

Nontrivial weak solutions of a quasilinear equation involving p -Laplace operator

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Abstract. In the present paper, using direct variational approach, and the monotone operator method, the existence of nontrivial solutions for a quasilinear elliptic equation involving the p -Laplacian is obtained.

Keywords. p -Laplacian; variational approach; monotone operator method; quasilinear elliptic equation; Sobolev spaces

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1 Introduction

We study the existence and nontrivial solutions of the Dirichlet boundary problem

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (\mathbf{P})$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $1 < p < N$.

The importance of problem $(\mathbf{P})$ arises from the existence of the p -Laplacian or the p -Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. Obviously, when $p = 2$, $\Delta_2 = \Delta$ is the usual Laplace operator. However, in case $p \neq 2$ the situation is very crucial, as for example, one encounters the lack of the Hilbert structure of the space $W_0^{1,p}(\Omega)$ when passing from $p = 2$ to $p \neq 2$.

It is well known that problems involving the p -Laplacian appear in many contexts. Some of these problems come from different areas of applied mathematics and physics. For example, they may be found in the study of non-Newtonian fluids, nonlinear elasticity, and reaction diffusions. For discussions about problems modelled by these boundary value problems, see, for example, [4]. One of the most widely used results for solving problem $(\mathbf{P})$ is the mountain pass theorem. In order to apply this theorem, it is necessary that the Euler-Lagrange functional associated to the problem has the Palais-Smale property. One way to ensure this is to assume that f satisfies some Ambrosetti-Rabinowitz-type condition (see, e.g., [2, 6]). Another technique used for obtaining solutions of problem $(\mathbf{P})$ is the blowup method due to Gidas and Spruck [8]. In order to use any of the techniques above, it is necessary that the nonlinearity f has subcritical growth.

The object of this paper is to study problem $(\mathbf{P})$ for nonlinearities f which do not necessarily satisfy the classical conditions, such as Ambrosetti-Rabinowitz condition. Instead, they must satisfy some specific conditions.

$L^p(\Omega)$, for $1 \leq p < \infty$, is the Lebesgue space of measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that $\int_{\Omega} |u|^p dx < +\infty$, while $L^\infty(\Omega)$ is the space of measurable functions such that $\operatorname{esssup}_{x \in \Omega} |u(x)| < +\infty$. Let define the norms

$$|u|_p = \left(\int_{\Omega} |u(x)|^p dx \right)^{1/p} \quad \text{and} \quad |u|_\infty = \operatorname{esssup}_{x \in \Omega} |u(x)|,$$

which make $L^p(\Omega)$ and $L^\infty(\Omega)$ Banach spaces, respectively. Let $W^{1,p}(\Omega)$ be the usual Sobolev space, i.e., $W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)\}$ which is endowed with norm

$$\|u\| = \left(\int_{\Omega} (|\nabla u(x)|^p + |u(x)|^p) dx \right)^{1/p}. \quad (1.1)$$

Then $W^{1,p}(\Omega)$ is a Banach space. $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. In $W_0^{1,p}(\Omega)$ we use the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_{\Omega} |\nabla u(x)|^p dx \right)^{1/p}. \quad (1.2)$$

Thanks to the Poincaré inequality, it is not difficult to see that (1.1) and (1.2) are equivalent. Therefore, in the sequel the norm in $W_0^{1,p}(\Omega)$ will be denoted by $\|\cdot\|$.

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Let Ω be an open and bounded subset of $\mathbb{R}^N$, with $N \geq 3$. Then $W_0^{1,p}(\Omega)$ is embedded continuously in $L^q(\Omega)$, denoted by $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, for every $q \in [1, p^*]$, where $p^* = Np/N - p$. The embedding is compact if and only if $q \in [1, p^*)$ (see [1]).

$u \in W_0^{1,p}(\Omega)$ is a weak solution of $(\mathbf{P})$ if any $\phi \in W_0^{1,p}(\Omega)$,

$$\int_{\Omega} \left((|\nabla u|^{p-2} \nabla u, \nabla \phi) + |u|^{p-2} u \phi \right) dx - \int_{\Omega} f(x, u) \phi dx = 0, \quad (1.3)$$

where “ $(\cdot, \cdot)$ ” is the standard inner product in $\mathbb{R}^N$. The energy functional corresponding to problem $(\mathbf{P})$ is defined as $J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$,

$$J(u) = \frac{1}{p} \int_{\Omega} (|\nabla u|^p + |u|^p) dx - \int_{\Omega} F(x, u) dx,$$

where $F(x, t) = \int_0^t f(x, s) ds$. By using the condition $(\mathbf{f}_1)$ (see below) together with standard arguments, one can easily shows that J is well-defined on $W_0^{1,p}(\Omega)$ and is of class $C^1(W_0^{1,p}(\Omega), \mathbb{R})$. Moreover, the derivative of J is the mapping $J' : W_0^{1,p}(\Omega) \rightarrow (W_0^{1,p}(\Omega))^*$ given by the formula

$$\langle J'(u), v \rangle = \int_{\Omega} \left((|\nabla u|^{p-2} \nabla u, \nabla v) + |u|^{p-2} uv \right) dx - \int_{\Omega} f(x, u) v dx,$$

for any $u, v \in W_0^{1,p}(\Omega)$. From the variational setting of problem $(\mathbf{P})$, i.e. (1.3), and definition of the derivative of J , it is obvious that weak solutions of $(\mathbf{P})$ correspond to critical points of J .

Let λ_1 be the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$, that is,

$$\lambda_1 = \inf_{0 \neq u \in W_0^{1,p}(\Omega)} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx},$$

it is well known that $\lambda_1 > 0$ (see [3]).

2 Variational approach

In this section, we give the first result of the present paper, based on the variational approach.

Theorem 1 Assume that $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and for all $x \in \Omega$, such that

$$\limsup_{|t| \rightarrow \infty} \frac{f(x, t)}{|t|^{p-1}} < \lambda_1 \text{ uniformly a.e. } x \in \Omega. \quad (\mathbf{f}_1)$$

Then problem $(\mathbf{P})$ has at least one solution. If in addition f also satisfies

$$\liminf_{t \rightarrow 0^+} \frac{f(x, t)}{t^{p-1}} > \lambda_1 \text{ uniformly a.e. } x \in \Omega, \quad (\mathbf{f}_2)$$

then problem $(\mathbf{P})$ has at least one nontrivial solution.

To obtain the result of Theorem 1, we need the following lemma.

(i) The functional J is coercive.

(ii) The functional J is weakly lower semicontinuous.

Proof. (i) From $(\mathbf{f}_1)$, there exists $a > 0$ and $b \in (0, \lambda_1)$ such that

$$|f(x, t)| \leq a + b |t|^{p-1} \quad \forall t \in \mathbb{R}.$$

Integrating, we get

$$|F(x, t)| \leq a |t| + \frac{b}{p} |t|^p \quad \forall t \in \mathbb{R}. \quad (2.1)$$

Then, using the definition of the first eigenvalue and Poincaré inequality, we have

$$\left| \int_{\Omega} F(x, u) dx \right| \leq a \int_{\Omega} |u| dx + \frac{b}{p} \int_{\Omega} |u|^p dx \leq c \|u\| + \frac{b}{p\lambda_1} \|u\|^p.$$

Therefore,

$$\begin{aligned} J(u) &= \frac{1}{p} \|u\|^p - \int_{\Omega} F(u) dx \geq \frac{1}{p} \|u\|^p - c \|u\| - \frac{b}{p\lambda_1} \|u\|^p \\ &\geq \frac{1}{p} \left(1 - \frac{b}{\lambda_1} \right) \|u\|^p - c \|u\|. \end{aligned}$$

Since $\lambda_1 > b$, J is coercive.

(ii) Let $\{u_k\} \subset W_0^{1,p}(\Omega)$ be a minimizing sequence for J . Since J is coercive, $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exists $u \in W_0^{1,p}(\Omega)$ such that passing to a subsequence, still denoted by $\{u_k\}$, we have

$$u_k \rightharpoonup u \text{ in } W_0^{1,p}(\Omega);$$

$$u_k \rightarrow u \text{ in } L^p(\Omega);$$

$$u_k(x) \rightarrow u(x) \text{ a.e. in } \Omega;$$

and there exists $w \in L^p(\Omega)$ such that $|u_k(x)| \leq w(x)$ a.e. in Ω and for all $k \in \mathbb{N}$.

Since F is continuous, we have $F(x, u_k(x)) \rightarrow F(x, u(x))$ a.e. in Ω . Moreover, from (2.1), we also have

$$|F(x, u_k(x))| \leq a |u_k(x)| + \frac{b}{p} |u_k(x)|^p \leq c(|w(x)| + |w(x)|^p) \in L^1(\Omega)$$

a.e. in Ω and for all $k \in \mathbb{N}$. Consequently, by the dominated convergence theorem, we obtain that

$$\int_{\Omega} F(x, u_k) dx \rightarrow \int_{\Omega} F(x, u) dx. \quad (2.2)$$

On the other hand, since the norm $\|\cdot\|$ is weakly lower semicontinuous we have

$$\|u\|^p \leq \liminf_k \|u_k\|^p. \quad (2.3)$$

Thus, considering the relations (2.2) and (2.3), we get

$$\begin{aligned} J(u) &= \frac{1}{p} \|u\|^p - \int_{\Omega} F(u) dx \\ &\leq \liminf_k \frac{1}{p} \|u_k\|^p - \lim_k \int_{\Omega} F(x, u_k) dx \\ &\leq \liminf_k \left(\frac{1}{p} \|u_k\|^p - \int_{\Omega} F(x, u_k) dx \right) = \liminf_k J(u_k). \end{aligned}$$

Therefore J is weakly lower semicontinuous. ■

Proof of Theorem 1. From Lemma 2, we know that the functional J is coercive and weakly lower semicontinuous. Therefore, it has a global minimum u on $W_0^{1,p}(\Omega)$, which is a critical point (see [10]). We now show that under condition (f_2) , u is not identically zero. From (f_2) , there exists $\mu > \lambda_1$ and $\delta > 0$ such that

$$f(x, t) \geq \mu t^{p-1} \quad \forall t \in [0, \delta],$$

and hence

$$F(x, t) \geq \frac{\mu}{p} t^p \quad \forall t \in [0, \delta].$$

Let φ_1 be the first eigenfunction corresponding to λ_1 . Since $\varphi_1 \in L^\infty(\Omega)$, there exists $t > 0$ sufficiently small such that $t\varphi_1(x) < \delta$ for a.e. $x \in \Omega$. Therefore,

$$\begin{aligned} J(t\varphi_1) &= \frac{1}{p} \|t\varphi_1\|^p - \int_{\Omega} F(t\varphi_1) dx \leq \frac{t^p}{p} \|\varphi_1\|^p - \frac{\mu t^p}{p} \int_{\Omega} |\varphi_1|^p dx \\ &= \frac{t^p \lambda_1}{p} \int_{\Omega} |\varphi_1|^p dx - \frac{t^p \mu}{p} \int_{\Omega} |\varphi_1|^p dx = \frac{t^p}{p} (\lambda_1 - \mu) \int_{\Omega} |\varphi_1|^p dx < 0. \end{aligned}$$

Further, let u be the solution that minimizes J . Then

$$J(u) = \min_{v \in W_0^{1,p}(\Omega)} J(v) \leq J(t\varphi_1) < 0.$$

Since $J(0) = 0$, it follows that $u \neq 0$. This completes the proof. ■

3 Monotone operator method

In this section, we use another method, named monotone operator method, for problem **(P)**. Therefore, we recall the following theorem, which is the key tool to get our second result.

Lemma 2 (Browder) *Let X be a reflexive real Banach space. Moreover, let $T : X \rightarrow X^*$ be an operator satisfying the conditions:*

- (i) T is bounded;
- (ii) T is demicontinuous;
- (iii) T is coercive;
- (iv) T is monotone on the space X , i.e., for all $u, v \in X$ we have

$$\langle T(u) - T(v), u - v \rangle \geq 0. \quad (3.1)$$

Then the equation

$$T(u) = h^* \quad (3.2)$$

has at least one solution $u \in X$ for every $h^* \in X^*$. If, moreover, the inequality (3.1) is strict for all $u, v \in X$, $u \neq v$, then the equation (3.2) has precisely one solution $u \in X$ for every $h^* \in X^*$.

We now proceed for the next result of the present paper. For this, we will establish the operator equations corresponding to problem **(P)**.

Let us define the operators $J, F : W_0^{1,p}(\Omega) \rightarrow (W_0^{1,p}(\Omega))^*$ by

$$\langle J(u), v \rangle = \int_{\Omega} (|\nabla u|^{p-2} (\nabla u, \nabla v) + |u|^{p-2} uv) dx \quad \forall u, v \in W_0^{1,p}(\Omega),$$

$$\langle F(u), v \rangle = \int_{\Omega} f(x, u) v dx \quad \forall u, v \in W_0^{1,p}(\Omega),$$

and set

$$T := J - F.$$

Then, from the monotone operator theory, the solution function $u \in W_0^{1,p}(\Omega)$ of **(P)** satisfying the operator equation

$$T(u) := J(u) - F(u) = 0 \quad (3.3)$$

is also the solution of the integral equation

$$\int_{\Omega} (|\nabla u|^{p-2} (\nabla u, \nabla v) + |u|^{p-2} uv) dx - \int_{\Omega} f(x, u) v dx = 0 \quad \forall v \in W_0^{1,p}(\Omega). \quad (3.4)$$

Namely, the existence of weak solution of problem **(P)** is equivalent to the existence of solution of the operator equation (3.3) (see [5, 7]).

Theorem 3 *Assume that the following assertions holds:*

- (f₃) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, and assume that there exist $c_1, c_2 > 0$ such that

$$|f(x, t)| \leq c_1 + c_2 |t|^{q-1},$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$, where $1 < q < p^*$, with $p > q$;

- (f₄) $f(x, 0) = 0$ and $(f(x, t) - f(x, s))(t - s) \leq 0$ for all $s, t \in \mathbb{R}$ and a.e. $x \in \Omega$.

Then problem **(P)** has an unique solution.

Proof. It is obvious from $(\mathbf{f}_3)$ that T is well defined, bounded and continuous (and hence demicontinuous).

From $(\mathbf{f}_4)$, for sufficiently large $\|u\|$, we have

$$\langle T(u), u \rangle = \int_{\Omega} (|\nabla u|^p + |u|^p) dx - \int_{\Omega} f(x, u) u dx \geq m_0 \|u\|^p.$$

This shows that T is coercive. Let us show the monotonicity of T . If $u = v$, the assertion of theorem is obvious. For the case $u \neq v$, from $(\mathbf{f}_4)$, it reads

$$\begin{aligned} \langle T(u) - T(v), u - v \rangle &\geq \int_{\Omega} \left(|\nabla u|^{p-2} (\nabla u, \nabla u - \nabla v) + |u|^{p-2} u (u - v) \right) dx \\ &\quad - \int_{\Omega} \left(|\nabla v|^{p-2} (\nabla v, \nabla u - \nabla v) + |v|^{p-2} v (u - v) \right) dx \\ &= \int_{\Omega} \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla u - \nabla v \right) dx \\ &\quad + \int_{\Omega} \left(|u|^{p-2} u - |v|^{p-2} v \right) (u - v) dx. \end{aligned}$$

Now, we apply the following well-known vectorial inequality:

For all $\xi, \eta \in \mathbb{R}^N$, it holds [9]

$$\begin{aligned} \left(|\xi|^{r-2} \xi - |\eta|^{r-2} \eta, \xi - \eta \right) &\geq 2^{1-r} |\xi - \eta|^r, \quad r \geq 2, \\ \left(|\xi|^{r-2} \xi - |\eta|^{r-2} \eta, \xi - \eta \right) &\geq (r-1) \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-r}}, \quad 1 < r < 2, \end{aligned}$$

Then, we get

$$\langle T(u) - T(v), u - v \rangle \geq 2^{1-p} \int_{\Omega} (|\nabla u - \nabla v|^p + |u - v|^p) dx > 0, \quad p \geq 2, \quad (3.5)$$

$$\langle T(u) - T(v), u - v \rangle \geq (p-1) \int_{\Omega} \left(\frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p}} + \frac{|u - v|^2}{(|u| + |v|)^{2-p}} \right) dx > 0, \quad 1 < p < 2, \quad (3.6)$$

This implies the monotonicity of T . As a consequence of Lemma 3, the equation

$$T(u) = J(u) - F(u) = h^*$$

has at least one solution $u \in W_0^{1,p}(\Omega)$ for every $h^* \in (W_0^{1,p}(\Omega))^*$. Moreover, since inequalities (3.5), (3.6) are strict, it follows then from Lemma 3 that there is a unique solution of (3.3), which in turn is a unique weak solution of $(\mathbf{P})$. ■

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