

Some Fully Symmetric Quadrature Rules for Approximation of Complex Line Integral

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Abstract: This paper is concerned with the construction of some quadrature rules of algebraic degree of precision nine, thirteen and seventeen for the approximate evaluation of integrals of an analytic function on a line segment in the complex plane. The nodes in each of the rules constructed are symmetrically situated about the contour of integration. Asymptotic error estimate of each rule has been derived and a comparative study regarding the accuracy in approximations obtained by the rules constructed has been studied. Some standard integrals of analytic functions have been numerically integrated by the quadrature rules constructed in this paper to substantiate the conclusion drawn on the relative efficiency of the rules.

Key Words: Analytic function, Quadrature rule, degree of precision, asymptotic error estimate.

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1. INTRODUCTION

Birkhoff-Young [2,1950] have formulated first, a five point quadrature rule :

$$R_1(f) = \frac{h}{15} [24f(z_0) + 4\{f(z_0 + h) + f(z_0 - h)\} - \{f(z_0 + ih) + f(z_0 - ih)\}]; \quad i = \sqrt{-1}, \quad (1)$$

for the numerical integration of

$$I(f) = \int_L f(z) dz, \quad (2)$$

where L is a directed line segment joining the points $z_0 - h$ and $z_0 + h$. Further the function $f(z)$ is assumed to be analytic in the disc

$$\Omega = \{z: |z - z_0| \leq r; r > |h|\} \text{ containing the line segment } L.$$

The degree of precision of this rule is five and its error term is given by

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$$|E_1(f)| \leq \frac{|h|^7}{1890} \max_{z \in S} |f^{(6)}(z)|, \quad (3)$$

where S denotes the square with vertices $z_0 + i^k h$, $k=0,1,2,3$ (Ref [17] and [3,p.136]). This error estimate is about four tenth as large as the corresponding error

$$|E_1^{ES}(f)| \leq \frac{|h|^7}{756} |f^{(6)}(\eta)|; \quad 0 < \frac{\eta - (z_0 - 2h)}{4h} < 1$$

associated with the so called extended Simpson rule [Scarbour, p.124] :

$$R_{ES} = \frac{h}{90} [114f(z_0) + 34\{f(z_0 + h) + f(z_0 - h)\} - \{f(z_0 + 2h) + f(z_0 - 2h)\}], \quad (4)$$

Later on, Lether [4, 1976] using the transformation:

$$z = z_0 + ht; \quad -1 \leq t \leq 1$$

transformed the integral (2) to the integral:

$$I(f) = h \int_{-1}^1 f(z_0 + ht) dt \quad (5)$$

and then suggested to numerically integrate the integral (5) by Gauss-Legendre three point rule to get the approximate value of the integral (2). It is shown analytically that the numerical integration of (2) by this procedure recommended by Lether produces better result than that obtained by the rule $R_1(f)$ although algebraic degree of precision of both the rules is five. This procedure is preferred to numerical integration by the rule $R_1(f)$ since, it requires the evaluation of the function at three nodes: $z_0, z_0 \pm \sqrt{0.6}h$ only and the truncation error is less than that incurred in approximation by the rule (1).

Tosic [16,1978] considering a different set of nodes proposed the following five point degree seven rule for the approximation of integral (2) which is given as

$$R_T(f) = \frac{16}{15}f(z_0) + \frac{h}{6} \left(\frac{7}{5} + \sqrt{\frac{7}{3}} \right) \{f(z_0 + \alpha h) + f(z_0 - \alpha h)\} + \frac{h}{6} \left(\frac{7}{5} - \sqrt{\frac{7}{3}} \right) \{f(z_0 + i\alpha h) + f(z_0 - i\alpha h)\}, \quad (6)$$

$$\alpha^4 = \frac{3}{7}.$$

Also, it is shown that the truncation error $E_T(f)$ incurred in approximation of the integral (2) by the rule (6) is

$$|E_T(f)| \sim \frac{|h|^9}{793800} |f^{(8)}(z_0)|.$$

The rule (6) is obtained as a special case of one parameter rule formulated by Tosic [16] i.e

$$R_T(\alpha) = h \left\{ 2 \times \left(1 - \frac{1}{5\alpha^4} \right) f(z_0) + \left(\frac{1}{6\alpha^2} + \frac{1}{10\alpha^4} \right) [f(z_0 + \alpha h) + f(z_0 - \alpha h)] + \left(-\frac{1}{6\alpha^2} + \frac{1}{10\alpha^4} \right) [f(z_0 + i\alpha h) + f(z_0 - i\alpha h)] \right\} \quad (7)$$

for the parameter $\alpha^4 = \frac{3}{7}$ and the rule $R_T(f)$ has been referred to as the formula of **maximum accuracy**. It may be noted here that, the rule given in equation (7) reduces to Birkhoff-Young rule given in (1) for $\alpha = 1$ and for $\alpha = \sqrt{3/5}$ the same rule given in equation (7) becomes three Point Gauss-Legendre transformed rules recommended by Lether [4].

Milovanović and Dordević [11] have further generalized the modified Birkhof-Young rule (R_T) to a two parameter rule (R_{MD}) of precision 9 with the corresponding truncation error E_{MD} in general and have obtained a rule of precision 13 corresponding to specific values of the parameter (r_1, r_2) given by

$$r_1^4 = \frac{63+4\sqrt{114}}{143} \quad \text{and} \quad r_2^4 = \frac{63-4\sqrt{114}}{143}$$

which are obtained by solving

$$E_{MD}[(z - z_0)^{10}] = 0 = E_{MD}[(z - z_0)^{12}]$$

for numerical integration of the integral (2).

Quadrature formulae of Birkhoff-Young type for analytic functions have been investigated in several papers [1], [9], [10], [12], [13]. These formulas can also be used to integrate real harmonic function (see [2]). In addition, we mention also that Lyness and Delves [5] and Lyness and Moler [6] and later Lyness [7] developed formulae for numerical integration and numerical differentiation of complex functions. Milovanovic et al [14] have also constructed quadrature rule of the Birkhoff-Young type for approximation of complex definite integral with a weight function.

As far as we know, quadrature rules of Newton Cotes type (meant for approximation of real definite integral) have not been formulated so far for approximate evaluation of complex line integrals of an analytic/ singular function choosing nodes symmetrically situated with respect to the line segment in the complex plane. Hence, our aim in this paper is to construct some such rules of precision nine, thirteen and seventeen choosing suitable nodes. It may be noted here that Birkhoff-Young rule $R_1(f)$ given in equation (1) is one such rule which is analogous to Simpson 1/3rd rule meant for approximation of real definite integrals.

2. CONSTRUCTION OF QUADRATURE RULE OF PRECISION NINE

We choose the nodes:

$$z_0, z_0 \pm h, z_0 \pm ih, z_0 \pm i\frac{h}{2},$$

of which the first five nodes are the nodes of the Birkhoff-Young rule (R_1) for the formulation of rule of precision 9 for approximate evaluation of the integral (2).

The proposed rule is denoted by $R_2(f)$ and is given by

$$\begin{aligned} R_2(f) = & A_{21}f(z_0) + B_{21}\{f(z_0 + h) + f(z_0 - h)\} + B_{22}\left\{f\left(z_0 + \frac{h}{2}\right) + f\left(z_0 - \frac{h}{2}\right)\right\} \\ & + C_{21}\{f(z_0 + ih) + f(z_0 - ih)\} + C_{22}\left\{f\left(z_0 + i\frac{h}{2}\right) + f\left(z_0 - i\frac{h}{2}\right)\right\}. \end{aligned} \quad (8)$$

The weights $A_{21}, B_{21}, B_{22}, C_{21}$ and C_{22} are to be determined so that

$$I[(z - z_0)^{2k}] = R_2[(z - z_0)^{2k}]; \quad k = 0, 1, 2, 3, 4. \quad (9)$$

From the conditions stated in equation (9) on the proposed quadrature formula (8), we get the following set of linear equations in $A_{21}, B_{21}, B_{22}, C_{21}$ and C_{22} :

$$A_{21} + 2B_{21} + 2B_{22} + 2C_{21} + 2C_{22} = 2h,$$

$$4B_{21} + B_{22} - 4C_{21} - C_{22} = \frac{4h}{3},$$

$$16B_{21} + B_{22} + 16C_{21} + C_{22} = \frac{16h}{5},$$

$$64B_{21} + B_{22} - 64C_{21} - C_{22} = \frac{64h}{7}$$

and $256B_1 + B_2 + 256C_1 + C_2 = \frac{256h}{9}.$

On solving the above equations, we have

$$A_{21} = \frac{-5880}{4725}h = -1.244\ 444\ 444\ 444\ 444\ 4h,$$

$$B_{21} = \frac{556}{4725}h = 0.117\ 671\ 957\ 671\ 958\ h,$$

$$B_{22} = \frac{5504}{4725}h = 1.164\ 867\ 724\ 867\ 725\ h,$$

$$C_{21} = \frac{-59}{4725}h = -0.012\ 486\ 772\ 486\ 772\ h$$

and $C_{22} = \frac{1664}{4725}h = 0.352\ 169\ 312\ 169\ 312\ h.$

Thus the quadrature rule proposed in equation (8) is now given by

$$\begin{aligned} R_2(f) = h & \left[\frac{-5880}{4725} f(z_0) + \frac{556}{4725} \{f(z_0 + h) + f(z_0 - h)\} + \frac{5504}{4725} \left\{ f\left(z_0 + \frac{h}{2}\right) + f\left(z_0 - \frac{h}{2}\right) \right\} \right. \\ & \left. - \frac{59}{4725} \{f(z_0 + ih) + f(z_0 - ih)\} + \frac{1664}{4725} \left\{ f\left(z_0 + i\frac{h}{2}\right) + f\left(z_0 - i\frac{h}{2}\right) \right\} \right] \end{aligned} \quad (10)$$

which is a 9 point rule of degree of precision at least 9.

Now,

$$I(z - z_0)^{10} = \frac{2 \times h^{11}}{11}$$

and $R_2(z - z_0)^{10} = \frac{2 \times 11h^{11}}{84}.$

Thus, $I(z - z_0)^{10} \neq R_2(z - z_0)^{10}$

which implies the degree of precision of $R_2(f)$ is exactly 9.

In the same vain (as the procedure is a mere repetition), we have constructed the rules $R_3(f)$ and $R_4(f)$ having degree of precisions thirteen and seventeen respectively and the rules are given by

$$\begin{aligned} R_3(f) = h & \left[\frac{932}{65} f(z_0) + \frac{1426361}{19519500} \{f(z_0 + h) + f(z_0 - h)\} - \frac{4038822}{1001000} \left\{ f\left(z_0 + \frac{h}{3}\right) + f\left(z_0 - \frac{h}{3}\right) \right\} + \right. \\ & \frac{1527579}{1626625} \left\{ f\left(z_0 + \frac{2h}{3}\right) + f\left(z_0 - \frac{2h}{3}\right) \right\} - \frac{23546}{4879875} \{f(z_0 + ih) + f(z_0 - ih)\} - \\ & \left. \frac{3333312}{1001000} \left\{ f\left(z_0 + i\frac{h}{3}\right) + f\left(z_0 - i\frac{h}{3}\right) \right\} + \frac{306099}{1626625} \left\{ f\left(z_0 + i\frac{2h}{3}\right) + f\left(z_0 - i\frac{2h}{3}\right) \right\} \right] \end{aligned} \quad (11)$$

and

$$\begin{aligned}
 R_4(f) = h \left[\frac{-47548744}{268515} f(z_0) + \frac{3819749452}{73226278125} \{f(z_0 + h) + f(z_0 - h)\} \right. \\
 + \frac{762997907456}{14645255625} \left\{ f\left(z_0 + \frac{h}{4}\right) + f\left(z_0 - \frac{h}{4}\right) \right\} - \frac{4554955904}{861485625} \left\{ f\left(z_0 + \frac{h}{2}\right) + f\left(z_0 - \frac{h}{2}\right) \right\} \\
 + \frac{10380713984}{12922284375} \left\{ f\left(z_0 + \frac{3h}{4}\right) + f\left(z_0 - \frac{3h}{4}\right) \right\} - \frac{180872771}{73226278125} \{f(z_0 + ih) + f(z_0 - ih)\} \\
 + \frac{656033964032}{14645255625} \left\{ f\left(z_0 + i\frac{h}{4}\right) + f\left(z_0 - i\frac{h}{4}\right) \right\} - \frac{2618398592}{861485625} \left\{ f\left(z_0 + i\frac{h}{2}\right) + f\left(z_0 - i\frac{h}{2}\right) \right\} \\
 \left. + \frac{1554440192}{12922284375} \left\{ f\left(z_0 + i\frac{3h}{4}\right) + f\left(z_0 - i\frac{3h}{4}\right) \right\} \right] \quad (12).
 \end{aligned}$$

Next we consider

3. ERROR ANALYSIS

Let $E_2(f)$ denote the truncation error associated with the rule $R_2(f)$ as given in (8) i.e.

$$E_2(f) = I(f) - R_2(f). \quad (13)$$

Here we assume that the function $f(z)$ is analytic in the disc:

$$\Omega = \{z \in \mathbb{C} : |z - z_0| \leq \rho = r|h| ; r > 1\} \quad (14)$$

Under this assumption, $f(z)$ can be expanded in terms of the Taylor's series about the point $z = z_0$ in the disc Ω as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (15)$$

where

$$a_n = f^{(n)}(z_0)/(n)!$$

are the Taylor's coefficients.

As the series given in (15) is uniformly convergent in Ω , integrating the series (15) term by term and then equating it to the integral of f we obtain:

$$\begin{aligned}
 I(f) = 2hf(z_0) + \frac{2h^3}{3(2!)} f^{(2)}(z_0) + \frac{2h^5}{5(4!)} f^{(4)}(z_0) + \frac{2h^7}{7(6!)} f^{(6)}(z_0) \\
 + \frac{2h^9}{9(8!)} f^{(8)}(z_0) + \frac{2h^{11}}{11(10!)} f^{(10)}(z_0) + \dots \quad (16)
 \end{aligned}$$

Again by expanding each term of the rule $R_2(f)$ given in (10) by Taylor's series expansion about the point $z = z_0$ in the disc Ω and then after simplification we obtain

$$\begin{aligned}
 R_2(f) = 2hf(z_0) + \frac{2}{3(2!)} h^3 f^{(2)}(z_0) + \frac{2}{5(4!)} h^5 f^{(4)}(z_0) + \frac{2}{7(6!)} h^7 f^{(6)}(z_0) \\
 + \frac{2}{9(8!)} h^9 f^{(8)}(z_0) + \frac{11 \times 2}{84(10!)} h^{11} f^{(10)}(z_0) + \frac{19 \times 2}{180(12!)} h^{13} f^{(12)}(z_0) \dots \quad (17)
 \end{aligned}$$

So from the equations (13), (16) and (17) we have

$$E_2(f) = \frac{37}{462(11!)} h^{11} f^{(10)}(z_0) - \frac{67}{462(13!)} h^{13} f^{(12)}(z_0) + \dots,$$

from which we have the following

Theorem 2.1: If $f(z)$ is analytic in Ω then

$$|E_2(f)| \sim \frac{37|h|^{11}}{462 \times 11!} |f^{(10)}(z_0)|.$$

In the same vein (as in the case of the rule $R_2(f)$) the respective error terms associated with the rules $R_3(f)$ and $R_4(f)$ can be obtained and are stated below

$$E_3(f) = -\frac{35357504}{1280674395(14!)} h^{15} f^{(14)}(z_0) - \frac{111949376}{5024184165(16!)} h^{17} f^{(16)}(z_0) \dots$$

and

$$E_4(f) = \frac{119732489722}{10237775708160(18!)} h^{19} f^{(18)}(z_0) \dots$$

Hence we have

Theorem 2.2: If $f(z)$ is analytic in Ω then

$$|E_3(f)| \sim \frac{35357504}{1280674395(14!)} |h|^{15} |f^{(14)}(z_0)|.$$

and

$$|E_4(f)| \sim \frac{119732489722}{10237775708160(18!)} |h|^{19} |f^{(18)}(z_0)|.$$

4. NUMERICAL VERIFICATION

For the approximate evaluation of Complex Integral, the following integrals are considered:

$$I_1 = \int_{-i}^i e^z dz$$

and

$$I_2 = \int_{-1}^1 e^x dx.$$

They have been integrated numerically by the rules $R_1(f)$, $R_2(f)$, $R_3(f)$ and $R_4(f)$ and the results of approximations are depicted in Table-1 and Table-2 respectively.

All the computations have been performed by using programs in MATLAB in format long.

Table-1

Rules	Approximate values of I_1	Absolute Error
R_1	1.682 417 145 154 309 i	0.000 524 824 461 484 i
R_2	1.682 941 947 664 176 i	0.000 000 021 951 617 i
R_3	1.682 941 969 615 479 i	0.000 000 000 000 314 i
R_4	1.682 941 969 615 771 i	0.000 000 000 000 022 i
Exact Value	1.682 941 969 615 793 i	

Table-2

Rules	Approximate values of I_2	Absolute Error
R_1	2.350 936 031 119 045	0.000 533 643 831 443
R_2	2.350 402 409 478 330	0.000 000 022 190 728
R_3	2. 350 402 387 287 922	0.000 000 000 000 320
R_4	2.350 402 387 287 581	0.000 000 000 000 021
Exact Value	2.350 402 387 287 602	

CONCLUSION: From the table of approximate values of the integrals I_1 and I_2 integrated by quadrature rules of precision 9, 13 and 17 we note that, the rule $R_2(f)$ which is a rule of precision 9 provides approximate values correct up to exactly seven decimal places where as, the approximate values of the integrals obtained by the rules of precision 13 and 17 agree with the exact values of the text integrals I_1 and I_2 correct up to twelve and thirteen decimal point respectively. Hence, one can safely accept the value of the integrals (whose value cannot be obtained analytically) obtained corresponding to the quadrature rule of highest precision without any risk, provided the result required is correct up to thirteen or less decimal places in any work of Science and Technology. In the nodes $z_0 \pm \alpha h$ and $z_0 \pm i\alpha h$ associated with the quadrature rule constructed by Tosic [16], Milovanovic [10], Dordevic [11] which may produce round off error when the approximation is carried out on a computer of limited precision; on the contrary the parameter α in rules of precision 9, 13 and 17 constructed in this paper are simple rational numbers, which is a positive advantage over the rules constructed by the predecessors (other researchers) .

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REFERENCES

1. Acharya, M., Acarya, B.P., Pati, S., Numerical evaluation of integrals of analytic functions, *Internat. J. Comput. Math.* 87, No.12, 2747-2751 (2010).
2. Birkhoff, G., Young, D.M., Numerical quadrature of analytic and harmonic functions, *J. Math. Phys.* 29, 217-221 (1950).
3. Davis, P.J., Rabinowitz, P., *Methods of numerical integration*, New York: Academic press, (1975).
4. Lether, F., On Birkhoff-Young quadrature of analytic functions, *J. Comput. Appl. Math.* 2, 81-84 (1976).
5. Lyness, J.N., Delves, L. M., on numerical contour integration round a closed contour, *Math. Comp.* 21, 561-577 (1967).
6. Lyness, J.N., Moler, C.B., Numerical differentiation of analytic functions, *SIAM J. Numer. Anal.* 4, 202-210 (1967).
7. Lyness, J.N., Quadrature method based on complex function values, *Math. Comp.* 23, 601-619 (1969).
8. Mastroianni, G., Milovanovic, G. V., *Interpolation Processes-Basic Theory and applications*, Springer Monographs in mathematics, Berlin-Heidelberg: Springer- Verlag, 2008.
9. McGregor, M.T., On a modified Birkhoff-Young quadrature formula for analytic functions, *Univ. Beograd. Publ. Electrotehn. Fak. Ser. Math.*, 3, 13-16 (1992).
10. Milovanovic, G. V., Cvetkovic, A.S., Stanic, M., A generalized Birkhoff-Young-chebyshev quadrature formula for analytic functions, *Appl. Math. Comput.* 218, 944-948 (2011).
11. Milovanovic, G. V., Dordevic, R.Z., On a generalization of modified Birkhoff-Young quadrature formula, *Univ. Beograd. Publ. Electrotehn. Fak. Ser. Math. Fiz.*, 762, 130-134 (1982).
12. Milovanovic, G. V., Generalized quadrature formulae for analytic functions, *Appl. Math. Comput.* 218, 8537-8551 (2012).
13. Milovanovic, G. V., Numerical quadratures and orthogonal polynomials, *Stud. Univ. Babes-Bolyai Math.* 56, 449-64 (2011).
14. Milovanovic, G. V., Tomic, D.D., Albjanic, M., Numerical integration of analytic function, *Numerical Analysis and Applied mathematics*, AIP. Conf. Proc. 1479, 1046-1049 (2012).
15. Scarborough, J.B., *Numerical Mathematical Analysis*, Baltimore: The Johns Hopkins Press, 1930.
16. Tomic, D. D., A modification of Birkhoff-Young quadrature formula for analytic functions, *Univ. Beograd. Publ. Electrotehn. Fak. Ser. Math., Fiz.* No. 602- No. 633, 73-77 (1978).
17. Young, D.M., An error bound for the numerical quadrature of analytic functions, *J. Math. Phys.* 31, 42-44 (1952).