

A $GI^{(A, B/(A, Q))}/D/1/Q_{\max}$ Queue Approach for the Estimation of the Proportion of a Deteriorated Length of a Road Network

Abstract

This article proposes a simple methodology for the classification of road segments of a highway into a set $S=\{0, 1, 2, \dots, Q_{\max}\}$ of states. A $GI^{(A, B/(A, Q))}/D/1/Q_{\max}$ queueing approach, estimates the length of a deteriorated/damaged portion of a road network which requires reconstruction for a better riding comfort which is measured in terms of multiples of the service period. One unit of service time is defined by a pavement management system (PMS) monitoring the queue aspects as an hour or a day needed to reconstruct a length of 'D' kilometres (km) of a damaged portion of a road segment; i.e. the service time here is deterministic. The entire coded length of the highway in terms of the maximum discrete service period is $Q_{\max}$ which means that the real length of the highway under study is $(DQ_{\max})$ km. Here, the inter-arrival time 'A' between consecutive batch arrival of damages and the size of a number 'B' of units of damaged portion arrived are measured as multiples of the single service time. Treating the sequence $\{t_n; n \in [0, 1, 2, \dots]\}$ of arrival epochs as renewal points of the queue occupancy process $Q(t)$, the queueing number of damaged units of road length at time 't', this queueing problem is investigated further using the embedded Markov Chain (MC) methods.

Key words: Pavement structure, queue occupancy, state stationary distribution.

1. INTRODUCTION

A pavement management system (PMS) provides procedures to monitor conditions of road lengths that deteriorate due to alligator cracking, rutting, corrugation and ravelling once the road is opened to traffic. A major assumption is that a road is considered to have deteriorated and to have reached its service life limit when its roughness reaches a specified trigger value.

Generally a new pavement structure of road networks deteriorates gradually once each road in good condition opens to traffic. The rate of pavement distress, due to alligator cracking, rutting, corrugation and ravelling, increases along different road segments of highways with increases in cumulative passage of vehicle traffic. The traffic users on such bad and damaged road segments would feel an increasing roughness and reduced riding comfort and safety. In order to provide the best riding comfort and low pavement distress at an acceptable level, there must be a well-equipped PMS.

A classical service facility of our everyday life is where successive customers arrive at random epochs, possibly have to wait for service in a queueing area, are then served and

40 finally leave the system. But telephone networks and call centres do not fit into that classical
41 framework. Hence mathematicians and computer scientists have developed models and a
42 theory to open complex and discrete time Markov chain (DTMC) models to other areas of
43 science.

44 Most of the discrete time queueing models emerge as an effective technique for Broadband
45 Integrated Services Digital Networks (B-ISDN) that operate on Asynchronous Transfer Mode
46 (ATM) switches. These ATM switches are on the cell level and are modelled by discrete time
47 queues as they are assumed to work synchronously on the basis of the smallest time unit. The
48 time scale is prescribed by the time needed to transmit just one cell. Because of this discrete
49 time development, researchers have dedicated their work on discrete time queueing systems
50 to the fields of computer science, electrical engineering, operations research and
51 mathematics. In particular DTMCs are often the basis for performance evaluation in today's
52 telecommunication and manufacturing systems. PMS is another branch of technical
53 development where slotted time mechanisms can be successfully utilized.

54 The Markov Chain methods that are applied by Tjan et al. (2005), Butt et al. (1994) and
55 Ortiz-Garcis et al. (2006) have been used as a tool of some PMS to predict network
56 deterioration over a period of time.

57 The success of a PMS depends on its ability to predict future pavements conditions of road
58 segments of highways under investigation and its capability to provide reconstruction process
59 towards better riding comfort for road users. If one treats the events of deterioration as
60 departure and the reconstruction as arrival, then observed data with the passage of time on the
61 length of a damaged portion of a road can be viewed as the queue length process of an
62 appropriate queueing model. This provides more scope for the estimation of future pavement
63 conditions of road segments using the method of queueing theory. In this paper, the authors
64 consider one such queueing methodology for the prediction of road network deterioration
65 over a period of time.

66 In section 2 we formulate an embedded MC representing the number of damaged units of
67 road length from the data bases of a well-equipped PMS which monitors pavement distress
68 conditions of road segments of a highway continuously. Section 3 discusses the issues about
69 the determination of the state transition probability matrix via the state specification transition
70 function. Section 4 deals with the numerical evaluation of the state stationary distribution of
71 the MC and also discusses ways of estimating the proportion of damaged road of the highway
72 under investigation. In section 5 is the conclusion as well as some scope for future reaserch.

74 2. FORMULATION OF AN EMBEDDED MARKOV CHAIN

76 Consider a PMS which monitors pavement distress conditions of road surfaces of a highway
77 continuously and keeps data bases of such events. Assume that this PMS uses a simple
78 scaling method for the classification of road segments into a set $S=\{0, 1, 2, \dots, Q_{\max}\}$ of

states of a $GI^{(A, B/(A, Q))}/D/1/Q_{\max}$ queue. It is based on the length of damaged road which requires reconstruction for a better riding comfort and is measured in terms of multiples of service period. The duration of time such as an hour or a day needed to reconstruct a length of 'D' kilometres of damaged portion of a road segment is defined as one unit of service time of a PMS monitoring the queue aspects which ensures that the service time is deterministic. The entire length of the highway coded in terms of the service period is $Q_{\max}$ which means that the real length is $(D Q_{\max})$ km. According to the similar scaling process, the inter-arrival time 'A' between consecutive batch arrival of damages and the size of a number 'B' of damages arrived are measured as multiples of the single service time. Treating the sequence $\{t_n; n \in \{0, 1, 2, \dots\}\}$ of arrival epochs as renewal points of the queue occupancy $Q(t)$ process (i.e. the number of damaged units of road length at time 't' in the queue), then a PMS measures such a queue occupancy value $Q(t_n)$ just before each arrival epoch 't_n'.

Let the sequence $A = \{A_n = t_n - t_{n-1}; n = 1, 2, 3, \dots\}$ of inter arrival times be general independent (GI) and identically distributed random variables. If the random size B_n of the number of deteriorated road length arrived at the end of the inter arrival time A_n is b_n and the queue occupancy just before 't_n' is $Q_n = Q(t_n^-)$ then

$$1 \leq b_n \leq \min\{B_{\max}, B_{n, \max} = Q_{\max} - \max(Q_n - A_{\min}, 0)\} \quad \dots (1)$$

The sequence $\{B_n\}$ is assumed to be independent or conditionally dependent on the queue occupancy value Q_n , $Q_{\max}$ and A_n such that the future status of the resulting embedded process $\{Q_n\}$ depends on the current queue occupancy state, and

$$Q_{n+1} = \max(Q_n - A, 0) + B_n \quad \dots \dots \dots (2)$$

Thus it is seen that the embedded process $\{Q_n\}$ forms a discrete time Markov chain (DTMC) on the state space $S = \{0, 1, 2, \dots, Q_{\max}\}$. Some new aspects concerning this DTMC, subject to a few specific assumptions made on the probability distributions of the input variables A and B (or B_n) and the maximum queue occupancy value $Q_{\max}$, are discussed in the subsequent sections. More details on DTMC and MC can be found in Sivasamy and Pugalenth (2009) and Prabhu et al. (1994).

In this paper, we propose a simple DTMC methodology for the classification of road segments of a highway into a set $S = \{0, 1, 2, \dots, Q_{\max}\}$ of states. We also propose a formula to compute the stationary distribution of the underlying Markov chain. The paper provides a numerical algorithm that has an advantage over the standard numerical algorithms in terms of the computational complexity as in Menth (2004).

3. TRANSITION PROBABILITY MATRIX AND TRANSITION FUNCTIONS

For an arbitrary random variable X, some notations and their meaning are as follows:

$\mathbb{R}$ = Discrete space of X =	$X_{\min}$ = minimum value of X , and $X_{\max}$ = maximum value of X
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$\{X_{\min}, X_{\min} + 1, \dots, X_{\max}\}$	
$x, x[i]$	Probability distribution of X and $\text{Prob}(X = i)$
$\bar{X} = \sum_{i=X_{\min}}^{X_{\max}} i x[i]$	Mean value of X
$X(Y = j)$	Conditional random variable X given $Y = j$
$x(Y = j), x[i](Y = j)$	Distribution and probability conditioned on $Y = j$
$P(Y = j)$	Probability conditioned on $Y = j$
$p = \sum_{j=Y_{\min}}^{Y_{\max}} P(Y = j)Y[j]$	Conditioning the conditional $P(Y = j)$ by y

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118 Determination of transition probabilities (i.e., the probability that a road at a certain level of
119 deterioration will transit to another level of deterioration) of any MC can be made simple if
120 sensitive input variables arising from various scenarios are well examined to know about how
121 they influence such transitions from one state to another at a point of time. In response to this
122 issue, the concept of the specification tool of the MC that has the same modelling power as
123 the traditional state transition matrix is developed in this section.

124 A comprehensive probability argument or algorithm is needed for every practitioner to
125 analyse various aspects of the above Markov model $\{Q_n\}$ formulated on the state space $\mathbb{S}$
126 looking into pavement deterioration of a road network over a period of time. Assuming time
127 homogeneity and denoting the state transition probability from state i just before an arrival
128 epoch t_n to state j just before the next arrival epoch t_{n+1} by $p_{i,j}$, the following statements can be
129 established:

130 $p_{i,j} = P(Q_{n+1}=j/Q_n=i), \text{ for all } i, j \in \mathbb{S} \dots\dots\dots (3)$

131 State transition probability Matrix $P = (p_{i,j}) \dots\dots\dots (4)$

132 Probability distribution $\sum_{j \in \mathbb{S}} p_{i,j} = 1, \text{ } i \in \mathbb{S} \dots\dots\dots (5)$

133 The probability that the system is in state 'i' initially at time t_0 is given by

134 $Q_0[i] = \text{Prob}(Q_0=i) \dots\dots\dots (6)$

The MC $\{Q_n\}$ is completely determined once the statements (3) through (6) are made known by some methodology either from historical data or by investigating the behaviour of internal and external factors which pull or push the transitions of the model.

For easy understating and to keep things simple, a numerical example is considered to illustrate the determination of the model completely. This is done by setting $Q_{\min}=0$ and $Q_{\max}=5$ and using the following unconditional distributions of Table-1 providing the inter-arrival time A and the batch size B of the number arrived at the end of A.

Table-1:Inter arrival time A and batch size B arrived subject to a policy $A_{\min}+B_{\max}\leq Q_{\max}$ forGI ^{(A, B/(A,Q))} /D/ 1/ $Q_{\max}$								
A=i	A _{min} =1	3	A _{max} =4		B=i	B _{min} =1	2	B _{max} =3
a[i]	0.2	0.6	0.2		b[i]	0.6	0.3	0.1

Since $\bar{A} = 2.8$, and $\bar{B} = 1.5$, the offered load of the system is $\rho = \frac{\bar{B}}{\bar{A}} = 0.5357$.

Discrete Service Time: The reconstruction time for a fixed length of ‘D’ km of a road network is measured, in terms of queuing context, as one unit of service time. The associated real time units needed to repair or reconstruct the same length of the damaged road could be an hour or a day or a month. This ordinal scale has been chosen for the sake of discretising the service time and to employ the methodology of GI^{(A, B/(A,Q))}/D/ 1/ $Q_{\max}$ queue for the estimation of the proportion of a deteriorated length of a road network.

Embedded Process $\{(Q_n, t_n): n=0, 1, 2, \dots\}$: Assume that the continuous time queue occupancy process $\{Q(t)\}$ is observed just before and after each discontinuity point of the renewal sequence $\{t_n: n=0, 1, 2, \dots\}$. Since a new arrival of deteriorated size B of random road length occurs at each t_n , this embedding mechanism leads to an SMP (Semi Markov Process) i.e. $\{(Q_n, t_n): n=0, 1, 2, \dots\}$ with an underlying MC $\{Q_n=Q(t_n^-) : n=0, 1, 2, \dots\}$ on the discrete state space $\mathbb{S}=\{0, 1, 2, 3, 4, Q_{\max}=5\}$.

Arrival Process and Input Variables: The nature of state transitions of the MC $\{Q_n\}$ is determined by the magnitudes (multiples of service time) of the two input variables, the inter arrival time ‘A’ and the size B of the batch arrived at the end of ‘A’ and the value of $Q_{\max}=5$. The sequence of inter arrival times $\{A_n = t_n - t_{n-1} : n=1, 2, \dots\}$ is assumed to be general independent (GI) while the observed size b_n of $B=B_n$ of the batch arrived at the end of A_n is independent of Q_n and A if

$$(Q_{\max} - \max(Q_n - A_{\min}, 0) = B_{n,\max}) \geq (B_{\max}=3) \quad \dots \quad (7)$$

For the case of Table-1 and $0 \leq Q_n \leq 3$, A and $B=B_n$ are independent variables since $B_{n,\max}$ is $Q_{\max} - \max(Q_n - A_{\min}, 0) = 5 - 2 = 3$. For either $Q_n=4$ or $Q_n=5$, the condition (7) is violated, so $B=B_n$ is conditionally dependent on the observed values of A_n , $Q_{\max}$ and Q_n as displayed in Table-2.

Table-2: Distribution of $B=B_n$ subject to $A_{\min} + B_{\max} \leq Q_{\max}$ when $Q_{\max}=5$, $A \in \{1,3,4\}$, and $B \in \{1,2, B_{\max}=3\}$ given $0 \leq Q_n \leq Q_{\max}$							
$\text{Prob}(B_n/A, Q_n)$	$Q_n=0$	$Q_n=1$	$Q_n=2$	$Q_n=3$	$Q_n=4$	$Q_n=5$	relationship
$b_n[1](A_{\min}=1, Q_n)$	0.6	0.6	0.6	0.6	0.6/0.9*	1*	Dependent*
$b_n[2](A_{\min}=3, Q_n)$	0.3	0.3	0.3	0.3	0.3/0.9*	---	
$b_n[3](A_{\min}=4, Q_n)$	0.1	0.1	0.1	0.1	----	-----	

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169 **State Transition Function:** Let the joint effect of the input variables be $Y=(A, B_n/(A, Q_n))$
170 and its distribution be denoted by $P(Y=(i, j/(i, q_n)))$:

171
$$P(Y=(i, j/(1, q_n)))=a[i]b[j](i, q_n) \quad \dots \quad (8)$$

172 Let $\mathbb{Y} = \{1=(A_{\min}, B_{\min}), 2=(1+A_{\min}, B_{\min}), 3=(2+A_{\min}, B_{\min}), \dots, k=(A_{\max}, B_{\max})\}$ be the set
173 of all m embers of the product set $(A \times B)$. To describe the transition behaviour of the
174 pavement/queueing system under investigation, define a state transition function $f: \mathbb{S} \times \mathbb{Y} \rightarrow$
175 $\mathbb{S}$ such that

176
$$Q_{n+1} = f(Q_n, (A, B_n/A, Q_n)) = \text{Max}(Q_n - A, 0) + B_n \quad \dots \quad (9)$$

177 Let $Z(Q_n=i, Q_{n+1}=j) = \{k: \text{ranging from } k_{\min} \text{ to } k_{\max}\}$ be the set of pre-images of 'f' falling in
178 $\mathbb{Y}$ such that those values of Q_{n+1} and Q_n satisfy the rule 'f(i, k)=j' defined by (9) and thus
179 $\bigcup_{j \in \mathbb{S}} Z(i, j) = \mathbb{Y}$ for $i \in \mathbb{S}$. The partition sets $Z(i, j)$ of $\mathbb{Y}$ for each given 'i' are reported in
180 Table-3.

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Table-3: Partitions $Z(i, j)$ of $\mathbb{Y}$ for each given $Q_n=i$ which moves to $Q_{n+1}=j$ in one transition, using $B_{n, \max}=2$ when $Q_n=4$ and $B_{n, \max}=1$ when $Q_n=5$.						
$Q_n \backslash Q_{n+1}$	0	1	2	3	4	5
0	$\mathbb{Y} - \{(1,3)\}$					
1	$\mathbb{Y} - \{(1,2), (3,3)\}$		$\{(1,3)\}$			
2	$\mathbb{Y} - \{(1,1), (3,2), (4,1)\}$		$\{(1,2), (3,3)\}$	$\{(1,3)\}$		
3	$\mathbb{Y} - \{(3,1), (4,2)\}$		$\{(1,1), (3,2), (4,1)\}$	$\{(1,2), (3,3)\}$	$\{(1,3)\}$	
4		$\{(4,1^*)\}$	$\{(3,1^*), (4,2^*)\}$	$\{(3,2^*)\}$	$\{(1,1^*)\}$	$\{(1,2^*)\}$
5			$\{(4,1^*)\}$	$\{(3,1^*)\}$		$\{(1,1^*)\}$

182

183 State Transition Matrix P

184 Using the set of probabilities $\{y[j] ; j=1, 2, \dots, k\}$ obtained by (8) that represents the
185 distribution of $Y=(A, B_n/(A, Q_n))$, and the above pre-images $Z(I, j)$ of Table-3, the state
186 transition probabilities of the MC $\{Q_n\}$ can be computed for a given (i,j) and $i, j \in \mathbb{S}$:

187
$$p_{i,j} = P(Q_{n+1}/Q_n=i) = \sum_{\{k \in Z(i, j): f(i, k)=j\}} y[k] \quad \dots \quad (10)$$

These transition probabilities are given in Table-4.

Table-4: State transition Probability matrix $P=(p_{ij})$						
$Q_n \backslash Q_{n+1}$	0	1	2	3	4	5
0	0.98	0.02				
1	0.86	0.12	0.02			
2	0.54	0.32	0.12	0.02		
3	0.12	0.42	0.32	0.12	0.02	
4		1.2/9	4.2/9	1.8/9	1.2/9	0.6/9
5			0.2	0.6		0.2

Thus the above transition matrix is obtained via the state transition function in a systematic way by applying the arguments such as embedded method using renewal points, defining discrete distributions of input variables and discrete service time.

4. THE STATIONARY DISTRIBUTION

Let the stationary distribution of the state transition matrix $P=(p_{ij})$ of Table-4 be denoted by a row vector $\pi=(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ of size 6. Let $\mathbf{e}=(1, 1, 1, 1, 1, 1)'$ be a column vector of unit elements of size 6. Using the standard classification of states of a MC method, it can be shown that all states of the transition matrix P communicate with each other and each state has period unity and so it is an irreducible MC.

The numerical evaluation of π can be done by solving the following matrix equations (11):

$$\pi P = \pi, \quad \pi \mathbf{e} = 1 \quad \dots\dots\dots (11)$$

The following $\pi=(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ is the solution of the equations (11):

$$\pi=(0.97707584, 0.02239864, 0.00051348, 0.00001175, 0.00000027, 0.00000002)$$

From the above numerical value $\pi=(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ obtained, it is concluded that the MC $\{Q_n\}$ will be in state 0 just before an arrival point with probability 0.977075864, (i.e. almost zero unit of service time is needed to reconstruct the current length of the deteriorated road or the deteriorated length =23 km). The Q =length of the deteriorated road is in state 1 (1xD km) just before an arrival point with probability 0.02239864, Q is in state 2 (i.e. 2xD km of road length is in a damaged condition) with probability 0.00051348, Q is in state 3 (i.e. 3xD km of road length is in a damaged condition) with probability 0.00001175, Q is in state 4 (i.e. 4xD km of road length is in a damaged condition) with probability 0.00000027, and Q is in state 5 (i.e. 5xD km of road length is in a damaged condition) with probability 0.00000002 in the long run. If the deterioration process $Q=\{Q_n\}$ is affected according to the input process of Table-1, then in the long run it is expected just before an arrival point that there is a 97.71 % chance that zero number of kilometres of the road network under study would be in a damaged condition while there is a 2.24% confidence that 2xD km would be in a deteriorated condition and so on.

4.1 Finite Quasi Birth and Death Process (QBD):

Most research in DTMC applications (Sivasamy and Pugalenth (2009)) has been done to obtain analytical solutions based on special assumptions in relation to arrival process and service time distribution. If the $Q_{\max}$ value of the above MC $\{Q_n\}$ of the PMS is very large, then using the advantages of matrix analytical methods of Latouche and Ramaswami (1999), stationary distribution of large sized DTMCs can be computed at a faster rate.

Assume that $S = \{0, 1, 2, \dots, Q_{\max}=101\}$ is the state space of the above MC $\{Q_n\}$ and define 'j' level be $l(j) = \{3j, 3j+1, 3j+2\}$ for $j=0, 1, \dots, 33$ so that there will be 34 levels. The initial level $l(0) = \{0, 1, 2\}$ will include the first three states and the last level (34^{th}), $l(33) = \{99, 100, 101\}$ will include the last three states of the MC $\{Q_n\}$ considered on the state space $S = \{0, 1, 2, \dots, Q_{\max}=101\}$. Thus the partitioned form of the corresponding transition matrix P is given by

$$P = \begin{pmatrix} & l(0) & l(1) & l(2) & l(3) & & l(31) & l(32) & l(33) \\ l(0) & B_0 & A_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ l(1) & A_2 & A_1 & A_0 & 0 & \dots & 0 & 0 & 0 \\ l(2) & 0 & A_2 & A_1 & A_0 & \dots & 0 & 0 & 0 \\ l(3) & 0 & 0 & A_2 & A_1 & \ddots & 0 & 0 & 0 \\ & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ l(32) & 0 & 0 & 0 & 0 & \dots & A_2 & A_1 & A_0 \\ l(33) & 0 & 0 & 0 & 0 & 0 & \dots & B_2 & B_1 \end{pmatrix} \dots \quad (12)$$

where B_0, B_1, B_2, A_0, A_1 and A_2 represent square matrices of order 3 as defined below:

$$B_0 = \begin{pmatrix} 0.98 & 0.02 & 0 \\ 0.86 & 0.12 & 0.02 \\ 0.54 & 0.32 & 0.12 \end{pmatrix}, B_1 = \begin{pmatrix} 0.12 & 0.02 & 0 \\ 1.8/9 & 1.2/9 & .6/9 \\ 0 & 0 & 0.2 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0.12 & 0.42 & 0.32 \\ 0 & 1.2/9 & 4.2/9 \\ 0.6 & 0 & 0.2 \end{pmatrix}$$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.02 & 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0.12 & 0.02 & 0 \\ 0.32 & 0.12 & 0.02 \\ 0.42 & 0.32 & 0.12 \end{pmatrix}, \text{ and } A_2 = \begin{pmatrix} 0.12 & 0.42 & 0.32 \\ 0 & 0.12 & 0.42 \\ 0 & 0 & 0.12 \end{pmatrix}.$$

Linear Level Reduction: In order to compute the stationary distribution of (12), linear level reduction (LLR) method discussed by Latouch and Ramaswami (1999) which is inspired from Gaussian elimination and proceeds in two phases may be used when the partitioned

state space $S = S_0 = \bigcup_{i=1}^M l(i)$ is restricted to sets $S_{(i)} = \bigcup_{i=1}^M l(i)$ for $i=1, 2, \dots, M$ where M is finite.

During the first phase, we construct restricted process 'i' with transition matrix P(i) on the state space $S_{(i)}$ so that for $i=0, 1, 2, \dots, (M-1)$

$$P_{(i)} = \begin{pmatrix} & l(i) & l(i+1) & l(i+2) & l(i+3) & & l(31) & l(32) & l(M=33) \\ l(i) & C_i & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ l(i+1) & A_2 & A_1 & A_0 & 0 & \cdots & 0 & 0 & 0 \\ l(i+2) & 0 & A_2 & A_1 & A_0 & \cdots & 0 & 0 & 0 \\ l(i+3) & 0 & 0 & A_2 & A_1 & \ddots & 0 & 0 & 0 \\ & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ l(32) & 0 & 0 & 0 & 0 & \cdots & A_2 & A_1 & A_0 \\ l(M=33) & 0 & 0 & 0 & 0 & 0 & \cdots & B_2 & B_1 \end{pmatrix} \quad \dots \quad (13)$$

and $P_{(M)}=C_M$, where the matrix C_i records the probability of returning to the level $l(i)$ starting from the same level $l(i)$ before reaching the level $l(i+1)$; the matrix $(I-C_i)^{-1} A_0$ records the first passage probabilities from level $l(i)$ to $l(i+1)$. Further for $0 \leq i \leq (M-1)$, matrices C_i are recursively defined by

$$C_0=B_0, C_i= A_1+ A_2 (I-C_{i-1})^{-1}A_0 \text{ for } 1 \leq i \leq M-1 \text{ and } C_M=B_1+A_2 (I-C_{M-1})^{-1}A_0 \quad \dots \quad (14)$$

The stationary vector $\boldsymbol{\pi}=(\pi_0, \pi_1, \dots, \pi_{31}, \pi_{32}, \pi_{33})$ is determined by the equations

$$\boldsymbol{\pi}_M = \boldsymbol{\pi}_M C_M, \pi_i = \pi_{i+1} A_2 (I-C_i)^{-1} \text{ for } 0 \leq i \leq M-1 \text{ and } \sum_{0 \leq i \leq M} \pi_i \mathbf{e} = 1 \quad \dots \quad (15)$$

The stationary distribution so computed by the linear level reduction (15) is reported in Table-5.

Table - 5: The stationary distribution $\pi=(\pi_0, \pi_1, \dots \pi_{31}, \pi_{32}, \pi_{33})$, when $Q_{\max}=101$

π_0	0.976699249 0.022390009 0.000513268995	π_{17}	2.31590394E-084 5.3090146E-086 1.21704685E-087
π_1	1.17662591E-005 2.69731574E-007 6.183369E-009	π_{18}	2.78997733E-089 6.39578789E-091 1.46618047E-092
π_2	1.41748523E-010 3.24946542E-012 7.44912561E-014	π_{19}	3.36109516E-094 7.70502737E-096 1.76631259E-097
π_3	1.70764926E-015 3.91464201E-017 8.97398689E-019	π_{20}	4.04912274E-099 9.28227262E-101 2.12788277E-102
π_4	2.05721086E-020 4.71598251E-022 1.08109924E-023	π_{21}	4.8779919E-104 1.11823853E-105 2.56346759E-107
π_5	2.4783289E-025 5.68136013E-027 1.30240393E-028	π_{22}	5.87653339E-109 1.34714575E-110 3.0882181E-112
π_6	2.98565124E-030 6.84435383E-032 1.56901043E-033	π_{23}	7.07947971E-114 1.62291106E-115 3.72038682E-117
π_7	3.59682417E-035 8.24541627E-037 1.89019219E-038	π_{24}	8.52867322E-119 1.95512646E-120 4.48196262E-122
π_8	4.33310628E-040 9.93328096E-042 2.27712094E-043	π_{25}	1.02745216E-123 2.35534747E-125 5.39943558E-127
π_9	5.22010784E-045 1.19666573E-046 2.74325532E-048	π_{26}	1.2377751E-128 2.83749509E-130 6.50471835E-132
π_{10}	6.28868162E-050 1.44162726E-051 3.30480899E-053	π_{27}	1.49115186E-133 3.41833996E-135 7.83625627E-137
π_{11}	7.57599609E-055 1.73673325E-056 3.98131461E-058	π_{28}	1.79639571E-138 4.11808575E-140 9.44036451E-142
π_{12}	9.12682819E-060 2.09224844E-061 4.79630321E-063	π_{29}	2.164124E-143 4.96107188E-145 1.13728392E-146
π_{13}	1.09951209E-064 2.52053879E-066 5.77812275E-068	π_{30}	2.60712752E-148 5.97662012E-150 1.37008979E-151
π_{14}	1.32458594E-069 3.03650164E-071 6.96092408E-073	π_{31}	3.14081537E-153 7.20005453E-155 1.65055182E-156
π_{15}	1.59573318E-074 3.65808382E-076 8.38584885E-078	π_{32}	3.78375094E-158 8.67392862E-160 1.98846952E-161
π_{15}	1.92238517E-079 4.40690599E-081 1.01024606E-082	π_{33}	4.54925641E-163 1.04580607E-164 9.15080312E-166
		sum= 0.999614569	

For a different application of discrete time queues together with matrix analytic methods, one may refer to Bindu Abraham and Jose (2003), in which they analysed DAR(1)/D/s queue with discrete autoregressive arrivals having Discrete Mittag-Leffler as marginal distribution and obtained the stationary distribution of the system size and the waiting time distribution of an arbitrary packet.

5. CONCLUSION AND FUTURE SCOPE

The above analysis gives a discrete time queuing approach to estimate the proportion of deteriorated road length of a pavement management problem using embedded method of formulating DTMC. It is easy to apply to determine the state transition probability matrix of the DTMC via a state transition function by paying attention on its pre-images associated with 'from and to states' of the MC. There is greater scope to extend the above methodology to estimate similar proportions of other branches of human activity such as spacer traffic of a network, multiplexing Voice over ATM (asynchronous transfer mode)/IP (Internet Protocol) etc.

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