The Smooth Representations of the p-adic Compact Group $U(1)$

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Abstract

In this article, we parametrize all smooth representations of the Unitary group in one variable. We will use the locally compact properties of the local fields to exhaust all representations of this group.

Keywords Unitary group; Smooth representation; Character; local Field.

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1 Introduction

Let F be a local *p-adic* field where p is an odd prime number. Let $O = O_F$ be the ring of integers in F, and $P = P_F$ be the maximal ideal in O. Let E/F be a quadratic extension and $x \to \bar{x}$ be the nontrivial element of the Galois group $\Gamma(E/F)$, and set $W_1 = E$, with the form $(x, y)_1 = x\bar{y}$. Then $(W_1, (,)_1)$ is a Hermitian space over E. The isometry group of this space, $G_1 = U(1)$ is the unitary group in one variable. This is the simplest abelian compact group. Since E is locally compact its unit element has a system of neighborhoods, we will use the characters of these neighborhoods to get the representations of $G_1 = U(1)$. In fact each irreducible representations of $G_1 = U(1)$ is an extension of some of these characters, and this is because E is abelian and $\mathbb{C}$ is an injective $Z-$module.

2 Preliminaries

Let $\nu(x) = x\bar{x}$ and $Tr(x) = x + \bar{x}$ be norm and trace maps, respectively. Let π be a uniformizer of E and $\varpi = \varpi_F$ a uniformizer of F. We may and will assume $\pi = \varpi$ when E/F is unramified, and $\varpi = \pi^2$ when E/F is ramified.

Let $k = k_F = O_F/P_F$ be the residual field of F with q elements and $k_E = O_E/P_E$ be the residual field of E, where O_E and P_E are the ring of integers and the maximal ideal of E, respectively. Also let χ be a non trivial character of F^+ with conductor O, i.e. $\chi_{|O}$ is trivial but $\chi_{|P^{-1}}$ is nontrivial. The structure of $U(1)$ is very simple, because any E- isomorphism from E to itself is a scalar, the *isometry group* of W_1 is $U(1) = \{a \in E \mid \nu(a) = 1\} = E^1$.

For any integer r, $r \geq 1$, set $E_r^1 = \left\{x \in E^1 \mid x \equiv 1 \,(\mathrm{mod}\, P_E^r)\right\}$ and let O_{E° be the set of all zero trace elements in O_E , and $P_{E^\circ}^r = O_{E^\circ} \cap P_E^r$, where $P_E^r = \{x \in E \mid x = a\pi^r, \text{ for some } a \in O_E\}$.

Lemma 1 *Let r and n be two positive integers with $\frac{n}{2} \leq r < n$, then we have:*

$$P_{E^\circ}^r / P_{E^\circ}^n \cong E_r^1 / E_n^1$$

Proof. Let $a\pi^r \in P_{E^\circ}^r$. Define $C : P_{E^\circ}^r \to E_r^1 / E_n^1$ by the Cayley transformation as follows:

$$C(a\pi^r) = \frac{1 - a\pi^r}{1 + a\pi^r} E_n^1$$

Then, C is a homomorphism because:

$$C(a\pi^r + b\pi^r) = \frac{1 - (a\pi^r + b\pi^r)}{1 + (a\pi^r + b\pi^r)} E_n^1 = \frac{1 - (a+b)\pi^r}{1 + (a+b)\pi^r} E_n^1$$

and

$$\begin{aligned}
C(a\pi^r)\, C(b\pi^r) &= \frac{1 - a\pi^r}{1 + a\pi^r} E_n^1 \frac{1 - b\pi^r}{1 + b\pi^r} E_n^1 = \frac{(1 - a\pi^r)(1 - b\pi^r)}{(1 + a\pi^r)(1 + b\pi^r)} E_n^1 \\
&= \frac{1 - (a+b)\pi^r + a\pi^r b\pi^r}{1 + (a+b)\pi^r + a\pi^r b\pi^r} E_n^1 \\
&= \frac{1 - (a+b)\pi^r}{1 + (a+b)\pi^r} \cdot \frac{1 + (a+b)\pi^r}{1 - (a+b)\pi^r} \cdot \frac{1 - (a+b)\pi^r + a\pi^r b\pi^r}{1 + (a+b)\pi^r + a\pi^r b\pi^r} E_n^1 \\
&= \frac{1 - (a+b)\pi^r}{1 + (a+b)\pi^r} \cdot \frac{1 - ((a+b)\pi^r)^2 + (a+b)\pi^r (a\pi^r b\pi^r)}{1 - ((a+b)\pi^r)^2 - (a+b)\pi^r (a\pi^r b\pi^r)} E_n^1 \\
&= \frac{1 - (a+b)\pi^r}{1 + (a+b)\pi^r} E_n^1
\end{aligned}$$

because $\frac{1-((a+b)\pi^r)^2+(a+b)\pi^r(a\pi^r b\pi^r)}{1-((a+b)\pi^r)^2-(a+b)\pi^r(a\pi^r b\pi^r)} \in E_n^1$, so

$$C(a\pi^r + b\pi^r) = C(a\pi^r)\, C(b\pi^r)$$

One can check easily that $\ker C = P_{E^\circ}^n$. To show C is onto, let $y = 1 + b\pi^r \in E_r^1$,

then $x = \frac{\frac{b}{2}\pi^r}{1+\frac{b}{2}\pi^r} \in O_E$. Further, one can show $C(x) = y$. And,

$$\begin{aligned} Tr(x) &= \frac{\frac{b}{2}\pi^r}{1+\frac{b}{2}\pi^r} + \frac{\overline{\frac{b}{2}\pi^r}}{1+\overline{\frac{b}{2}\pi^r}} = \frac{\frac{b}{2}\pi^r\left(1+\overline{\frac{b}{2}\pi^r}\right) + \left(\overline{\frac{b}{2}\pi^r}\right)\left(1+\frac{b}{2}\pi^r\right)}{\nu\left(1+\frac{b}{2}\pi^r\right)} \\ &= \frac{\frac{b}{2}\pi^r + \frac{b}{2}\pi^r\left(\overline{\frac{b}{2}\pi^r}\right) + \left(\overline{\frac{b}{2}\pi^r}\right) + \left(\overline{\frac{b}{2}\pi^r}\right)\frac{b}{2}\pi^r}{\nu\left(1+\frac{b}{2}\pi^r\right)} \\ &= \frac{\frac{1}{2}Tr(b\pi^r) + \frac{1}{2}\nu(b\pi^r)}{\nu\left(1+\frac{b}{2}\pi^r\right)} \end{aligned}$$

Now since $\nu(y) = \nu(1+b\pi^r) = 1 + Tr(b\pi^r) + \nu(b\pi^r) = 1$, we deduce that the numerator of the last fraction is zero, and the proof is completed. ■

Lemma 2 *Let $\alpha \in O_{E^\circ}$ with $\upsilon_E(\alpha) = -n-1$, where n is a positive integer, and let r be the positive integer $r = \left[\frac{n+1}{2}\right]$, Where $[,]$ denotes the greatest integer part function. Define $\chi_\alpha : E_r^1 \to \mathbb{C}^\times$ as follows:*

$$\chi_\alpha(h) = \chi(Tr(\alpha(h-1))), \forall h, h \in E_r^1$$

Then, χ_α is a character of E_r^1.

Proof. Let $h_1 = 1 + a_1\pi^r$ and $h_2 = 1 + a_2\pi^r$ be two elements of E_r^1, then we have:

$$h_1h_2 = 1 + (a_1+a_2)\pi^r + a_1\pi^r a_2\pi^r$$

now by selection of r, one can check that $Tr(\alpha a_1\pi^r a_2\pi^r) \in O$ so :

$$\begin{aligned} \chi_\alpha(h_1h_2) &= \chi(Tr(\alpha(a_1+a_2)\pi^r)) \\ &= \chi(Tr(\alpha a_1\pi^r))\chi(Tr(\alpha a_2\pi^r)) \\ &= \chi_\alpha(h_1)\chi_\alpha(h_2). \end{aligned}$$

■

Lemma 3 1. *Let E/F be ramified. Then, for any positive integer m, we have: $Tr(P_E^m) = P^{\left[\frac{m+1}{2}\right]}$. Where $[,]$ denotes the greatest integer part function.*

2. *Let E/F be unramified. Then, for any positive integer m, we have $Tr(P_E^m) = P^m$.*

Proof. (1). *Let $x = a\pi^m \in P_E^m$, and $m = 2l$ be even, then:*

$$\begin{aligned} Tr(x) &= a\pi^m + \bar{a}\pi^m = (a+\bar{a})\pi^{2l} = Tr(a)\left(\pi^2\right)^l \\ &= Tr(a)\varpi^l = Tr(a)\varpi^{\left[\frac{m+1}{2}\right]} \end{aligned}$$

If $m = 2l+1$ is odd, then for $x = a\pi^m \in P_E^m$,

$$Tr(x) = a\pi^m - \bar{a}\pi^m = (a - \bar{a})\pi^{2l+1},$$

and $(a - \bar{a}) \in P_E$, i.e. $Tr(x) \in P_E^{2l+2} = \left(P_E^2\right)^{l+1} = P^{l+1}$, and $l + 1 = \left[\frac{m+1}{2}\right]$.
(2). Since E/F is unramified we have $\pi = \varpi$, now same argument as in part 1 will finish the proof.(a general form of this lemma can be found in [8].) ■

Lemma 4 *Notations are as in lemma 2. Then χ_α is trivial on E_r^1 if and only if $\chi(Tr(\alpha y)) = 1$ for any $y \in P_E^r$.*

Proof. Suppose χ_α is trivial on E_r^1, and let $y \in P_E^r$ be arbitrary, then $\frac{1+y}{1+\bar{y}} \in E_r^1$, and there exists $z \in P_E^{2r}$ such that:

$$\frac{1+y}{1+\bar{y}} = 1 + y - \bar{y} + z$$

From here we can derive:

$$1 = \chi_\alpha\left(\frac{1+y}{1+\bar{y}}\right) = \chi(Tr(\alpha(y - \bar{y}))) \tag{1}$$

On the other hand, since $y + \bar{y} \in F$ and $Tr(\alpha) = 0$, we have

$$\chi(Tr(\alpha(y + \bar{y}))) = 1 \tag{2}$$

from 1 and 2 we will obtain:

$$\chi(Tr(2\alpha y)) = 1$$

Now, since 2 is unit from the last equality, we deduce:

$$\chi(Tr(\alpha y)) = 1, \forall y, y \in P_E^r$$

The other side is obvious. ■

Corollary 5 *If E/F is unramified then the conductor of χ_α is $n+1 = -\upsilon_E(\alpha)$, and if E/F is ramified then the conductor of χ_α is $n = -\upsilon_E(\alpha) - 1$.*

Proof. By lemma 4, and definition of χ_α, h is in the conductor if and only if $Tr(\alpha(h-1)) \in O = O_F$. ■

Lemma 6 *Let E/F be unramified, then $|(O_{E^\circ}/P_{E^\circ})| = | k | = q$.*

Proof. Since E/F is unramified, $[k_E : k] = 2$ so $| k_E | = q^2$. Now define: $f : k_E \to O_E^\circ/P_E^\circ$ by $f(a + P_E) = a - \bar{a} + P_{E^\circ}$. One can check f is a well-defined homomorphism and Hilbert's 90 shows that f is onto. Further, $\ker f = k$ and thus $|(O_{E^\circ}/P_{E^\circ})| = |(k_E/k)| = \frac{q^2}{q} = q$. ■

Lemma 7 *Let E/F be ramified, then:*

1. $E_r^1 = E_{r+1}^1$, *When r is even*
2. $E_r^1/E_{r+1}^1 \cong k$ *When r is odd.*

Proof. (1). Let $\Re$ be the set of representative elements of k in O, and let $a\pi^r \in P_{E^\circ}^r$. There exist elements $a_\circ \in \Re$ and $a_1 \in O_E$ such that $a = a_\circ + a_1\pi$. Since r is even $a = -\bar{a}$ and from here $a_\circ + a_1\pi = -(a_\circ - a_1\pi)$, so π must divide $2a_\circ$, $\pi \mid 2a_\circ$, but 2 is unit and hence $a_\circ = 0$ i.e. $a\pi^r = a_1\pi^{r+1} \in P_{E^\circ}^{r+1}$ and now we can apply Lemma1.
(2). Let r be a positive odd integer. Then, for $a\pi^r \in P_{E^\circ}^r$, we have $a = \bar{a}$ i.e. $a \in O$. Now define: $f : P_{E^\circ}^r \to k$ by $f(a\pi^r) = a + P$. It is easy to check that f is onto and $\ker f = P_{E^\circ}^{r+1}$, so Lemma 1 will finished the proof. ■

3 Representations of $U(1)$

Proposition 8 *Let $n \geq 2$ be an integer and let $r = \left[\frac{n+1}{2}\right]$. Then, any character of E_r^1 is in the χ_α form, for some $\alpha \in O_{E^\circ}$.*

Proof. Define $\lambda : O_{E^\circ}. \to \left(E_r^1/E_n^1\right)^\wedge$ by $\lambda(\alpha) = \chi_\alpha$, where $(.)^\wedge$ is Pontryagin Dual. Using Lemma 1 we have $\ker\lambda = P_{E^\circ}^{(n-r)}$. Since E_r^1/E_n^1 is finite , by Lemma 1, Corollary5, and Lemmas 4, and 6, λ is onto. ■

Corollary 9 *Any irreducible representation of E^1 with conductor $n, n \geq 2$ is an extension of some $\chi_\alpha, \alpha \in O_{E^\circ}$.*

Proof. Since $\mathbb{C}^\times$ is an injective $Z-$*module*, we can extend any $\chi_\alpha, \alpha \in O_{E^\circ}$ from E_r^1 to E^1. ■

3.1 Other Representations of E^1.

Now we will determine the other representations of E^1, i.e. those with conductor equal to one. These are determined by the following lemma.

Lemma 10
1. *E^1/E_1^1 is a cyclic group of order $q+1$, when E/F is unramified,*
2. *$E^1/E_1^1 \cong Z_2$, when E/F is ramified.*

Proof. (1). Define $f : E^1 \to k_E^\times$ by $f(a) = a + P_E$. Then f is a homomorphism and one can show $\ker f = E_1^1$ and the image of f is equal to the set of norm one elements in $k_E^\times$. Now since $k_E^\times$ is a multiplicative subgroup of a finite field so is cyclic. Next lemma will show the order of this group is $q+1$. ■

(2). When E/F is ramified we have $k_E = k$, and the only norm one elements in k are ± 1, now apply part (1). ■

Lemma 11 *Let $\mathcal{R}$ be the set of norm one elements in $k_E^\times$ when E/F is unramified, then $|\mathcal{R}| = q+1$.*

Proof. Define $f : k_E^\times \to \mathcal{R}$ by $f(a) = \frac{a}{\bar{a}}$. By Hilbert's 90 f is onto, and the $\ker f = k^\times$, so $k_E^\times / k^\times \cong \mathcal{R}$, and

$$|\mathcal{R}| = \left| k_E^\times / k^\times \right| = \frac{q^2-1}{q-1} = q+1.$$

■

Proposition 12 *Any primitive $|\mathcal{R}|-th$ root of unity, ς say, determines a character of E^1 with the conductor one, and any character of E^1 with the conductor one will be determined by a primitive $|\mathcal{R}|-th$ root of unity.*

Proof. Let ς be a given primitive $|\mathcal{R}|-th$ root of unity, and let a be a generator of E^1/E_1^1. Now define

$$\xi : E^1/E_1^1. \to \mathbb{C}^\times$$

by

$$\xi\left(a^i\right) = \varsigma^i, \text{ for any } i, 0 \le i \le |\mathcal{R}|$$

Then one can check that ξ is a character of E^1/E_1^1. Now by inflation we can define ξ on E^1. Conversely, let ξ be a character of E^1 with conductor one, and let a be a generator of E^1/E_1^1. Now set $\varsigma = \xi(a)$. ■

Proposition 13 *Any non trivial irreducible representation of E^1 is an extension of some χ_α, or is a character, ξ say, of cyclic group E^1/E_1^1.*

Proof. Since E^1 is compact, for any irreducible representation, φ say, of E^1 there exists a positive integer n such that this representation is trivial on E_n^1. Let n be the least integer with this property. If $n = 1$ then the representation is in the form of some ξ because of proposition 12, and if $n > 1$ then by proposition 8, and its corollary the representation is an extension of some χ_α. ■

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