\varepsilon(v_i)} + \sum_{\varepsilon(v_i)=1} 2d(v_i)x^{(\varepsilon(v_i)+1)} \right] \\ &= 4\left[\sum_{\varepsilon(v_i) \geq 2} d(v_i)x^{\varepsilon(v_i)} + \sum_{\varepsilon(v_i)=1} d(v_i)x^{\varepsilon(v_i)} + \sum_{\varepsilon(v_i)=1} d(v_i)x^1 \right] \\ &= 4[\xi^c(G, x) + x(n-1)\|n-1\|] \end{aligned}$$

□

Theorem 2.2. *For the subdivision graph of the double graph $S(G')$, the Eccentric connectivity index and Eccentric connectivity polynomial are bounded below as follows.*

$$\xi^c(S(G')) \geq 8[\xi^c(G) + (n-1)\|n-1\|].$$

and

$$\xi^c(S(G', x)) \geq 4[\xi^c(G, x^2) + x^2(n-1)\|n-1\|].$$

Proof. In $S(G')$, $d_{S(G')}(x_i) = d_{S(G')}(y_i) = 2d_G(v_i)$ and $\varepsilon_{S(G')}(x_i) = \varepsilon_{S(G')}(y_i) = 2\varepsilon_G(v_i)$, when $\varepsilon_G(v_i) \geq 2$ and $\varepsilon_{S(G')}(x_i) = \varepsilon_{S(G')}(y_i) = 2[\varepsilon_G(v_i) + 1]$, when $\varepsilon_G(v_i) = 1$.

Let $z_1, z_2, \dots, z_m$ be the m subdivision vertices in $S(G')$. Then $d_{S(G')}(z_i) = 2$ and $\varepsilon_{S(G')}(z_i) \geq 1$

$$\begin{aligned} \xi^c(S(G')) &= \sum_{i=1,2,\dots,n} d_{S(G')}(x_i)\varepsilon_{S(G')}(x_i) + \sum_{i=1,2,\dots,n} d_{S(G')}(y_i)\varepsilon_{S(G')}(y_i) + \sum_{i=1,2,\dots,m} d_{S(G')}(z_i)\varepsilon_{S(G')}(z_i) \\ &= 2\left[\sum_{\varepsilon_G(v_i) \geq 2} 2d_G(v_i)2\varepsilon_G(v_i) + \sum_{\varepsilon_G(v_i)=1} 2d_G(v_i)(2\varepsilon_G(v_i) + 1) \right] + 2 \sum_{i=1,2,\dots,m} \varepsilon_{S(G')}(z_i) \\ &= 8\left[\sum_{\varepsilon(v_i) \geq 2} d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i)=1} d(v_i) \right] + \sum_{i=1,2,\dots,m} \varepsilon_{S(G')}(z_i) \\ &\geq 8[\xi^c(G) + (n-1)\|n-1\|]. \end{aligned}$$

Similarly,

$$\begin{aligned} \xi^c(S(G', x)) &= \sum_{i=1,2,\dots,n} d_{S(G')}(x_i)x^{\varepsilon_{S(G')}(x_i)} + \sum_{i=1,2,\dots,n} d_{S(G')}(y_i)x^{\varepsilon_{S(G')}(y_i)} + \sum_{i=1,2,\dots,m} d_{S(G')}(z_i)x^{\varepsilon_{S(G')}(z_i)} \\ &= 2\left[\sum_{\varepsilon_G(v_i) \geq 2} 2d_G(v_i)x^{2\varepsilon_G(v_i)} + \sum_{\varepsilon_G(v_i)=1} 2d_G(v_i)x^{(2\varepsilon_G(v_i)+1)} \right] + 2 \sum_{i=1,2,\dots,m} x^{\varepsilon_{S(G')}(z_i)} \\ &= 4[\xi^c(G, x^2) + x^2(n-1)\|n-1\|] + 2 \sum_{i=1,2,\dots,m} x^{\varepsilon_{S(G')}(z_i)} \\ &\geq 4[\xi^c(G, x^2) + x^2(n-1)\|n-1\|]. \end{aligned}$$

□

2.2. Eccentric connectivity index and polynomial of extended double cover of a graph and its subdivision graph. Let $G = (V, E)$ be a simple connected graph with $V = v_1, v_2, \dots, v_n$. The extended double cover of G , introduced by Alon [2] in 1986, is denoted by G'' . It is the bipartite graph with bipartition (X, Y) where $X = x_1, x_2, \dots, x_n$ and $Y = y_1, y_2, \dots, y_n$ in which x_i and y_j are adjacent if and only if either v_i and v_j are adjacent in G or $i = j$. For example, the extended double cover of the complete graph is the complete bipartite graph.

Theorem 2.3. *The eccentric connectivity index of the extended double cover graph G'' satisfies the inequality*

$$\xi^c(G'') \geq 2[\xi^c(G) + n\|n - 1\|]$$

and eccentric connectivity polynomial of the extended double cover graph G'' satisfies the inequality

$$[\xi^c(G''), x] \geq 2x[\xi^c(G, x) + x\|n - 1\|].$$

Proof. Let G be a simple connected graph with n vertices and m edges. Then its extended double cover graph will consists of $2n$ vertices and $n + 2m$ edges. In G'' , $d_{G''}(x_i) = d_{G''}(y_i) = d_G(v_i) + 1$ and $\varepsilon_{G''}(x_i) = \varepsilon_{G''}(y_i) = \varepsilon_G(v_i) + 1$, for $i = 1, 2, \dots, n$.

$$\begin{aligned} \xi^c(G'') &= \sum_{i=1,2,\dots,n} d_{G''}(x_i)\varepsilon_{G''}(x_i) + \sum_{i=1,2,\dots,n} d_{G''}(y_i)\varepsilon_{G''}(y_i) \\ &= 2\left[\sum_{\varepsilon(v_i) \geq 2} (d(v_i) + 1)(\varepsilon(v_i) + 1) + \sum_{\varepsilon(v_i) = 1} 2(d(v_i) + 1)(\varepsilon(v_i) + 1) \right] \\ &\geq 2\left[\sum_{\varepsilon(v_i) \geq 2} d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i) = 1} (d(v_i) + 1)(\varepsilon(v_i) + 1) \right] \\ &\geq 2\left[\sum_{\varepsilon(v_i) \geq 1} d(v_i)\varepsilon(v_i) + \sum_{\varepsilon(v_i) = 1} d(v_i) \right] + \sum_{\varepsilon(v_i) = 1} \varepsilon(v_i) + \sum_{\varepsilon(v_i) = 1} 1 \\ &\geq 2[\xi^c(G) + (n - 1)\|n - 1\|] + 2\|n - 1\| \\ &\geq 2[\xi^c(G) + n\|n - 1\|]. \end{aligned}$$

The eccentric connectivity polynomial of the extended double cover graph G'' is given by,

$$\begin{aligned} \xi^c(G'', x) &= \sum_{i=1,2,\dots,n} d(x_i)x^{\varepsilon(x_i)} + \sum_{i=1,2,\dots,n} d(y_i)x^{\varepsilon(y_i)} \\ &= 2\left[\sum_{\varepsilon(v_i) \geq 2} (d(v_i) + 1)x^{(\varepsilon(v_i)+1)} + \sum_{\varepsilon(v_i) = 1} (d(v_i) + 1)x^{(\varepsilon(v_i)+1)} \right] \\ &\geq 2x\left[\sum_{\varepsilon(v_i) \geq 2} d(v_i)x^{\varepsilon(v_i)} + \sum_{\varepsilon(v_i) = 1} d(v_i)x^{\varepsilon(v_i)} \right] + 2x\left[\sum_{\varepsilon(v_i) \geq 1} x^{\varepsilon(v_i)} \right] \\ &\geq 2x[\xi^c(G, x) + x\|n - 1\|]. \end{aligned}$$

REFERENCES

- [1] Alaeiyan, M., Asadpour, J., and Mojarad, R. (2013). *A Study of an Infinite Family of Carbon Nanocones by the Eccentricity Connectivity Polynomial*. Fullerenes, Nanotubes and Carbon Nanostructures, 21(10), 849–860.
- [2] Alon, N., Eigenvalues and expanders, Combinatorica 6(1986) 8396.
- [3] Ashrafi, A., Ghorbani, M., and Hossein-Zadeh, M. (2011). *The eccentric connectivity polynomial of some graph operations*. Serdica Journal of Computing, 5(2), pp. 101–116.
- [4] Das, K. C., and Trinajstić, N. (2011). *Relationship between the eccentric connectivity index and Zagreb indices*. Computers and Mathematics with Applications, 62(4), 1758–1764.
- [5] De, N. (2012). *On eccentric connectivity index and polynomial of thorn graph*. Applied Mathematics, 3, 931.
- [6] Dolic, T. O. M. I. S. L. A. V., Saheli, M., and Vukic'evic, D. A. M. I. R. (2010). *Eccentric connectivity index: extremal graphs and values*. Iranian J. Math. Chem, 1(2), 45 – 56.
- [7] Ghorbani, M., and Iranmanesh, M. A. (2013). *Computing eccentric connectivity polynomial of fullerenes*. Fullerenes, Nanotubes and Carbon Nanostructures, 21(2), 134–139.
- [8] Gutman, I. (2011). *Eccentric connectivity index of chemical trees*. arXiv preprint arXiv:1104.3206.
- [9] H. Hua, S. Zhang, K. Xu, *Further results on the eccentric distance sum*, Disc. Appl. Math. 160(2012) 170180.
- [10] Ilic, A., Yu, G., and Feng, L. (2011). *On the eccentric distance sum of graphs*. Journal of Mathematical Analysis and Applications, 381(2), 590–600.
- [11] Morgan, M. J., Mukwembi, S., and Swart, H. C. (2011) *On the eccentric connectivity index of a graph*. Discrete Mathematics, 311(13), 1229–1234.
- [12] Yarahmadi, Z., Moradi, S., and Dolic, T. (2014) *Eccentric connectivity index of graphs with subdivided edges*. Electronic Notes in Discrete Mathematics, 45, 167–176.
- [13] Yu, G., Feng, L., and Ilic, A. (2011). *On the eccentric distance sum of trees and unicyclic graphs*. Journal of Mathematical Analysis and Applications, 375(1), 99–107.