

On the Weyl approach to tensor representations of exceptional Lie groups: The case of G_2 and F_4

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Abstract

An attempt is made to approach the irreducible representations of the exceptional Lie groups G_2 and F_4 by symmetrization of some defining representations by means of Young tableaux, procedure that works rather well for most of the classical groups. For G_2 the program is completely successful, while it is not quite so for F_4 . As the five exceptional Lie groups are related to octonions, we also comment on the “octonion” character of these two groups, in particular the relation of F_4 to some other connected Spin groups.

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1 Introduction

W. Killing was the first to find the *exceptional* Lie groups in 1888, later confirmed and slightly rectified by E. Cartan in his Thesis (1894). Cartan also afterwards (1911) interpreted the lowest rank exceptional group G_2 as the automorphism group of the octonion division algebra $\mathbb{O}$; he went further and developed a method for finding all the irreducible complex (finite-dimensional) representations (*irreps* for short) for *all* semisimple Lie algebras (and hence also for the groups), including the exceptional ones. The method relies heavily on the root system, dual space of the Cartan subalgebra and highest weights of representations, and constitutes nowadays a completely understood situation [12, 14].

The classical Lie groups are by definition matrix groups, so that they come up with a natural, *faithful* minimal irreducible representation; for example, for the Lie algebra A_1 the *compact* representative is the group $SU(2)$, so if we denote $[1]$ as the *defining* representation, it turns out that all the *irreps* of $SU(2)$ can be obtained from this two-dimensional one by means of k^{th} tensor powers $[1]^k$ and reduction to standard symmetry types (partitions of the number k in Young Tableaux), following Weyl’s method [26]: symmetrization *commutes* with taking tensor products. For example, in this (rank one) simplest case, it is enough to take the full symmetric part of the tensor power, say $[k] \equiv \text{full symm. } [1]^k$, to obtain the series D_j of *irreps*, where $k = a \equiv 2j$

is an integer and $\dim D_j = (a + 1) = 2j + 1$. The *irrep* [1] hence corresponds to $j = \frac{1}{2}$, [2] is $j = 1$, [4] is $j = 2$, etc, (so $j = \frac{a}{2}$ is a natural number or **half-integer**, a standard notation to be slightly *modified* below). This method can be applied, in principle, to the full series of Cartan's list, but for the orthogonal and symplectic cases (series B_l, C_l and D_l) one has, besides tensor products and reduction to symmetry type, to take away traces to get the irreducible pieces: symmetric or skew-symmetric traces according to whether one deals with the orthogonal or symplectic cases; also in the case of the B_l, D_l series the presence of the $Spin(n)$ groups makes the procedure *not* completely automatic; in fact the dimensions of the spin groups are better obtained from the corresponding Clifford algebras.

The decomposition problem of representations of (products of) semisimple (complex) Lie algebras has a long history, and constitutes nowadays an important chapter within the representation theory of Lie algebras. Starting with the pioneering work of Cartan, Weyl, Dynkin and Kostant, among others (see e.g [14, 26] and references therein), the characterization of representations by means of highest weights provided an effective method to solve satisfactorily the decomposition problem, as well as deeply related questions like induced and subduced representations. In this context, efficient algorithms based on highest weights of representations allowed to describe decompositions in closed form, and further allowed to deduce important structural properties [17, 18, 23]. As a complement of these algebraic methods, representations of complex Lie algebras have been tabulated extensively [19], and various symbolic computer packages like LiE [11] or SCHUR [27, 28], among others, have been developed to solve the explicit decomposition of tensor products of representations, as well as the related problem of branching rules. In the context of physical applications, the latter package SCHUR turned out to be a valuable tool to analyze symmetric and skew-symmetric tensor product decompositions, an extremely important aspect in establishing selection rules and the understanding of conflicting symmetries observed in atomic spectroscopy [22].

In this work we try to apply this Weyl technique of tensor products and symmetrization (eventually taking the traceless parts) to obtain the unitary irreducible finite dimensional representations of the exceptional groups, working in this paper only with those of lowest rank: G_2 and F_4 , and focusing only on the compact form, so that unitarity of representations is guaranteed. As we shall show, the method encounters special difficulties for the group F_4 , and it seems, at first glance, that this program cannot be realized. The motivation for this approach, rather than the traditional algebraic method, lies in the fact that a direct inspection enables us to establish a connection with certain geometrical features, specially related to the octonion algebra and the geometry of spheres. In this respect, the precise structure of representations can be better visualized when taking into account the geometrical interpretation of the Lie groups as twisting products of spheres [6].

The plan of the work is the following ; in Section 2 we sum up the needed properties of G_2 ; in Section 3 we apply this to obtain the reduction of powers of the defining 7-dimensional *irrep*; in Section 4 we indicate how generic *irreps* of G_2 behave under this method. Then we try to apply the same technique for F_4 in Sections 5-6, advertising already that our results are not comparable in success to the G_2 case. In Section 7 we interpret these two exceptional Lie groups and some

$Spin(n)$ groups ($n = 7, 8, 9$) as groups related to the octonions $\mathbb{O}$, where the non-associativity of $\mathbb{O}$ makes the octonionic vector spaces to stop at dimension three. The last Section enumerates our conclusions and some future perspectives of the problem.

2 Characterization of G_2

The classical series A_l, B_l, C_l and D_l comprehend, as groups, the unimodular $SL(N)$, symplectic $Sp(N)$ and orthogonal groups $O(N)$, characterized respectively as stabilizers, i.e., as transformation groups leaving invariant a n -form (volume form τ for $SL(N)$), a 2-form ω (group $Sp(N)$) or a quadratic form Q (group $O(N)$). The reason for the absence of further series (in the real and complex case) relies mainly on the fact that a third rank tensor on a n -dimensional linear space grows in dimension generically like n^3 , whereas the equivalence under the general linear group $GL(n, \mathbb{R})$ depends only on n^2 . However, there is essentially a *single* (new) exception, which is *non-generic*, namely a 3-form Ω in dimension 7 that allows a stabilizer with $7^2 - 35 = 14$ degrees of freedom, the dimension of the rank two exceptional Lie group G_2 . This point was already considered in [8], see also [3, 9]: one face of G_2 is as the automorphism group of the octonions, the other as stabilizer of a 3-form.

Any algebra of vectors in $\mathbb{R}^n$ can be understood as generated by a mixed tensor $\Gamma \in T_2^1(\mathbb{R}^n)$, as $\Gamma \in T_1^1(\mathbb{R}^n)$ correspond to endomorphisms (or matrices), $\Gamma \in T_0^1(\mathbb{R}^n)$ to (contravariant) vectors and $\Gamma \in T_1^0(\mathbb{R}^n)$ to dual 1-forms θ . As we shall argue in a moment, G_2 also keeps invariant a quadratic form (giving the norm to the octonions), hence it lies inside some orthogonal group, actually $O(7)$. Now any regular (i.e. non-degenerate) quadratic form transforms a tensor $\Gamma \in T_2^1(\mathbb{R}^n)$ into a covariant tensor $\Gamma \in T_3^0(\mathbb{R}^n)$, which becomes a 3-form here by the skew-symmetric properties of the octonion product.

So, as happens for the linear groups, the exceptional G_2 (in its compact form) is naturally endowed with an irreducible real *defining* representation of dimension 7. It is a subgroup of the group $SO(7)$ (as G_2 is connected) and also preserves a 3-form Ω invariant. Having in mind these facts, we shall see that it is possible to compute, by means of tensor product decompositions, the irreducible representations (*irreps* in short) and see how these match the familiar highest weight method. However, before considering this problem it is convenient to recall more closely the relation of G_2 with the octonions (see e.g. [4]).

In searching for division algebras over the reals $\mathbb{R}$, one encounters the complex numbers $\mathbb{C}$ by incorporating an anti-involutive unit i as the unit vector in some $\mathbb{R}^2$ -space. The conjugation $\bar{z} = x - iy$ is an automorphism, and the norm $\mathcal{N}(z) = \bar{z} z$ is real and non-negative definite. Proceeding similarly to this Hamilton found in 1843 the skew-field $\mathbb{H}$ of the quaternions by considering *two* anti-involutive units i and j , which have to anticommute, and adding the product $k = ij$, also anticommuting with both. This enables us to write a quaternion q intrinsically as a scalar part u plus a 3-dimensional vector part $\mathbf{x}$: $q = u + \mathbf{x}$. Here the conjugation $\bar{q} = u - \mathbf{x}$ is an *anti*-automorphism, but the norm $\mathcal{N}(q) = \bar{q} q = u^2 + \mathbf{x} \cdot \mathbf{x}$ is still real and non-negative definite (for a recent treatment of intrinsic quaternions see [10]).

Starting with *three* algebraically independent units e_1, e_2 and e_3 , we need three binary products e_1e_2, e_1e_3 and e_2e_3 as well as a triple product $(e_1e_2)e_3$ (plus 1, of course) to obtain the octonions $\mathbb{O}$ (so $1+3+3+1$ units), that we write assuming already the non-associativity. This leads to a *division algebra* in $\mathbb{R}^8$ which is non-commutative and non-associative, but alternative, as $[e_1, e_2, e_3] := (e_1e_2)e_3 - e_1(e_2e_3)$ is fully skew-symmetric. The norm is real. If $o = v + \xi$ is a generic octonion, with v a real scalar and ξ a 7-vector, then $\bar{o} = v - \xi$ and $N(o) = \bar{o}o \geq 0, o^{-1} = \bar{o}/N(o)$.

To find the automorphism groups (*autos* in short) of these division algebras, we recall that $Aut(\mathbb{R}) = \text{Id}$ (which is rather obvious, as $0 \rightarrow 0$ and $1 \rightarrow 1$ under any automorphism), $Aut_{\mathbb{R}}(\mathbb{C}) = \mathbb{Z}_2$ (complex conjugation) and $Aut_{\mathbb{R}}(\mathbb{H}) = SO(3)$, the latter because any *auto* has to preserve the (real) norm (so it lies inside $O(3)$) and orientation (so it lies inside $SL(3, \mathbb{R})$); in both cases $\mathbb{C}$ and $\mathbb{H}$ the *Aut* group acts naturally (transitively!) on the imaginary norm-one elements, making up the spheres $\mathbb{S}^0$ (just two points) and $\mathbb{S}^2$ respectively.

For the octonions, we can conclude at once that $G_2 := Aut(\mathbb{O}) \subset SO(7)$: The unit imaginary octonions make up the six-sphere $\mathbb{S}^6$. But here, for $\mathbb{O}$, there is more to demand than in $\mathbb{H}$: the vector e_1 can go under some *auto* α to any place in the six-sphere (say, the north pole), while the second unit e_2 will have then $6 - 1 = 5$ degrees of freedom (say, it will go to the equator $\mathbb{S}^5$); but the third unit e_3 would have *only three* degrees, because $\alpha(e_1), \alpha(e_2)$ and $\alpha(e_1e_2) = \alpha(e_1)\alpha(e_2)$ are already fixed. As $6 + 5 + 3 = 14$, this is the dimension of the octonion automorphism group, as Lie group, whereas $\dim SO(7) = 21$. Hence G_2 comes into the world with a minimal irreducible real faithful representation of dimension 7: as $21 = 7 + 14$, it turns out naturally that the Lie algebra of $SO(7)$, seen from G_2 splits in $14 + 7$, i.e. the adjoint plus the defining *irrep*.

Also $Spin(7)$ (*irrep* Δ of $SO(7)$) acts again *transitively* on the 7-sphere (if Δ denotes the spinor representation, $\dim \Delta = 2^{(7-1)/2} = 8$ and real type). The stabilizer (little group) has dimension $21 - (8 - 1) = 14$ and acts on the equator $\mathbb{S}^6$, and this turns out to be another view of acting G_2 ; it can be imagined as $Aut(\mathbb{O})$ or as the stabilizer of $Spin(7)$ acting on the 7-sphere, $Spin(7)/G_2 \simeq \mathbb{S}^7$. By the way, the stabilizer of G_2 acting *transitively* on the 6-sphere is $SU(3)$: $14 - 6 = 8$. One wonders whether the repeated appearance of the group $SU(3)$ in *physics* (as flavor and/or color group, etc.) has any relation with this “octonionic” interpretation of $SU(3)$.

We would also like to recall that both G_2 and $Spin(7)$ can act as *exceptional holonomy groups* on Riemannian manifolds, an industry very much cultivated nowadays (see e.g. [15]): in fact, no other of the remaining exceptional groups can act as an holonomy group. In particular, a compact 7 dimensional manifold of G_2 holonomy is the natural choice [24] for stringlike compactifications in M-theory, which works in 11 dimensions.

3 Tensor products. Finding the *irreps* of G_2

In this section we embark on our program: to find the irreducible representations of the (compact real) form of G_2 by means of tensor products and splitting, starting from the defining (fundamental)

representation in dimension 7. To achieve this, *four* important facts have to be considered:

- G_2 is a *rotation* group, as $G_2 \subset SO(7)$ [implies taking traces out in symmetric tensors]
- G_2 leaves a 3-form Ω invariant, from the octonion product [also taking traces out]
- The *adjoint* representation of G_2 lies inside the Young tableau $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$ as $G_2 \subset O(7)$ the adjoint *irrep* of which lies in $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$, so that $21 = 14 + (21 - 14)$
- G_2 has *rank two*, thus it will be enough to consider two-row tableaux, while from the 7-dimensional *irrep* up to three will occur (or four, but $\binom{7}{3} = \binom{7}{4} = 35$).

With these conditions implicitly assumed, it is sort of mechanical to deduce the *irreps* of G_2 from the Weyl technique of splitting the powers of the defining *irrep* of dimension 7. We convene to denote the latter representation by $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$. The first step is the reduction of the square:

$$\begin{smallmatrix} \square \\ \square \end{smallmatrix} \otimes \begin{smallmatrix} \square \\ \square \end{smallmatrix} = \begin{smallmatrix} \square & \square \end{smallmatrix} + \begin{smallmatrix} \square \\ \square \end{smallmatrix} \quad \text{dimensions : } 7 \times 7 = 28 + 21; \quad (1)$$

It is obvious that the adjoint representation of $O(7)$ is within $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$, so that we know how the latter splits for G_2 : $21 = 14 + 7$ (Adj. + Def. representations). This enables us to recover the adjoint and defining representations of G_2 . As $G_2 \subset SO(7)$, we have to take the trace of the symmetric parts in the tensor products, leading to the decomposition $28 = 27 + 1$. Indeed, 1, 7, 14 and 27 are the smallest possible dimensions of the *irreps* of G_2 (see general formula below): $\bullet$ will denote the trivial one dimensional representation. The tensor product of the defining *irrep* leads to the tableaux ($7^2 = (27 + 1) + (7 + 14)$):

$$\begin{smallmatrix} \square \\ \square \end{smallmatrix} \otimes \begin{smallmatrix} \square \\ \square \end{smallmatrix} = \begin{smallmatrix} \square & \square \end{smallmatrix} + \begin{smallmatrix} \square \\ \square \end{smallmatrix} = \left(\begin{smallmatrix} \square & \square \end{smallmatrix}' + \bullet \right) + \left(\begin{smallmatrix} \square \\ \square \end{smallmatrix}' + \begin{smallmatrix} \square \\ \square \end{smallmatrix}' \right) \quad (2)$$

We have defined the $'$ *irreps* as one of the G_2 -irreducible parts, arbitrarily. This notation will be used in the following.

To “correspond” with the traditional method (with roots and weights), we just write the sixth order polynomial (see [14] p.140) yielding the dimensions of the *irreps* in terms of *two* integers a, b (since $\text{rank } G_2 = 2$) by the highest weight procedure. The polynomial is of sixth order because this is the number of positive roots, as $\frac{1}{2}(\dim G_2 - \text{rank } G_2) = \frac{14-2}{2} = 6$. The full formula reads:

$$\dim(a, b) = \frac{1}{5!} (a+1) (b+1) (a+b+2) (a+2b+3) (a+3b+4) (2a+3b+5) \quad (3)$$

where $a, b \in \mathbb{Z}^+$ are natural numbers or zero; from the Dynkin diagram of G_2 one easily deduces the last three terms above, in particular the sum of the two fundamental roots is a positive root (term $(a + b + 2)$ above).

We have at once

Symbol	•			
dim	1	7	14	27
(a, b)	(0, 0)	(1, 0)	(0, 1)	(2, 0)
Name	Id	Def(ining)	Adj(oint)	

in a self-explaining notation. This is precisely our program in the simplest cases. To pursue, here are more examples of decomposition of products of *irreps*:

$$\begin{array}{c} \text{single box} \otimes \text{two boxes stacked} = \text{two boxes side-by-side}' + \text{three boxes stacked} + \text{single box} \quad \text{or} \quad 7 \times 14 = 64 + 27 + 7; \end{array} \quad (4)$$

$$\begin{array}{c} \text{single box} \otimes \text{two boxes side-by-side}' = \text{three boxes side-by-side}' + \text{two boxes stacked}' + \text{two boxes side-by-side}' + \text{two boxes stacked} + \text{single box} \\ \text{or} \quad 7 \times 27 = 77 + 64 + 27 + 14 + 7; \end{array} \quad (5)$$

To rationalize these, we know that the *irrep*  of dimension $7 \times 8 \times 9/3! = 84$ becomes $(84 - 7) + 7$ taking out traces, so that ' is of dimension 77; the split of , of dimension 112, is more complicated, but still unique. As for  of dimension 35, it should contain the trivial representation because G_2 preserves the 3-form Ω , so that $35 - 1 = 34$ becomes obviously $27 + 7$:

three row Tableaux are irrelevant, so that in dimension 27 we have ' = , as we said above. In the following, $[M, N]$ will label any two-row *irrep* coming from tableaux; full equivalence to the (a, b) notation, which we shall also simplify a bit, will be exhibited just below. We should remark that there are non-equivalent representations with the same dimension. Above we have seen that ' has dimension 77; there is another one corresponding to the irreducible part of .

4 Expression of the *irreps* of G_2 using Tableaux

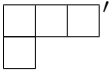
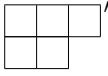
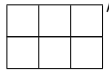
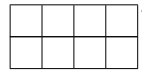
We just limit ourselves to show the lowest order *irreps*, with the corresponding tableaux (with the additional ' label), the dimension and the correspondence with the classical (a, b) notation. Our

starting point is the trivial representation $\bullet$, the *defining* representation $(1, 0)$ of dimension 7, and the *adjoint* representation $(0, 1)$ of dimension 14. As they are codified in the Dynkin diagram, we tend to call them either *primordial irreps* or *fundamental irreps*.

Now, applying directly the general formula (3), we obtain the following values

Symbol	$\bullet$					
dim	1	7	14	27	64	77
$[a, b]$	$(0, 0)$	$(1, 0)$	$(0, 1)$	$(2, 0)$	$(1, 1)$	$(0, 2)$

(6)

Symbol						
dim	77	189	286	273	182	748
$[a, b]$	$(3, 0)$	$(2, 1)$	$(1, 2)$	$(0, 3)$	$(4, 0)$	$(0, 4)$

(7)

The general trend is self-explanatory: two row Young tableaux with $[M, N]$ labels correspond precisely to the *irrep*

$$[M = a + b, N = b] \quad (8)$$

(or (a, b) in the classical highest weight notation). As said, it is enough to consider two-row tableaux, because the rank of G_2 is two. The tableau with a *single* row $[M, N = 0]$, as we already suspected, corresponds exactly to the $(a, b = 0)$ *irrep* in the classical highest weight notation; also the full two-row tableau $[M, M]$ corresponds to the $(0, b)$ case. In particular, $(0, 2)$ is the *second irrep* of dimension 77 (sometimes denoted $\overline{77}$) while the first was $(3, 0)$. There are higher dimensional non-equivalent *irreps* having the same dimension, like $(2, 3)$ and $(8, 0)$ with $d = 2079$, $(1, 5)$ and $(7, 1)$ with $d = 4928$, etc.

For completeness in the exposition, we note that this procedure is a direct adaptation of the classical case, by adding the general Weyl-Freudenthal results for the remaining rank less than three cases (see e.g. [14]), but we change *slightly* the notation, writing now also natural numbers as $n, m, p, q, \dots$; More precisely: n for $a + 1$; m for $b + 1$, p for $c + 1$ and q for $d + 1$ (the last two in further cases). So obviously now $n, m, p, q \in \mathbb{N} = 1, 2, 3 \dots$ are natural numbers (which seems more customary).

A_1 .- Dynkin diagram $\bullet$.- Weyl group $\mathbb{Z}_2$.- Compact form $SU(2)$ of dimension 3.- Dimension of *irreps* $(a + 1) = n$.- Smallest *irreps* 1 (Id), 2 (Def.), 3 (Adj.), 4, 5... In terms of angular momentum we have $a = 2j$; $n = 2j + 1$.

$A_1 \oplus A_1$.- Dynkin diagram $\bullet \quad \bullet$.- The Weyl group is the Klein Vierergruppe $V = (\mathbb{Z}_2)^2$.- Compact form $Spin(4)$ of dimension 6.- Dimension of *irreps* $(a + 1)(b + 1) = n m$.- Smallest *irreps* $(1, 1)$ (Id), $(2, 1)$ of dimension 2, $(1, 2)$ of dimension 2, $(2, 2)$ of dimension 4 [$SO(4)$], $(3, 1) + (1, 3)$ of dimension 6 (adjoint: reducible as $O(4)$ is not simple), etc.

A_2 .- Dynkin diagram $\bullet \text{---} \bullet$.- Weyl group S_3 .- Compact form $SU(3)$ of dimension 8.- Dimension of *irreps* $\frac{1}{2}(a + 1)(b + 1)(a + b + 2) = \frac{1}{2}n m (n + m)$.- Smallest *irreps* 1 (Id), 3 (Def.), $\underline{3}$, $\underline{6}$, $\underline{6}$, 8 (Adj.), 10, etc.

$B_2 = C_2$.- Dynkin diagram $\bullet \implies \bullet$.- Weyl group D_4 (“octic group”).- Compact form $Spin(5) = Sp(2)$ of dimension 10.- Dimension of *irreps* $\frac{1}{3!}(a+1)(b+1)(a+b+2)(a+2b+3) = \frac{1}{3!}n \ m \ (n+m) \ (n+2m)$.- Smallest *irreps* 1 (Id), 4 ($Spin(5)=Sp(2)$), 5 = $\frac{1}{2}(4 \times 3) - 1$ ($SO(5)$), 10 (Adj.), etc.

The degree of the polynomial giving the dimension of the *irreps* is always the number of positive roots. For A_3 we would have $\frac{1}{2}(\dim A_3 - \text{rank } A_3) = \frac{1}{2}(15 - 3) = 6$, and also for G_2 we obtain a sixth order polynomial as $\frac{1}{2}(\dim G_2 - \text{rank } G_2) = \frac{1}{2}(14 - 2) = 6$.

5 The case of F_4

As happens for G_2 , the first step should be to have a consistent definition of the group. The *first* definition of F_4 is as a group acting by automorphisms on the Jordan algebra or set of Hermitean traceless 3×3 matrices over the octonions $\mathbb{O}$ ($J_3(Oct)$) see below):

$$\begin{pmatrix} x & o_1 & o_2 \\ \bar{o}_1 & y & o_3 \\ \bar{o}_2 & \bar{o}_3 & -x - y \end{pmatrix},$$

where $x, y \in \mathbb{R}$ and o_i are octonions. The total number of degrees of freedom is $3 \times 8 + 2 = 26$. The group F_4 acts on these matrices, maintaining the commutative symmetric product $M \circ M' = \frac{1}{2}(MM' + M'M)$ and yields the (defining) real irreducible representation of dimension 26.

There is, however, a *second* and more beautiful characterization of F_4 . Take $O(8)$ or rather $Spin(8)$ with the known triality. Among other things, this means that the three 8-dimensional *irreps* are permuted by this triality: the vector *irrep* $\square$ and the two spinor *irreps* $\triangle_{L,R}$ are real and isomorphic, as abstract group, to $Spin(8)/\mathbb{Z}_2 \simeq SO(8)$. The centre of $Spin(8)$ is isomorphic to the Klein *Viererguppe* $\mathbb{Z}_2^2$, generated by three involutions. It turns out that each one of them is sacrificed in the three quotients, $\triangle_{L,R}$. Notice that any $Spin(4n)$ has a centre isomorphic to $\mathbb{Z}_2^2$, with automorphism group S_3 , but in all dimensions except 8 the full outer automorphism group of $Spin(4n)$ is only $\mathbb{Z}_2$, permuting just the two *chiral* spinor *irreps*.

The group S_3 , permuting the three *irreps* of $Spin(8)$ can be seen as a group of *outer* automorphisms of $Spin(8)$, as follows clearly from the Dynkin diagram, and as the Weyl group of the latter has order $8 \times 24 = 192$ ($\mathcal{W}(O(8)) \simeq \mathbb{Z}_2^3 \triangleleft S_4$) and $192 \times 6 = 1152$, which is the order of the Weyl group of F_4 . As $\dim F_4 = 52$, it becomes clear from

$$\dim F_4 = \dim O(8) + 3 \times 8 \tag{9}$$

as it incorporates $SO(8)$ with the three 8 dimensional *irreps*. The *outer* automorphisms of $O(8)$ become inner automorphisms of F_4 . Incidentally, the Weyl group of F_4 is also the isometry group of the *octacube*, the combination of a hypercube and a hyperoctahedron, an exceptional regular polytope living *only* in dimension 4 found by Schläfli in 1851 (see e.g. [21]; in simplices it is 24, 96, 96, 24 (so self-dual)), and having *no* counterpart in any other dimension [7].

There is a *third* presentation of F_4 , albeit related to the first one: the Octonions are non-associative, so no vector spaces of dimension higher than three make sense. In particular, Ruth Moufang found in 1932 the projective plane $\mathbb{OP}^2$ of (real) dimension 16, so that F_4 acts *transitively* on it by isometries¹ with little group (stabilizer) $Spin(9)$:

$$\mathbb{OP}^2 \simeq F_4 / Spin(9) : 16 = 52 - 36. \quad (10)$$

Indeed, one can extract the 16 idempotents of $\mathbb{OP}^2$ from the first characterization of F_4 , as acting on $J_3(Oct)$ with 27 elements (arbitrary trace): so that this presentation hinges with the first one. Under the subgroup $Spin(9)$, the 26 dimensional *irrep* of F_4 splits as $26 = 16 + vect(9) + 1$, while under $Spin(8)$ it splits as $26 = 8 \times 3 + 2 \times 1$. Notice that all groups $SO(8)$, $SO(9)$ and F_4 have rank four, which facilitates the inclusions below.

There exists an interesting chain relating $G_2 \subset Spin(7)$ and $F_4 \subset E_6$ to some of the spin groups (ranks 2, 3, 4, 4, 4, 6):

$$G_2 \subset Spin(7) \subset Spin(8) \subset Spin(9) \subset F_4 \subset E_6 \quad (11)$$

For example, $21 = 14 + 7$ for $Spin(7)$, $14 = \text{Adj. of } G_2, \text{ Adj.} + (\text{vect}(\text{def.}) = 7)$; it matches with $78 = 52 + 26$ for E_6 , $\text{Adj}(52)$ of F_4 , $\text{Adj.} + (\text{def}(26))$. But there is a better chain: the defining *irrep* of $Spin(7)$ is of dimension 8, and acts *transitively* on the sphere $\mathbb{S}^7$ with little group G_2 . An analogous situation happens for E_6 and F_4 . While the defining *irrep* of E_6 is complex of dimension 27, the defining *irrep* of F_4 is of dimension 26. Adams [1, 2] comments on all this and expands the chain up to E_8 . For example, he finds

$$\dim E_8 = 248 = \dim F_4 + \dim G_2 + 26 \times 7, \quad (12)$$

the latter term being the product of the dimensions of the defining *irrep* of F_4 and G_2 . This enables us to concoct the longer chain

$$SU(2) \subset SU(3) \subset G_2 \subset Spin(7) \subset Spin(8) \subset Spin(9) \subset F_4 \subset E_6 \subset [E_7] \subset E_8. \quad (13)$$

We hope that these inclusions and their consequences will be better understood in the future, as well as the rôle that they play in physical applications.

As a consequence of all this, the situation of the simple Lie group F_4 is rather different from the previous considered group G_2 . Hence, the problem of characterizing *irreps* also presents rather different features.

¹The exceptional group E_6 also acts on this plane by collineations.

6 *Irreps of F_4*

The highest weight procedure works for F_4 as for any other Lie group, so that we just include the formula for completeness. We write the dimension formula in dependence of n, m, p, q where, like before, $n = a + 1$, etc. We also adopt the convention that the *irrep* (0001) of dimension 26 is written as [1112] in our notation, (0010) of dimension 273 as [1121], (0100) of dimension 1274 as [1211] and (1000) of dimension 52 as [2111]. Then the formula reads [12]:

$$\begin{aligned} \dim [n m p q] = & n m p q (n + m) (2m + p) (p + q) (2n + 2m + p) (2m + p + q) \\ & (m + p) (2n + 2m + p + q) (n + m + p) (2m + 2p + q) (2n + 2m + 2p + q) \\ & (n + 2m + p) (m + p + q) (n + m + p + q) (4n + 8m + 4p + q) (n + 2m + p + q) \\ & (2n + 4m + 3p + q) (2n + 4m + 3p + 2q) (n + 2m + 2p + q) (n + 3m + 2p + q) \\ & (2n + 3m + 2p + q) / (5893965000 \times 2^{13}). \end{aligned} \quad (14)$$

If we try to understand the defining representation [1112] in analogy to the procedure for G_2 , then the wedge product should reproduce the representation [1121], which does not happen as $\frac{1}{2}(26 \times 26) = 273 + 2 \times 26$. Also $52 = 2 \times 26$ does not arise easily from wedge products, so that we abandon the attempt, and place F_4 in the same “box” as the B-D groups, for which the Spin *irreps* spoil the Weyl procedure.

However, some features of the F_4 *irreps* are understandable, for example in the product

$$25 \times 26 = 351 + 325 = (324 + 26 + 1) + (273 + 52). \quad (15)$$

The trivial representation (1) is to be expected, as our group should be an orthogonal group (as isometry of the Cayley plane, for example). We also expect the square of the adjoint representation to include a copy of itself:

$$\begin{array}{ccccccccc} [2111] \otimes [2111] = & [3111] & + & [1211] & + & [1113] & + & [2111] & + & [1111] \\ 52^2 = & 1053 & & 1274 & & 324 & & 52 & & 1 \end{array} \quad (16)$$

The symmetric part of $[2111] \otimes [2111]$ splits as $[3111] + [1113] + [1111]$, while the skew-symmetric part contains $[1211] + [2111]$.

However, for the remaining products there is no apparent pattern as seen for G_2 , and hence the procedure cannot be carried further without introducing additional assumptions. These features point out that the procedure must be somewhat generalized in order to cover the case of F_4 as effectively as it was for G_2 . Instead of trying more cases, in the following section we indulge in comparing these “octonionic” groups with the other (compact) cases: orthogonal ($\mathbb{R}$), unitary ($\mathbb{C}$) and symplectic-unitary ($\mathbb{H}$).

7 Octonionic groups

Classical Lie groups are naturally characterized as transformation groups, which enlighten their structure and make them naturally available for applications. In this context, it is well known that the Hermitean forms over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ give rise to the orthogonal, unitary and symplectic-unitary Lie groups $O(n)$, $U(n)$ and $SpU(n)$ respectively. For the last groups we will use the alternative notation $Sq(n)$, as $Sp(n)$ refers to groups preserving a skew-symmetric bilinear form. A natural question that arises from this series is whether there are groups directly connected to the octonion division algebra. Even if it fails (from the algebraic point of view) to have properties as nice as the underlying fields, its geometrical richness suggests that some groups should be more or less naturally associated to the Octonions.

Recall first a result of Hopf stating that the Betti numbers of semisimple Lie groups are given by those of odd spheres (see e.g. [6]). For example, the following naked spheres correspond to three groups as manifolds:

$$O(1) \simeq \mathbb{S}^0; \quad U(1) = SO(2) \simeq \mathbb{S}^1; \quad \widetilde{SO}(3) = Spin(3) \simeq \mathbb{S}^3 \quad (17)$$

While the bundle $\mathbb{S}^3 \simeq \mathbb{S}U(2) \rightarrow SU(3) \rightarrow \mathbb{S}^5$ implies that $SU(3) \simeq \mathbb{S}^3 \times \mathbb{S}^5$, where $\times$ just means the twisting product of spheres, in general we have the following correspondences [6]:

$$U(n) \simeq \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{S}^5 \times \dots \times \mathbb{S}^{2n-1} \quad (18)$$

$$O(2n+1) \simeq \mathbb{S}^0 \times \mathbb{S}^3 \times \mathbb{S}^7 \times \dots \times \mathbb{S}^{4n-1} \quad (19)$$

$$O(2n) \simeq \mathbb{S}^0 \times \mathbb{S}^3 \times \mathbb{S}^7 \times \dots \times \mathbb{S}^{4n-5} \quad (20)$$

In fact, the odd sphere expansion above just means identity of the Betti numbers. For each odd sphere in the product there is an invariant I_m :

$$I_2 \text{ corresponds to } \mathbb{S}^3 : 2 = \frac{1}{2}(3+1) \text{ is the Cartan-Killing metric} \quad (21)$$

$$I_n \text{ corresponds to } \mathbb{S}^{2n-1} : n = \frac{1}{2}(2n-1+1) \quad (22)$$

In fact, as I_2 is an invariant for any semisimple group, the sphere expansion always starts by a sphere $\mathbb{S}^3$. Incidentally, the sequence has to be “palindromic” to ensure that the Euler number χ (alternate sum of Betti numbers) is zero, as it should be as Lie groups are parallelizable. For example, for the group $SO(8)$ the sphere dimensions are (3, 7, 7, 11).

All this is still valid for the exceptional Lie groups. For G_2 we have $G_2 \simeq \mathbb{S}^3 \times \mathbb{S}^{11}$, where $8 = 11 - 3$ reflects the octonion number 8. This difference is equal to 4 for the $Sq(n)$ and orthogonal groups, and just 2 for the unitary series. On the other hand, the sphere $\mathbb{S}^7$ *would* correspond to the unit octonionic group, *but* the non-associativity prevents $\mathbb{S}^7$ from being a legitimate group (in modern mathematical terminology this corresponds to a *loop*, the underlying manifold being parallelizable due to the octonion product!). As Ramond explains in detail [20], the sphere $\mathbb{S}^7$ gets

“stabilized” to a true (i.e. associative) group by incorporation of the 3- and 11- spheres (we do not recount the argument, see [20] for details), with the result that some spin groups ($n = 7, 8, 9$) acquire a certain “octonionic” character. We just include the full list:

1. $\mathbb{S}^7$ is like “ $Oct(1)$ ” group: as it is not a group, so the “”
2. $G_2 \simeq \mathbb{S}^3 \times \mathbb{S}^{11}$ is like $SOct(1)$ group (it lacks the sphere $\mathbb{S}^7$)
3. $Spin(7) \simeq \mathbb{S}^3 \times \mathbb{S}^7 \times \mathbb{S}^{11}$ is like $Oct(1)$
4. $Spin(8) \simeq \mathbb{S}^3 \times \mathbb{S}^7 \times \mathbb{S}^7 \times \mathbb{S}^{11}$, similar to $O(4) \simeq \mathbb{S}^3 \times \mathbb{S}^3$, we write $Oct(1)^2$
5. $Spin(9) \simeq \mathbb{S}^3 \times \mathbb{S}^7 \times \mathbb{S}^{11} \times \mathbb{S}^{15}$: we call it $Oct(2)$.
6. $F_4 \simeq \mathbb{S}^3 \times \mathbb{S}^{11} \times \mathbb{S}^{15} \times \mathbb{S}^{23}$: we call it $SOct(3)$ (S because the absence of $\mathbb{S}^7$).

These are just our *definitions*, so that no proof is required. Notice also the sphere dimension differences: they oscillate between 4 and 8, which once more relate real orthogonal groups with “octonionic” groups.

Just a final remark: the projective plane $\mathbb{K}\mathbb{P}^2$ over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ is a symmetric space [13]:

$$\mathbb{R}\mathbb{P}^2 \simeq SO(3)/O(2), \quad \mathbb{C}\mathbb{P}^2 \simeq SU(3)/U(2), \quad \mathbb{H}\mathbb{P}^2 \simeq Sq(3)/q(2) \quad (23)$$

with real dimensions 2, 4 and 8. For the octonions (Moufang plane) we have a similar situation

$$\mathbb{O}\mathbb{P}^2 = F_4/Spin(9) = SOct(3)/Oct(2) \quad (24)$$

of dimension $52 - 36 = 16$, still being a symmetric space.

8 Final comments

In this work, a direct approach to the representations of the exceptional Lie groups G_2 and F_4 based on the Weyl technique has been considered, avoiding the traditional method of highest weight theory, trying to understand the representations in terms of symmetry properties of Young tableaux.

A lot of features concerning the exceptional Lie groups still remain to be properly understood, specially in connection with the E -series and its geometrical implementation into High Energy Physics [20]. We believe that G_2 is fairly well understood, and have tried to dispel one of the remaining uncertainties about this group: as any of the classical groups, the *irreps* can be understood in terms of symmetry types, starting from the defining seven dimensional representation and taking into account the natural action of G_2 as the automorphism group of the unit imaginary octonions.

Even if for the group F_4 there remain, we feel, some aspects to be worked out, the fact that the plane $\mathbb{O}\mathbb{P}^2$ was identified with the symmetric space $F_4/Spin(9)$ by Borel [5] provides fresh and

interesting insights. As the group F_4 plays a considerable role in various models of atomic physics [16, 25], it would be desirable to find a more compact geometrical description of the irreducible representations, besides its purely algebraic description. The direct adaptation of Weyl's technique has been shown to be unsuccessful, albeit we are confident in finding some generalization of the procedure in order to cover this Lie group and, if possible, the groups of the E -series. In this context, fermion realizations could be a suitable alternative. Whether such a generalization can provide an effective answer to some questions left open by the algebraic method, like the explanation of unsuspected symmetries underlying the vanishing of matrix elements satisfying the usual selection rules, is still unknown. We hope however to find a satisfactory solution, at least partially, to these problems in the near future.

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