

Chebyshev polynomial solutions of twelfth-order boundary-value problems

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Abstract

In this paper, Chebyshev method is applied to solve twelfth-order boundary-value problems. The numerical results obtained with minimum amount of computation are compared with differential transformation method to show the efficiency of the method. The results show that the Chebyshev method is high accuracy, more convenient and efficient for solving twelfth-order boundary-value problems.

Keywords: Chebyshev polynomials, Twelfth-order, Nonlinear, Numerical solution.

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1 Introduction

Twelfth-order boundary-value problem arises in an infinite horizontal layer of fluid is heated from below, with the assumption that a uniform magnetic field is also applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, instability sets in. When instability sets in as ordinary convection, it is modelled by tenth- order boundary-value problems, when instability sets in as over stability, it is modelled by twelfth-order boundary-value problems [(1)]. Agarwal's book [(2)] contains theorems which detail the conditions for existence and uniqueness of solutions of the higher-order boundary-value problems.

Many numerical methods have been developed for solving the twelfth-order boundary-value problem (1.1) numerically. Each of these methods has its inherent advantages and disadvantages and the search for alternative, more general, easier and more accurate methods is a continuous and ongoing process. Next, we present a selective review of the main and the most recent methods. These methods include: finite difference method [(3; 4)], differential transform method [(5)], modified decomposition method [(6)], splines [(7; 8)], non-polynomial spline technique [(9)] and variational iteration method [(10)].

In this study we consider the twelfth-order boundary-value problem of the type:

$$u^{(12)}(x) + q(x)u(x) = \eta(x, u), \quad x \in J = [a, b] \quad (1.1)$$

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subject to the boundary conditions

$$u^{(2n)}(a) = \alpha_{2n}, \quad u^{(2n)}(b) = \beta_{2n}, \quad n = 0, 1, \dots, 5. \quad (1.2)$$

where α_{2n} and β_{2n} ($n = 0, 1, 2, \dots, 5$) are finite real constants and the functions $q(x)$ and $\eta(x, u)$ are continuous on $[a, b]$.

In recent years, a lot of attention has been devoted to the study of the Chebychev method to investigate various scientific models. It is possible to solve linear differential equations [(11)], nonlinear [(12; 13)], integral equations [(14)], second and fourth-order elliptic equations [(15)], nonlinear system of second-order boundary value problems [(16)], fourth-order Sturm-Liouville problems [(17)], Troesch's problem [(18)], linear partial differential equations [(19)], nonlinear optimal control problems [(20)] and integro differential equations [(21)] by using this method.

The organization of the paper is as follows. Section 2 contains notation, definitions and some results of Chebychev polynomial. In Section 3 the Chebychev is developed for linear twelfth-order boundary value problems. The method is developed for nonlinear twelfth-order boundary value problems in Section 4. Some numerical examples are presented in Section 5. Finally, Section 6 provides conclusions of the study.

2 Some Properties of Chebychev Polynomials

The well known Chebychev polynomials are defined on the interval $[-1, 1]$ and the following definitions are necessary for this step [(22)]

Definition 2.1. Chebychev polynomial of degree n is defined as

$$T_n(x) = \cos(n \arccos(x)), \quad n = 0, 1, \dots, \quad x \in [-1, 1]$$

or, in a more instructive form,

$$T_n(x) = \cos n\theta, \quad x = \cos \theta, \quad \theta \in [0, \pi]$$

One sees at once that, on $[-1, 1]$,

1 T_n takes its maximal value with alternating signs $(n + 1)$ times:

$$\|T_n\| = 1, \quad T_n(x_k) = (-1)^k,$$

$$x_k = \cos\left(\frac{\pi k}{n}\right), \quad k = 0, 1, \dots, n \quad (2.1)$$

2 T_n has n distinct zeros:

$$T_n(t_k) = 0, \quad t_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), \quad k = 1, 2, \dots, n.$$

Lemma 2.1. Chebychev polynomials T_n satisfy the recurrence relation

$$\begin{aligned} T_0(x) &= 1, & T_1(x) &= x, \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), & n &\geq 1. \end{aligned}$$

In particular, T_n is indeed an algebraic polynomial of degree n with the leading coefficient 2^{n-1} . Also, orthogonality

$$\begin{aligned} \int_{-1}^1 T_m(x) T_n(x) \frac{dx}{\sqrt{1-x^2}} &= 0, & n &\neq m \\ \int_{-1}^1 T_n^2(x) \frac{dx}{\sqrt{1-x^2}} &= \begin{cases} \frac{\pi}{2}, & n > 0; \\ \pi, & n = 0. \end{cases} \end{aligned} \quad (2.2)$$

In this paper we use orthonormal Chebychev polynomials, noting property (2.2).

Theorem 2.2. *On the interval $[-1, 1]$, among all polynomials of degree n with leading coefficient $a_n = 1$, the Chebychev polynomial $\frac{1}{2^{n-1}}T_n$ deviates least from zero, i.e.,*

$$\inf_{(a_i)} \|x^n + a_{n-1}x^{n-1} + \dots + a_0\| = \frac{1}{2^{n-1}} \|T_n\|.$$

We are concerned with the approximate solution by means of the Chebychev polynomials in the form

$$u(x) = \sum_{r=0}^N {}' a_r T_r(x), \quad -1 \leq x \leq 1 \quad (2.3)$$

in which a summation symbol with single prime denotes a sum with first term halved and $T_r(x)$ denotes Chebychev polynomials of the first kind of degree r , a_r are unknown Chebychev coefficients and N is chosen any positive integer such that $N \geq 2$. Let us assume that the first, second, ..., and twelfth derivatives of the function $u(x)$ have truncated Chebychev series expansion of the form

$$u^{(k)}(x) = \sum_{r=0}^N {}' a_r^{(k)} T_r(x), \quad k = 1, 2, \dots, 12. \quad (2.4)$$

Then the solution expressed by (2.3) and its derivatives can be written in the matrix forms

$$\mathbf{A}^{(k)} = 2^k \mathbf{M}^k \mathbf{A},$$

$$\mathbf{U}^{(k)} = 2^k \mathbf{T} \mathbf{M}^k \mathbf{A},$$

where

$$\mathbf{T} = \begin{pmatrix} T_0(x_0) & T_1(x_0) & \dots & T_N(x_0) \\ T_0(x_1) & T_1(x_1) & \dots & T_N(x_1) \\ T_0(x_2) & T_1(x_2) & \dots & T_N(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_N) & T_1(x_N) & \dots & T_N(x_N) \end{pmatrix},$$

$$\mathbf{U}^{(k)} = \begin{pmatrix} u^{(k)}(x_0) \\ u^{(k)}(x_1) \\ \vdots \\ u^{(k)}(x_N) \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix},$$

$$\mathbf{A} = \left[\frac{a_0}{2}, a_1, \dots, a_N \right]^T$$

and

$$\mathbf{M} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \dots & m_1 \\ 0 & 0 & 2 & 0 & 4 & 0 & \dots & m_2 \\ 0 & 0 & 0 & 3 & 0 & 5 & \dots & m_3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{(N+1) \times (N+1)}. \quad (2.5)$$

where

$$m_1 = \frac{N}{2}, \quad m_2 = 0, \quad m_3 = N \quad \text{if } N \text{ is odd}$$

$$m_1 = 0, \quad m_2 = N, \quad m_3 = 0 \quad \text{if } N \text{ is even}$$

The method can be developed for the problem defined in the domain $[a, b]$.

Definition 2.2. On the interval $[a, b]$, the shifted Chebychev polynomial is given by

$$T_n^*(x) = T_n(y), \quad y = \frac{2}{b-a} \left(x - \frac{a+b}{2} \right).$$

Notice that its leading coefficient is equal to $2^{n-1} \left(\frac{2}{b-a} \right)^n$.

To obtain the solution in terms of shifted Chebychev polynomials $T_r^*(x)$ in the form

$$u(x) = \sum_{r=0}^N a_r^* T_r^*(x), \quad a \leq x \leq b,$$

where $T_r^*(x) = T_r \left(\frac{2}{b-a} \left(x - \frac{a+b}{2} \right) \right)$.

It is followed the previous procedure using the collocation points defined by

$$x_i = \frac{b-a}{2} \left[\left(\frac{a+b}{b-a} \right) + \cos \left(\frac{i\pi}{N} \right) \right], \quad i = 0, 1, \dots, N, \quad (2.6)$$

and the relation

$$A^{*(k)} = \left(\frac{4}{b-a} \right)^k \mathbf{M}^k \mathbf{A}^*, \quad k = 0, 1, \dots, 12.$$

where

$$\mathbf{A}^* = \left[\frac{a_0^*}{2}, a_1^*, \dots, a_N^* \right]^\tau$$

It is easily obtained $T = T^*$, because of the properties of Chebychev polynomials.

3 The description of Chebychev scheme

3.1 Linear Two-Point Boundary Value Problems

Consider a linear, two-point boundary value problem of the form

$$u^{(ii)}(x) + q(x)u(x) = f(x), \quad x \in J = [a, b]. \quad (3.1)$$

The approximate solution for $u(x)$ is represented by the formula

$$u(x) = \sum_{r=0}^N a_r^* T_r^*(x), \quad a \leq x \leq b, \quad (3.2)$$

where $T_r^*(x) = T_r \left(\frac{2}{b-a} \left(x - \frac{a+b}{2} \right) \right)$ presents the shifted Chebychev polynomials of the first kind of degree r and a_r for $r = 0, 1, \dots, N$ are the undetermined Chebychev coefficients and the Chebychev collocation points in $[a, b]$ as equation (2.6). And its derivatives have truncated Chebychev series expansion of the form

$$u^{(\nu)}(x) = \sum_{r=0}^N (a_r^*)^\nu T_r^*(x), \quad \nu = 0, 1, 2, \dots, 12. \quad (3.3)$$

Then the solution expressed by (3.2) and its derivatives can be written in the matrix forms

$$\begin{aligned} u(x) &= \mathbf{T}^*(x) \mathbf{A}^* \\ u^{(\nu)}(x) &= \mathbf{T}^*(x) \mathbf{A}^\nu = 4^\nu \mathbf{T}^*(x) \mathbf{M}^\nu \mathbf{A}^*. \end{aligned} \quad (3.4)$$

Using (3.4) and substituting $x = x_i$ in (2.6) and applying the collocation to it, we eventually obtain the following theorem

Theorem 3.1. *If the assumed approximate solution of the problem (3.1) is (3.2), then the discrete Chebychev system is given by*

$$u^{(12)}(x_i) + q(x_i) u(x_i) = f(x_i). \quad \square \quad (3.5)$$

This system is written in the matrix form

$$\mathbf{W} \mathbf{A}^* = \mathbf{F}, \quad (3.6)$$

where

$$\mathbf{W} = \left(\frac{4}{b-a} \right)^{12} \mathbf{T}^* \mathbf{M}^{12} + \mathbf{Q} \mathbf{T}^*,$$

and

$$\mathbf{Q} = \begin{bmatrix} Q(x_0) & 0 & \dots & 0 \\ 0 & Q(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q(x_N) \end{bmatrix}, \quad \text{and} \quad \mathbf{F} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix}.$$

The boundary condition is derived from equation (1.2) and matrices for conditions are

$$\begin{aligned} \left(\frac{4}{b-a} \right)^j T^*(a) \mathbf{M}^j \mathbf{A}^* &= \alpha_{2j}, \\ \left(\frac{4}{b-a} \right)^j T^*(b) \mathbf{M}^j \mathbf{A}^* &= \beta_{2j}, \quad j = 0, 1, \dots, 5. \end{aligned} \quad (3.7)$$

Consequently, replacing twelfth rows of the augmented matrix by the equation (3.7), we have

$$\tilde{\mathbf{W}} \mathbf{A}^* = \mathbf{F}. \quad (3.8)$$

Now we have a linear system of $N + 1$ equations of the $N + 1$ unknown coefficients. We can obtain the coefficients of the approximate solution by solving this linear system. The system (3.8) may be easily solved by a variety of methods. In this paper we used the Q-R method. The solution $\mathbf{A}^*$ gives the coefficients in the approximate Chebychev solution.

3.2 Non-Linear Two-Point Boundary Value Problems

Consider a non-linear , two-point boundary value problem of the form

$$u^{xii}(x) + q(x) u(x) + P(x) u^\sigma(x) = f(x), \quad x \in J = [a, b]. \quad (3.9)$$

The approximate solution for $u(x)$ is represented by the formula

$$u(x) = \sum_{r=0}^N a_r^* T_r^*(x), \quad a \leq x \leq b.$$

We need the following lemma

Lemma 3.2. : *[(18)] The following relation holds*

$$\begin{aligned} \begin{pmatrix} u^\sigma(x_0) \\ u^\sigma(x_1) \\ \vdots \\ u^\sigma(x_N) \end{pmatrix} &= \begin{pmatrix} u(x_0) & 0 & \dots & 0 \\ 0 & u(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u(x_N) \end{pmatrix}^{\sigma-1} \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix} \\ &= (\bar{\mathbf{U}})^{\sigma-1} \mathbf{U} \\ &= (\bar{\mathbf{T}}^* \mathbf{A}^*)^{\sigma-1} \mathbf{T}^* \mathbf{A}^* \end{aligned} \quad (3.10)$$

where

$$\overline{\mathbf{T}}^* = \begin{pmatrix} T(x_0) & 0 & \dots & 0 \\ 0 & T(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T(x_N) \end{pmatrix}, \quad \overline{\mathbf{A}}^* = \begin{pmatrix} \mathbf{A}^* & 0 & \dots & 0 \\ 0 & \mathbf{A}^* & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{A}^* \end{pmatrix}.$$

Proof. Substituting the Chebychev collocation points into the $u^\sigma(x)$, we obtain the matrix representation. $\square$

Theorem 3.3. If the assumed approximate solution of the problem (3.9) is (3.2), then the discrete Chebychev system is given by

$$u^{(12)}(x_i) + q(x_i)u(x_i) + p(x_i)u^\sigma(x_i) = f(x_i). \quad \square \quad (3.11)$$

This system is written in the matrix form

$$\mathbf{Q}\mathbf{A}^* = \mathbf{F} \quad (3.12)$$

where

$$\mathbf{Q} = \left(\frac{4}{b-a} \right)^{12} \mathbf{T}^* \mathbf{M}^{12} + \mathbf{Q}\mathbf{T}^* + \mathbf{P} \left(\overline{\mathbf{T}}^* \mathbf{A}^* \right)^{\sigma-1} \mathbf{T}^*.$$

Replacing the boundary condition in equation (3.7) we have

$$\tilde{\mathbf{Q}}\mathbf{A}^* = \tilde{\mathbf{F}}. \quad (3.13)$$

Now we have a nonlinear system of $N + 1$ equations in the $N + 1$ unknown coefficients. We can obtain the coefficients in the approximate solution by solving this nonlinear system.

4 Examples and comparisons.

In this section, we will consider three examples to illustrate the accuracy and efficiency of the presented method. All examples which were collected from the available literatures [(5; 10; 8)]. All computations were carried out using **Matlab**(R2011a) on a personal computer.

In the examples, the maximum absolute error at Chebychev collocation points is taken as

$$\|E_{\text{Cheby}}\| = \max_{0 \leq i \leq N} |u_{\text{exact}}(x_i) - u_{\text{Chebychev}}(x_i)|.$$

Example 1: [(10; 8)] First, we consider the following BVP

$$u^{xii}(x) + u(x) = - (120 + 23x + x^3) e^x, \quad 0 \leq x \leq 1$$

subject to the boundary condition

$$\begin{aligned} u(0) &= 0, & u(1) &= 0 \\ u'(0) &= 1, & u'(1) &= -e \\ u''(0) &= 0, & u''(1) &= -4e \\ u'''(0) &= -3, & u'''(1) &= -9e \\ u^{(4)}(0) &= -8, & u^{(4)}(1) &= -16e \\ u^{(5)}(0) &= -15, & u^{(5)}(1) &= -25e \end{aligned}$$

whose exact solution is

$$u(x) = x(1-x)e^x.$$

The maximum absolute error, $\|E_{\text{Cheby}}\|$, is reported in **Table 1** as N increases from $N = 12$ to $N = 22$.

Table 1 Maximum absolute errors for Example 1

N	$\ E_{\text{Cheby}}\ $
12	3.122E-011
14	3.104E-012
16	4.324E-013
18	1.279E-013
20	2.312E-014
22	4.153E-015

Maximum absolute error are tabulated in **Table 2** for Chebychev polynomial together with the analogous results of [(5)].

Table 2 Comparison of maximum absolute error for Example 1

$\ E_{\text{Cheby}}\ $	Results of [(5)]
4.153E-15	1.2791E-10

Example 2: [(10; 8)] Next, we consider the following BVP

$$u^{(10)}(x) - u(x) = -24x \cos x - 132 \sin x, \quad -1 \leq x \leq 1$$

subject to the boundary condition

$$\begin{aligned} u(-1) &= u(1) = 0 \\ u'(-1) &= u'(1) = 2 \sin(1) \\ u''(-1) &= -u''(1) = -4 \cos(1) - 2 \sin(1) \\ u'''(-1) &= u'''(1) = 6 \cos(1) - 6 \sin(1) \\ u^{(4)}(-1) &= -u^{(4)}(1) = 8 \cos(1) + 12 \sin(1) \\ u^{(5)}(-1) &= -u^{(5)}(1) = -20 \cos(1) + 10 \sin(1) \end{aligned}$$

whose exact solution is

$$u(x) = (x^2 - 1) \sin x.$$

Maximum absolute errors at different N are tabulated in **Table 3** for the Chebychev method.

Table 3 Maximum absolute errors for Example 2

N	$\ E_{\text{Cheby}}\ $
12	6.465E-11
14	1.154E-12
16	5.654E-13
18	3.274E-14
20	1.128E-14
22	2.113E-15

Maximum absolute error are tabulated in **Table 4** for Chebychev polynomial together with the analogous results of [(5)].

Table 4 Comparison of maximum absolute error for Example 2

$\ E_{\text{Cheby}}\ $	Results of [(5)]
2.113E-15	2.2839E-09

Example 3: [(5; 28)] Now we consider a nonlinear 12^{th} -order boundary value problem:

$$u^{(12)}(x) - u''' = 2e^x u^2, \quad 0 \leq x \leq 1$$

subject to the boundary condition

$$\begin{aligned} u^{(k)}(0) &= (-1)^k, \\ u^{(k)}(1) &= (-1)^k e^{-1}, \quad k = 0, 1, \dots, 5. \end{aligned}$$

The numerical results are given in **Table 5** which are exactly the same as obtained by modified decomposition method (MDM)[(28)] and obtained by differential transformation method [(5)].

Table 5 Numerical results for Example 3

x	Solution by MDM [(28)]	Solution by DTM [(5)]	Solution of Chebychev
0.0	1.000000000000000	1.000000000000000	1.000000000000000
0.1	0.90483741803596	0.90483741803596	0.90483741803596
0.2	0.81873075307798	0.81873075307811	0.81873075307799
0.3	0.74081822068172	0.74081822068239	0.74081822068169
0.4	0.67032004603564	0.67032004603717	0.67032004603566
0.5	0.60653065971263	0.60653065971462	0.60653065971263
0.6	0.54881163609403	0.54881163609560	0.54881163609413
0.7	0.49658530379141	0.49658530379213	0.49658530379148
0.8	0.44932896411722	0.44932896411736	0.44932896411719
0.9	0.40656965974060	0.40656965974060	0.40656965974060
1.0	0.36787944117144	0.36787944117144	0.36787944117144

5 Conclusion

Chebychev method is remarkably effective for solving twelfth-order boundary-value problems. This method is very powerful, efficient and promoting, which will be certainly found wide applications.

Chebychev method was tested on three problems, two linear and one nonlinear. A comparison with modified decomposition method (MDM)[(28)] and differential transformation method [(5)] are made and it was seen that the Chebychev method yields good results.

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