

A Class of Implicit Six Step Hybrid Backward Differentiation Formulas for the Solution of Second Order Differential Equations.

Abstract

In this paper, we propose a class implicit six step Hybrid Backward Differentiation Formulas (HBDF) for the solution of second order Initial Value Problems (IVPs). The method is derived by the interpolation and collocation of the assumed approximate solution. The single continuous formulation derived is evaluated at grid point of $x = x_{n+k}$ and its second derivative at $x = x_{n+j}$, $j = 1, 2, \dots, k-1$ and $x = x_{n+\mu}$ respectively, where k is the step number of the methods. The interpolation and collocation procedures lead to a system of $(k+1)$ equations, which are solved to determine the unknown coefficients. The resulting coefficients are used to construct the approximate continuous solution from which the Multiple Finite Difference Methods (MFDMs) are obtained and simultaneously applied to provide the direct solution to IVPs. Numerical examples are given to show the efficiency of the method.

Keywords

Hybrid method; Backward Differentiation Formulas; Collocation; Interpolation; Second Order; Multiple Finite Difference.

Introduction

In recent times, the integration of Ordinary Differential Equations (ODEs) are investigated using some kind of block methods. This paper discusses the family of implicit Linear Multistep Method (LMM) for numerical integration of general second order ODEs which arise frequently in the area of science and engineering especially mechanical system, control theory and celestial mechanics and are generally written as: $y'' = f(x, y', y)$, $y(a) = y_0$, $y'(a) = \eta_0$

(1)

In practice the problems are reduced to systems of first order equations and any method for first order equations is used to solve them see Awoyomi [1]. It has been extensively discussed that due to the dimension of the problem after it has been reduced to a system of first order equations also, more often the reduced systems of ordinary differential equations (ODEs) is not well posed, unlike the given problem. The approach waste a lot of Computer time and human efforts, hence there is a need to develop algorithms to handle these classes of problems directly without any reduction to system of first order ODEs .

Development of LMM for solving ODE can be generated using methods such as taylor's series, numerical interpolation, numerical integration and collocation method, which are restricted by an assumed order of convergence. In this paper we will consider the contribution of multi step collocation technique introduced by Onumayi et al [2] by deriving our new method. Some researchers have attempted the solution of directly using linear multistep methods without reduction to system of first order ordinary differential equations the include Mohammed et al [3],Yusuph and Onumayi [4] and Onumayi et al [5].

Block methods for solving ODEs have initially been proposed by Milne [6].The Milne's idea of proceeding in blocks was developed by Rosser [7] for Runge-Kutta method. Also block Backward Differentiation Formulas (BDF) methods are discussed and developed by many researchers [8-16]. The method of collocation and interpolation of the power series approximate solution to generate continuous LMM has been adopted by many researchers among them are (Houwen et al. [17], Fatunla [18], Jiaxiang [19]).

In this paper we are suggested a construction of six step HBDF method, it is self-starting and can be applied for the numerical solution of IVPs (Cauchy problem) for second-order ODEs.

Material and Method

We seek an approximation of the form

$$Y(x) = \sum_{j=0}^{r+s-1} \ell_j x^j \quad (2)$$

Where ℓ_j are unknown coefficients to be determined and $1 \leq r < k$ and $s > 0$ are the number of interpolation and collocation points respectively. We then construct our continuous approximation by imposing the following conditions

$$Y(x) = y_{n+j}, \quad j = 0, 1, 2, \dots, k-1 \quad (3)$$

$$Y''(x_{n+k}) = f_{n+k} \quad (4)$$

We note that $y_{n+\mu}$ is the numerical approximation to the analytical solution $y(x_{n+\mu}), f_{n+\mu} = f(x_{n+\mu}, y_{n+\mu}, y'_{n+\mu})$.

Equations (3) and (4) lead to a system of $(k+1)$ equations which is solved by Cramer's rule to obtain ℓ_j . Our continuous approximation is constructed by substituting the values ℓ_j into equation (2). After some manipulation, the continuous method is expressed as

$$Y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_\mu(x) y_{n+\mu} + h^2 \beta_k(x) f_{n+k} \quad (5)$$

Where $\alpha_j(x), \beta_k(x)$ and $\alpha_\mu(x)$ are continuous coefficients. We note that since the general second order ordinary differential equation involves the first derivative, the first derivative formula

$$Y'(x) = \frac{1}{h} \left(\sum_{j=0}^{k-1} \alpha'_j(x) y_{n+j} + \alpha'_\mu(x) y_{n+\mu} + h^2 \beta'_k(x) f_{n+k} \right) \quad (6)$$

$$Y'(x) = \delta(x) \quad (7)$$

$$Y'(a) = \delta_0 \quad (8)$$

2.1 Specification of Methods

2.1.1 Six Step Methods with one- off -step point at interpolation.

To derive this methods, we use Eq.(5) to obtained a continuous 5-step HBDF method with the following specification : $r=7, s=1, k=6$. We also express $\alpha_j(x), \alpha_\mu(x)$ and $\beta_k(x)$ as a functions of t , where $t = \frac{x - x_n}{h}$ to obtain the continuous form as follows:

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} + \alpha_5 y_{n+5} + \alpha_{11} y_{n+\frac{11}{2}} + h^2 \beta_6 f_{n+6}$$

86 (9)

87 Where

88
$$\alpha_0(t) = 1 - \frac{2223121}{850080}t + \frac{3014973}{1133440}t^2 - \frac{3157433}{2266880}t^3 + \frac{186435}{453376}t^4 - \frac{156729}{2266880}t^5 + \frac{13999}{2266880}t^6 - \frac{773}{3400320}t^7$$

89
$$\alpha_1(t) = \frac{27775}{3864}t - \frac{179425}{15456}t^2 + \frac{2088853}{278208}t^3 - \frac{231881}{92736}t^4 + \frac{126277}{278208}t^5 - \frac{3961}{92736}t^6 + \frac{227}{139104}t^7$$

90
$$\alpha_2(t) = -\frac{4125}{368}t - \frac{244245}{10304}t^2 - \frac{1109779}{61824}t^3 + \frac{409541}{61824}t^4 - \frac{26609}{20608}t^5 + \frac{7909}{61824}t^6 - \frac{157}{30912}t^7$$

91
$$\alpha_3(t) = \frac{25619}{1932}t - \frac{389547}{12880}t^2 + \frac{648047}{25760}t^3 - \frac{51789}{5152}t^4 + \frac{10755}{5152}t^5 - \frac{5601}{25760}t^6 + \frac{347}{38640}t^7$$

92
$$\alpha_4(t) = -\frac{57805}{5152}t - \frac{546379}{20608}t^2 - \frac{8589991}{370944}t^3 + \frac{403209}{41216}t^4 - \frac{794791}{370944}t^5 + \frac{9641}{41216}t^6 - \frac{1865}{185472}t^7$$

93
$$\alpha_5(t) = \frac{2211}{280}t - \frac{21309}{1120}t^2 + \frac{114827}{6720}t^3 - \frac{10049}{1344}t^4 + \frac{3817}{2240}t^5 - \frac{1301}{6720}t^6 + \frac{29}{3360}t^7$$

94
$$\alpha_{\frac{11}{2}}(t) = -\frac{17516}{5313}t - \frac{30368}{3795}t^2 - \frac{247696}{34155}t^3 + \frac{7312}{2277}t^4 - \frac{5072}{6831}t^5 + \frac{976}{11385}t^6 - \frac{928}{239085}t^7$$

95
$$\beta_6(t) = \frac{1}{10304}(1320t - 3254t^2 + 3023t^3 - 1385t^4 + 335t^5 - 41t^6 + 2t^7)$$

96 Evaluating (31) at $x = x_{n+6}$ yields Hybrid Six step implicit method

97
$$y_{n+6} = \frac{257}{28336}y_n - \frac{51}{644}y_{n+1} + \frac{405}{1288}y_{n+2} - \frac{247}{322}y_{n+3} + \frac{3555}{2576}y_{n+4} - \frac{81}{28}y_{n+5} + \frac{768}{253}y_{n+\frac{11}{2}} + \frac{45}{644}h^2f_{n+6}$$

98 (10)

99 Taking the second derivative of equation of equation (9), thereafter, evaluating the
100 resulting continuous polynomial solution at $x = x_{n+2}, x = x_{n+3}, x = x_{n+4}, x = x_{n+5}, x = x_{n+\frac{11}{2}}$ we

101 generate five additional methods

102
$$y_{n+2} = -\frac{16829}{579590}y_n + \frac{81682}{142263}y_{n+1} + \frac{7724}{26345}y_{n+3} + \frac{30685}{94842}y_{n+4} - \frac{7498}{26345}y_{n+5} + \frac{960512}{7824465}y_{n+\frac{11}{2}} - \frac{2576}{5269}h^2f_{n+2} - \frac{26}{5269}h^2f_{n+6}$$

103 (11)

103
$$y_{n+3} = \frac{353}{85976}y_n - \frac{2915}{52758}y_{n+1} + \frac{6455}{11724}y_{n+2} + \frac{39035}{70344}y_{n+4} - \frac{391}{5862}y_{n+5} - \frac{3584}{290169}y_{n+\frac{11}{2}} - \frac{3220}{8793}h^2f_{n+3} + \frac{5}{17586}h^2f_{n+6}$$

(12)

$$y_{n+4} = -\frac{11601}{15281585} y_n + \frac{8468}{833541} y_{n+1} - \frac{20754}{277847} y_{n+2} + \frac{774072}{1389235} y_{n+3} + \frac{908316}{1389235} y_{n+5} \\ - \frac{6680576}{45844755} y_{n+\frac{11}{2}} - \frac{92736}{277847} h^2 f_{n+4} - \frac{540}{277847} h^2 f_{n+6} \quad (13)$$

$$y_{n+5} = \frac{2663}{3217148} y_n - \frac{16075}{1974159} y_{n+1} + \frac{5675}{146234} y_{n+2} - \frac{9514}{73117} y_{n+3} + \frac{1266205}{2632212} y_{n+4} \\ + \frac{13411328}{21715749} y_{n+\frac{11}{2}} - \frac{560}{3179} h^2 f_{n+5} - \frac{5}{4301} h^2 f_{n+6} \quad (14)$$

$$y_{n+\frac{11}{2}} = -\frac{5200659}{563806208} y_n + \frac{11352935}{140951552} y_{n+1} - \frac{90099405}{281903104} y_{n+2} + \frac{54767691}{70475776} y_{n+3} - \frac{773233395}{563806208} y_{n+4} \\ + \frac{259721451}{140951552} y_{n+5} + \frac{34155}{157312} h^2 f_{n+\frac{11}{2}} - \frac{7248285}{140951552} h^2 f_{n+6} \quad (15)$$

Since our method is design to simultaneously provide the values of

$y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+\frac{11}{2}}, y_{n+6}$ at a block point $x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}, x_{n+\frac{11}{2}}, x_{n+6}$, the six

equations (10)-(15) are not sufficient to provide the solution for seven unknown

$y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+\frac{11}{2}}, y_{n+6}$. Thus, we obtain an additional method from (8), given by

$$850080h\delta_0 + 2223121y_0 - 6110500y_1 + 9528750y_2 - 11272360y_3 + 9537825y_4 \\ - 6712596y_5 + 2805760y_{\frac{11}{2}} = 108900h^2 f_6 \quad (16)$$

The derivatives are obtained from (7) by imposing that

$\delta(x_{n+\mu}) = \delta_{n+\mu}, \mu = \{j, v\}, j = 0, \dots, 6$, thus, we have

$$h\delta_{n+1} = -\frac{39411}{283360} y_n - \frac{17177}{11592} y_{n+1} + \frac{8037}{2576} y_{n+2} - \frac{8919}{3220} y_{n+3} + \frac{10747}{5152} y_{n+4} - \frac{387}{280} y_{n+5} + \frac{45184}{79695} y_{n+\frac{11}{2}}$$

$$- \frac{27}{1288} h^2 f_{n+6}$$

$$h\delta_{n+2} = \frac{1803}{80960} y_n - \frac{1061}{3312} y_{n+1} - \frac{12865}{15456} y_{n+2} + \frac{327}{184} y_{n+3} - \frac{4441}{4416} y_{n+4} + \frac{143}{240} y_{n+5} - \frac{3776}{15939} y_{n+\frac{11}{2}} +$$

$$\frac{3}{368} h^2 f_{n+6}$$

$$h\delta_{n+3} = -\frac{2833}{340032} y_n + \frac{695}{7728} y_{n+1} - \frac{2895}{5152} y_{n+2} - \frac{6493}{19320} y_{n+3} + \frac{11695}{10304} y_{n+4} - \frac{57}{112} y_{n+5} + \frac{5056}{26565} y_{n+\frac{11}{2}}$$

$$- \frac{15}{2576} h^2 f_{n+6}$$

$$h\delta_{n+4} = \frac{309}{56672}y_n - \frac{607}{11592}y_{n+1} + \frac{633}{2576}y_{n+2} - \frac{2823}{3220}y_{n+3} + \frac{845}{15456}y_{n+4} + \frac{51}{56}y_{n+5} - \frac{22912}{79695}y_{n+\frac{11}{2}} + \frac{9}{1288}h^2f_{n+6}$$

$$h\delta_{n+5} = -\frac{117}{25760}y_n + \frac{475}{11592}y_{n+1} - \frac{1325}{7728}y_{n+2} + \frac{297}{644}y_{n+3} - \frac{16435}{15456}y_{n+4} - \frac{197}{840}y_{n+5} + \frac{1408}{1449}y_{n+\frac{11}{2}} - \frac{15}{1288}h^2f_{n+6}$$

$$h\delta_{n+6} = \frac{42859}{1700160}y_n - \frac{1693}{7728}y_{n+1} + \frac{4449}{5152}y_{n+2} - \frac{40189}{19320}y_{n+3} + \frac{37519}{10304}y_{n+4} - \frac{3849}{560}y_{n+5} + \frac{123328}{26565}y_{n+\frac{11}{2}} +$$

$$\frac{801}{2576}h^2f_{n+6}$$

$$h\delta_{n+\frac{11}{2}} = \frac{16209}{2072576}y_n - \frac{29315}{423936}y_{n+1} + \frac{184635}{659456}y_{n+2} - \frac{83259}{117760}y_{n+3} + \frac{258995}{188416}y_{n+4} - \frac{8217}{2048}y_{n+5} + \frac{498251}{159390}y_{n+\frac{11}{2}} +$$

$$\frac{1485}{47104}h^2f_{n+6}$$

Error Analysis and Zero Stability

Following Fatunla [18] and Lambert [20] we define the local truncation error associated with the conventional form of (5) to be the linear difference operator

$$L[y(x); h] = \sum_{j=0}^k \{\alpha_j y(x + jh)\} + \alpha_v y(x + \nu h) + h^2 \beta_v y''(x + jh) \quad (17)$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (17) as a Taylor series about the point x to obtain the expression

$$L[y(x); h] = C_0 y(x) + C_1 h y' + \dots + C_q h^q y^{(q)}(x) + \dots, \quad (18)$$

where the constant coefficients C_q , $q = 0, 1, \dots$ are given as follows: C_q , $q = 0, 1, \dots$

$$C_0 = \sum_{j=0}^k \alpha_j,$$

$$C_1 = \sum_{j=1}^k j \alpha_j,$$

.

.

$$C_q = \left[\frac{1}{q!} \sum_{j=1}^k j^q \alpha_j - q(q-1) \sum_{j=1}^k j^{q-2} \beta_j \right].$$

According to Henrici [21], method (5) has order p if

$$C_0 = C_1 = \dots = C_p = C_{p+1} = 0, \quad C_{p+2} \neq 0$$

Therefore, C_{p+2} is the error constant and $C_{p+2}h^{p+2}y^{(p+2)}(x_n)$ the principal local truncation error at the point x_n . It is established from our calculations that the HBDF have higher order and relatively small error constants as displayed in the Table 1.

In order to analyze the methods for zero stability, we normalize the HBDF schemes and write them as a block method from which we obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(RA^{(0)} - A^{(1)}) = R^k(R-1) \quad (19)$$

Where $A^{(0)}$ is the identity matrix of dimension $k+1$, $A^{(1)}$ is the matrix of dimension $k+1$. Case $k=6$. It is easily shown that (9)-(16) are normalized to give the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(RA^{(0)} - A^{(1)}) = R^6(R-1)$$

Where $A^{(0)}$ an identity matrix of is dimension seven and $A^{(1)}$ is a matrix of dimension seven given by

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Following Fatunla [18] the block method by combining $k+1$ HBDF is zero-stable, since from (19),

$\rho(R) = 0$ satisfy $|R_j| \leq 1$ $j = 1, \dots, k$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed 2. The block method by combining $k+1$ HBDF is consistent since HBDF have order $P > 1$. According to Henrici [21], we can safely ascertain the convergence of HBDF method.

Results

We report here a numerical example taken from the literature.

Problem 1

$$y'' - y' = 0, y(0) = 0, y'(0) = -1, h = 0.1$$

159 Exact Solution $y(x) = 1 - e^{-x}$

160

Source: Mohammed [8]

161 Problems 2

162 $y'' + y = 0, y(0) = 1, y'(0) = 1, h = 0.1$

163 Exact Solution $y(x) = \cos(x) + \sin(x)$

164

Source: Awari [22]

165

166

167 Table 1: Order and Error Constants For the HBDF methods

Step number	Method	order	Error constant
6	(9)	6	$-\frac{801}{144256}$
	(10)	6	$\frac{3446421}{56}$
	(11)	6	$-\frac{87945}{56}$
	(12)	6	$-\frac{141471}{56}$
	(13)	6	$-\frac{1378311}{56}$
	(14)	6	$-\frac{179315}{6924288}$
	(15)	6	$-\frac{31273}{288512}$

168

169

170 **Table 2: Showing exact solutions and the computed results from the proposed methods**
 171 **for problem 1**

x	Exact Solution	Proposed Method
0	0	0
0.1	-0.105170918	-0.1051709032
0.2	-0.221402758	-0.2214027199

0.3	-0.349858808	-0.3498587456
0.4	-0.491824698	-0.4918246118
0.5	-0.648721271	-0.6487211607
0.6	-0.822118800	-0.8221186664
0.7	-1.013752707	-1.013752553
0.8	-1.225540928	-1.225540746
0.9	-1.459603111	-1.459602901
1.0	-1.718281828	-1.718281590

Table 3: Comparing the absolute errors for Six step (HBDF) to errors in Mohammed [8]for problem 2

x	Error in K=6 (HBDF)	Error in K=5(BDF) Mohammed [8]
0	0.000000000E+00	0.000000000E+00
0.1	1.480000000E-08	2.198000000E-05
0.2	3.810000002E-08	6.070400000E-06
0.3	6.240000000E-08	1.005100000E-05
0.4	8.620000003E-08	1.402530000E-05
0.5	1.103000000E-07	1.799340000E-05
0.6	1.336000001E-07	2.161620000E-05
0.7	1.540000001E-07	2.799300000E-05
0.8	1.820000000E-07	3.456100000E-05
0.9	2.100000001E-07	4.111400000E-05
1.0	2.380000002E-07	4.765600000E-05

Table 4: Showing exact solutions and the computed results from the proposed methods for problem 2

x	Exact Solution	Proposed Method
0	1	1
0.1	1.094837582	1.094837655

0.2	1.178735909	1.178736102
0.3	1.250856696	1.250857010
0.4	1.310479336	1.310479768
0.5	1.357008100	1.357008646
0.6	1.389978088	1.389978742
0.7	1.409059874	1.409060598
0.8	1.414062800	1.414063636
0.9	1.404936878	1.404937018
1.0	1.38177329	1.381774327

Table 5: Comparing the absolute errors for Six step (HBDF) to errors in Awari [23]for problem 2

x	Error in K=6 (HBDF)	Error in K=6 (BDF) Awari [22]
0	0.000000000E+00	0.000000E-00
0.1	7.300000004E-08	1.157000E-07
0.2	1.930000000E-07	3.099000E-07
0.3	3.140000000E-07	5.055000E-07
0.4	4.320000000E-07	6.957000E-07
0.5	5.459999999E-07	8.789000E-07
0.6	6.540000002E-07	1.054000E-06
0.7	7.240000000E-07	1.008000E-06
0.8	8.360000001E-07	9.226000E-07
0.9	1.400000001E-07	8.261000E-07
1.0	1.037000000E-07	7.216000E-07

Discussion

The HBDF methods are implemented as simultaneous numerical integration for IVPs without requiring starting values and predictors (see Tables 2,3,4 and 5). We proceed by explicitly

obtaining initial conditions at x_{n+k} , $n=0, k, \dots, N-k$ using the computed values $Y(x_{n-k}) = y_{n+k}$ and $\delta(x_{n-k}) = \delta_{n+k}$ over sub-intervals $[x_0, x_k], \dots, [x_{N-K}, x_N]$ which do not overlap. We give examples to illustrate the efficiency of the methods. We report here a numerical example taken from the literature.

Conclusion(s)

A Collocation technique which yields a method with Continuous Coefficients has been presented for the approximate Solution of second Order ODEs with initial conditions. Two test examples have been solved to demonstrate the efficiency of the proposed methods and the Results Compare Favourably with the exact Solution and existing methods in the literature. Interestingly, all the discrete schemes used in the Block formulation were from a single continuous formulation (CF).

References

1. Awoyemi D.O., *A class of continuous linear multistep method for general second order initial value problem in ordinary differential equation*, Intern. J. Comp. Math., 1999, 72, p.29–37.
2. Onumanyi P., Awoyemi D.O., Jator S.N., Sirisena U.W., *New linear Multistep with Continuous Coefficients for First Order Initial Value Problems*, J. Nig. Math. Soc., 1994, 13, p.37-51.
3. Mohammed U., Jiya M., Mohammed A.A., *A class of six step block method for solution of general second order ordinary differential equations*, Pacific Journal of Science and Technology, 2010, 11(2), p.273-277.
4. Yusuph Y., Onumanyi P., *New Multiple FDMS through multi step collocation for $y'' = f(x, y)$* , Abacus, 2002, 29(2), p. 92-100.
5. Onumanyi P., Sirisena U.W and Jator S.N., *Continous Finite difference Approximation for Solving Differential Equations*, Inter.J. Comp Maths, 1999, 72(1), p. 15-27.

- 218 6. Milne W.E., *Numerical Solution of Differential Equations*, John Wiley & Sons: New
219 York, USA, 1953
- 220 7. Rosser J.B., *A Runge-Kutta for all seasons*. *Siam Rev*, 1967, 9(3), p.417–452.
- 221 8. Mohammed U., *A class of implicit five step block method for general second order*
222 *ordinary differential equations*. *Journal of Nigerian Mathematical Society (JNMS)*,
223 30, p 25-39
- 224 9. Ibrahim Z.B., Othman K.I., Suleiman M., *Implicit r-point block backward*
225 *differentiation formula for solving first-order stiff ODEs*, *Applied Mathematics and*
226 *Computation*, 2007, 186, p.558–565.
- 227 10. Majid Z.A., Suleiman M.B., *Implementation of four-point fully implicit block method for*
228 *solving ordinary differential equations*, *Applied Mathematics and Computation*, 2007,
229 184(2), p.514–522.
- 230 11. Akinfenwa O., Jator S., Yoa N., *An eighth order Backward Differentiation Formula with*
231 *Continuous Coefficients for Stiff Ordinary Differential Equations*, *International*
232 *Journal of Mathematical and Computer Sciences*, 2011, 7(4), p.171-176.
- 233 12. Akinfenwa O.A., Jator S.N., Yao N.M., *Continuous block backward differentiation*
234 *formula for solving stiff ordinary differential equations*, *Computers and Mathematics*
235 *with Applications*, 2013, 65, p.996–1005.
- 236 13. Yahaya Y.A., Mohammed U., *A Reformulation of Implicit Five Step Backward*
237 *Differentiation Formulae in Continuous Form for Solution of First Order Initial Value*
238 *Problem*, *Journal of General Studies*, 2009, 1(2), p. 134-144.
- 239 14. Yahaya Y.A., Mohammed U., *Fully Implicit Three Point Backward Differentiation*
240 *Formulae for Solution of First Order Initial Value Problems*, *International Journal of*
241 *Numerical Mathematics*, 2010, 5(3), p.384-398.

- 242 15. Mohammad U., Yahaya Y.A., *Fully Implicit four Point Block Backward Differentiation*
243 *Formulae for Solution of First Order Initial Value Problems*, Leonardo Journal of
244 Sciences, 2010, 16, p. 21-30.
- 245 16. Semenov D.E., Mohammed U., Semenov M.E., *Continuous multistep methods for*
246 *solving first order ordinary differential equations*, Proceeding of Third Postgraduate
247 Consortium International Workshop Innovations in Information and Communication
248 Science and Technology IICST , Tomsk. Russia, 2013, p.165-170.
- 249 17. Van der Houwen P.J., Sommeijer B.P., Cong Nguyen Huu., *Stability of collocation-*
250 *based Runge-Kutta-Nyström methods*, BIT, 1991,31, p.469–481.
- 251 18. Fatunla S.O., *Block Method for Second Order Differential Equation*. International
252 Journal Computer Mathematics, 1991, 41, p. 55-63.
- 253 19. Jiaxiang X., Cameron I.T., *Numerical solution of DAE systems using block BDF*
254 *methods*, Journal of Computation and Applied Mathematics, 1995, 62, p. 255-266.
- 255 20. Lambert J.D., *Computational Methods in Ordinary Differential Equations*, New York,
256 John Wiley and Sons, 1973.
- 257 21. Henrici P.,*Discrete Variable Methods in Ordinary Differential Equations*. John Wiley &
258 Sons: New York, USA.
- 259 22. Awari Y.S.,*Derivation and application of six point linear multistep numerical methods*
260 *for solution of second order initial value problems*, ISOR Journal of Mathematics,
261 2013, 7(2), p.23–29.

262
263