

Original Research Article

Some Fully Symmetric Quadrature Rules for Approximation of Complex Line Integral

Abstract: This paper is concerned with the construction of some quadrature rules of algebraic degree of precision nine, thirteen and seventeen for the approximate evaluation of integrals of an analytic function on a line segment in the complex plane. The nodes in each of the rules constructed are symmetrically situated about the contour of integration. Asymptotic error estimate of each rule has been derived and a comparative study regarding the accuracy in approximations obtained by the rules constructed has been studied. Some standard integrals of analytic functions have been numerically integrated by the quadrature rules constructed in this paper to substantiate the conclusion drawn on the relative efficiency of the rules.

Key Words: Analytic function, Quadrature rule, degree of precision, asymptotic error estimate.

2000 Mathematics Subject Classification Number: 65D30, 65D32.

1. INTRODUCTION

Birkhoff-Young [2,1950] have formulated first, a five point quadrature rule :

$$R_1(f) = \frac{h}{15} [24f(z_0) + 4\{f(z_0 + h) + f(z_0 - h)\} - \{f(z_0 + ih) + f(z_0 - ih)\}] + E_1(f) \quad (1)$$

for the numerical integration of

$$I(f) = \int_L f(z) dz . \quad (2)$$

where L is a directed line segment joining the points $z_0 - h$ and $z_0 + h$. Further the function $f(z)$ is assumed to be analytic in the disc

$$\Omega = \{z: |z - z_0| \leq r; r > |h|\} \text{ containing the line segment } L.$$

The degree of precision of this rule is five and its error term is given by

$$|E_1(f)| \sim \frac{|h|^7}{1890} \max_{z \in S} |f^{(6)}(z_0)|, \quad (3)$$

where S denotes the square with vertices $z_0 + i^k h$, $k=0,1,2,3$ (Ref [17] and [3,p.136]). This error estimate is about four tenth as large as the corresponding error

$$|E_1^{ES}(f)| \sim \frac{|h|^7}{756} |f^{(6)}(\eta)|; \quad 0 < \frac{\eta - (z_0 - 2h)}{4h} < 1$$

associated with the so called extended Simpson rule [Scarbour, p.124] :

$$R_{ES} = \frac{h}{90} [114f(z_0) + 34\{f(z_0 + h) + f(z_0 - h)\} - \{f(z_0 + 2h) + f(z_0 - 2h)\}] \quad (4)$$

33 Later on, Lether [4, 1976] using the transformation:

$$34 \quad z = z_0 + ht; \quad -1 \leq t \leq 1$$

35 transformed the integral (2) to the integral:

$$36 \quad I(f) = h \int_{-1}^1 f(z_0 + ht) dt \quad (5)$$

37 and then suggested to numerically integrate the integral (5) by Gauss-Legendre three point rule to get the
 38 approximate value of the integral (2). It is shown analytically that the numerical integration of (2) by this
 39 procedure recommended by Lether produces better result than that obtained by the rule $R_1(f)$ although
 40 algebraic degree of precision of both the rules is five. This procedure is preferred to numerical integration by
 41 the rule $R_1(f)$ since it requires the evaluation of the function at three nodes: $z_0, z_0 \pm \sqrt{0.6}h$ only and the
 42 truncation error is less than that incurred in approximation by the rule (1).

43 Tosic [16,1978] considering a different set of nodes proposed the following five point degree seven rule for
 44 the approximation of integral (2) which is given as

$$45 \quad R_T(f) = \frac{16}{15}f(z_0) + \frac{h}{6}\left(\frac{7}{5} + \sqrt{\frac{7}{3}}\right)\{f(z_0 + \alpha h) + f(z_0 - \alpha h)\} + \frac{h}{6}\left(\frac{7}{5} - \sqrt{\frac{7}{3}}\right)\{f(z_0 + i\alpha h) + f(z_0 - i\alpha h)\} \quad (6)$$

$$46 \quad \alpha^4 = \frac{3}{7}.$$

47 Also, it is shown that the truncation error $E_T(f)$ incurred in approximation of the integral (2) by the rule (6) is,

$$48 \quad |E_T(f)| \sim \frac{|h|^9}{793800} |f^{(8)}(z_0)|.$$

49 The rule (6) is obtained as a special case of one parameter rule formulated by Tosic [16] i.e

$$50 \quad R_T(\alpha) = h \left\{ 2 \times \left(1 - \frac{1}{5\alpha^4}\right) f(z_0) + \left(\frac{1}{6\alpha^2} + \frac{1}{10\alpha^4}\right) [f(z_0 + \alpha h) + f(z_0 - \alpha h)] + \left(-\frac{1}{6\alpha^2} + \frac{1}{10\alpha^4}\right) [f(z_0 + i\alpha h) + \right. \\ 51 \quad \left. f(z_0 - i\alpha h)] \right\} \quad (7)$$

52 for the parameter $\alpha^4 = \frac{3}{7}$ and the rule $R_T(f)$ has been referred to as the formula of **maximum accuracy**. It
 53 may be noted here that the rule given in equation (7) reduces to Birkhoff-Young rule given in (1) for $\alpha = 1$
 54 and for $\alpha = \sqrt{3/5}$ the same rule given in equation (7) becomes three point Gauss-Legendre transformed rule
 55 recommended by Lether [4].

56 Milovanović and Dordević [11] have further generalized the modified Birkhoff-Young rule (R_T) to a two
 57 parameter rule (R_{MD}) of precision 9 with the corresponding truncation error E_{MD} in general and have obtained
 58 a rule, of precision 13 corresponding to specific values of the parameter (r_1, r_2) given by

$$59 \quad r_1^4 = \frac{63+4\sqrt{114}}{143} \quad \text{and} \quad r_2^4 = \frac{63-4\sqrt{114}}{143}$$

60 which are obtained by solving

$$61 \quad E_{MD}[(z - z_0)^{10}] = 0 = E_{MD}[(z - z_0)^{12}]$$

62 for numerical integration of the integral (2).

Quadrature formulae of Birkhoff-Young type for analytic functions have been investigated in several papers [1], [9], [10], [12], [13]. These formulas can also be used to integrate real harmonic function (see [2]). In addition, we mention also that Lyness and Delves [5] and Lyness and Moler [6] and later Lyness [7], developed formulae for numerical integration and numerical differentiation of complex functions. Milovanovic et al [14] have also constructed quadrature rule of the Birkhoff-Young type for approximation of complex definite integral with a weight function.

As far as we know, quadrature rules of Newton Cotes type (meant for approximation of real definite integral) have not been formulated so far for approximate evaluation of complex line integrals of an analytic/ singular function choosing nodes symmetrically situated with respect to the line segment in the complex plane. Hence, our aim in this paper is to construct some such rules of precision nine, thirteen and seventeen choosing suitable nodes. It may be noted here that Birkhoff-Young rule $R_1(f)$ given in equation (1) is one such rule which is analogous to Simpson $1/3^{\text{rd}}$ rule meant for approximation of real definite integrals.

2. CONSTRUCTION OF QUADRATURE RULE OF PRECISION NINE

We choose the nodes

$$z_0, z_0 \pm h, z_0 \pm ih, z_0 \pm i\frac{h}{2}$$

of which the first five nodes are the nodes of the Birkhoff-Young rule (R_1).

for the formulation of rule of precision 9 for the approximate evaluation of the integral (2).

The proposed rule is denoted by $R_2(f)$ and is given by

$$\begin{aligned} R_2(f) = & A_{21}f(z_0) + B_{21}\{f(z_0 + h) + f(z_0 - h)\} + B_{22}\left\{f\left(z_0 + \frac{h}{2}\right) + f\left(z_0 - \frac{h}{2}\right)\right\} \\ & + C_{21}\{f(z_0 + ih) + f(z_0 - ih)\} + C_{22}\left\{f\left(z_0 + i\frac{h}{2}\right) + f\left(z_0 - i\frac{h}{2}\right)\right\}. \end{aligned} \quad (8)$$

The weights $A_{21}, B_{21}, B_{22}, C_{21}$ and C_{22} are to be determined so that

$$I[(z - z_0)^{2k}] = R[(z - z_0)^{2k}]; \quad k = 0, 1, 2, 3, 4 \quad (9)$$

From the condition stated in equation (9) on the proposed quadrature formula (8) we get the following set of linear equations in $A_{21}, B_{21}, B_{22}, C_{21}$ and C_{22}

$$A_{21} + 2B_{21} + 2B_{22} + 2C_{21} + 2C_{22} = 2h$$

$$4B_{21} + B_{22} - 4C_{21} - C_{22} = \frac{4h}{3}$$

$$16B_{21} + B_{22} + 16C_{21} + C_{22} = \frac{16h}{5}$$

$$64B_{21} + B_{22} - 64C_{21} - C_{22} = \frac{64h}{7}$$

$$256B_{21} + B_{22} + 256C_{21} + C_{22} = \frac{256h}{9}.$$

On solving the above equations we have

$$A_{21} = \frac{-5880}{4725}h = -1.244\ 444\ 444\ 444\ 4h$$

$$B_{21} = \frac{556}{4725}h = 0.117\ 671\ 957\ 671\ 958\ h$$

$$B_{22} = \frac{5504}{4725}h = 1.164\ 867\ 724\ 867\ 725\ h$$

$$C_{21} = \frac{-59}{4725}h = -0.012\ 486\ 772\ 486\ 772\ h$$

$$C_{22} = \frac{1664}{4725}h = 0.352\ 169\ 312\ 169\ 312\ h. \ .$$

Thus the quadrature rule proposed in equation (8) is now given by

$$\begin{aligned} R_2(f) = h & \left[\frac{-5880}{4725}f(z_0) + \frac{556}{4725}\{f(z_0 + h) + f(z_0 - h)\} + \frac{5504}{4725}\left\{f\left(z_0 + \frac{h}{2}\right) + f\left(z_0 - \frac{h}{2}\right)\right\} \right. \\ & \left. - \frac{59}{4725}\{f(z_0 + ih) + f(z_0 - ih)\} + \frac{1664}{4725}\left\{f\left(z_0 + i\frac{h}{2}\right) + f\left(z_0 - i\frac{h}{2}\right)\right\} \right] \end{aligned} \quad (10)$$

which is a 9 point rule of degree of precision at least 9.

Now

$$I(z - z_0)^{10} = \frac{2 \times h^{11}}{11}$$

$$\text{and} \quad R_2(z - z_0)^{10} = \frac{2 \times 11h^{11}}{84}.$$

$$\text{Thus} \quad I(z - z_0)^{10} \neq R_2(z - z_0)^{10}$$

which implies the degree of precision of $R_2(f)$ is exactly 9.

3. ERROR ANALYSIS

Let $E_2(f)$ denote the truncation error associated with the rule $R_2(f)$ as given in (8) i.e.

$$E_2(f) = I(f) - R_2(f). \quad (11)$$

Here we assume that the function $f(z)$ is analytic in the disc:

$$\Omega = \{z \in \mathbb{C} : |z - z_0| \leq \rho = r|h| ; r > 1\} \quad (12)$$

Under this assumption, $f(z)$ can be expanded in terms of the Taylor's series about the point $z = z_0$ in the disc Ω as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (13)$$

$$\text{Where} \quad a_n = f^{(n)}(z_0)/(n)!$$

are the Taylor's coefficients.

As the series given in (13) is uniformly convergent in Ω , by integrating both sides of the series (13) term by term we obtain

$$I(f) = 2hf(z_0) + \frac{2h^3}{3(2!)}f^{(2)}(z_0) + \frac{2h^5}{5(4!)}f^{(4)}(z_0) + \frac{2h^7}{7(6!)}f^{(6)}(z_0) \quad (14)$$

$$+ \frac{2h^9}{9(8!)} f^{(8)}(z_0) + \frac{2h^{11}}{11(10!)} f^{(10)}(z_0) + \dots$$

Again by expanding each term of the rule $R_2(f)$ given in (10) by Taylor's series expansion about the point $z = z_0$ in the disc Ω and then after simplification we obtain

$$R_2(f) = 2hf(z_0) + \frac{2}{3(2!)} h^3 f^{(2)}(z_0) + \frac{2}{5(4!)} h^5 f^{(4)}(z_0) + \frac{2}{7(6!)} h^7 f^{(6)}(z_0) \quad (15)$$

$$+ \frac{2}{9(8!)} h^9 f^{(8)}(z_0) + \frac{11 \times 2}{84(10!)} h^{11} f^{(10)}(z_0) + \frac{19 \times 2}{180(12!)} h^{13} f^{(12)}(z_0) \dots$$

So from the equations (11), (14) and (15) we have

$$E_2(f) = \frac{37}{462(11!)} h^{11} f^{(10)}(z_0) - \frac{67}{462(13!)} h^{13} f^{(12)}(z_0) \dots$$

Hence we have the following theorem.

Theorem 2.1: If $f(z)$ is analytic in Ω then

$$|E_2(f)| \sim \frac{37|h|^{11}}{462 \times 11!} |f^{(10)}(z_0)|.$$

In the same vain (as the procedure is a mere repetition) we have constructed the rules $R_3(f)$ and $R_4(f)$ having degree of precisions thirteen and seventeen respectively and the rules are given by

$$R_3(f) = h \left[\frac{932}{65} f(z_0) + \frac{1426361}{19519500} \{f(z_0 + h) + f(z_0 - h)\} - \frac{4038822}{1001000} \left\{ f\left(z_0 + \frac{h}{3}\right) + f\left(z_0 - \frac{h}{3}\right) \right\} + \right.$$

$$\frac{1527579}{1626625} \left\{ f\left(z_0 + \frac{2h}{3}\right) + f\left(z_0 - \frac{2h}{3}\right) \right\} - \frac{23546}{4879875} \{f(z_0 + ih) + f(z_0 - ih)\} -$$

$$\left. \frac{3333312}{1001000} \left\{ f\left(z_0 + i\frac{h}{3}\right) + f\left(z_0 - i\frac{h}{3}\right) \right\} + \frac{306099}{1626625} \left\{ f\left(z_0 + i\frac{2h}{3}\right) + f\left(z_0 - i\frac{2h}{3}\right) \right\} \right] \quad (17)$$

and

$$R_4(f) = h \left[\frac{-47548744}{268515} f(z_0) + \frac{3819749452}{73226278125} \{f(z_0 + h) + f(z_0 - h)\} \right.$$

$$+ \frac{762997907456}{14645255625} \left\{ f\left(z_0 + \frac{h}{4}\right) + f\left(z_0 - \frac{h}{4}\right) \right\} - \frac{4554955904}{861485625} \left\{ f\left(z_0 + \frac{h}{2}\right) + f\left(z_0 - \frac{h}{2}\right) \right\}$$

$$+ \frac{10380713984}{12922284375} \left\{ f\left(z_0 + \frac{3h}{4}\right) + f\left(z_0 - \frac{3h}{4}\right) \right\} - \frac{180872771}{73226278125} \{f(z_0 + ih) + f(z_0 - ih)\}$$

$$+ \frac{656033964032}{14645255625} \left\{ f\left(z_0 + i\frac{h}{4}\right) + f\left(z_0 - i\frac{h}{4}\right) \right\} - \frac{2618398592}{861485625} \left\{ f\left(z_0 + i\frac{h}{2}\right) + f\left(z_0 - i\frac{h}{2}\right) \right\}$$

$$\left. + \frac{1554440192}{12922284375} \left\{ f\left(z_0 + i\frac{3h}{4}\right) + f\left(z_0 - i\frac{3h}{4}\right) \right\} \right]$$

and the respective error terms are

$$E_3(f) = -\frac{35357504}{1280674395(14!)} h^{15} f^{(14)}(z_0) - \frac{111949376}{5024184165(16!)} h^{17} f^{(16)}(z_0) \dots$$

$$\text{and } E_4(f) = \frac{119732489722}{1023775708160(18!)} h^{19} f^{(18)}(z_0) \dots$$

Hence we have the following theorem.

Theorem 2.2: If $f(z)$ is analytic in Ω then

$$|E_3(f)| \sim \frac{35357504}{1280674395(14!)} |h|^{15} |f^{(14)}(z_0)|.$$

and

$$|E_4(f)| \sim \frac{119732489722}{10237775708160(18!)} |h|^{19} |f^{(18)}(z_0)|.$$

4. NUMERICAL VERIFICATION

For the approximate Evaluation of Complex Integral the following integrals are considered:

$$I_1 = \int_{-i}^i e^z dz$$

$$I_2 = \int_{-1}^1 e^x dz$$

They have been integrated numerically by the rules $R_1(f)$, $R_2(f)$, $R_3(f)$ and $R_4(f)$ respectively and the results of approximations are depicted in Table-1 and Table-2.

All the computation has been performed by using programs in MATLAB in format long.

Table-1

Rules	Approximate evaluation of I_1
R_1	1.682 417 145 154 309 i
R_2	1.682 941 947 664 176 i
R_3	1.682 941 969 615 479 i
R_4	1.682 941 969 615 771 i
Exact Value	1.682 941 969 615 793 i

Table-2

Rules	Approximate Value of I_2
R_1	2.350 936 031 119 045
R_2	2.350 402 409 478 330
R_3	2. 350 402 387 287 922
R_4	2.350 402 387 287 581
Exact Value	2.350 402 387 287 602

163

164 **CONCLUSION:** From the table of approximate values of the integrals I_1 and I_2 integrated by quadrature
165 rules of precision 9, 13 and 17, we note that the rule $R_2(f)$ which is a rule of precision 9 provides approximate
166 values correct up to exactly seven decimal places where as the approximate values of the integrals obtained
167 by the rules of precision 13 and 17 agree with the exact values of the text integrals I_1 and I_2 correct up to
168 twelve and thirteen decimal point respectively. Hence one can safely accept the value of the integrals (whose
169 value can not be obtained analytically) obtained corresponding to the quadrature rule of highest precision
170 without any risk, provided the result required is correct up to thirteen or less decimal places in any work of
171 Science and Technology. In the nodes $z_0 \pm \alpha h$ and $z_0 \pm i\alpha h$ associated with the quadrature rule constructed
172 by Tomic [16], Milovanovic [10], Dordevic [11] which may produced round of error when the approximation is
173 carried out on a computer of limited precision; on the contrary the parameter α in rules of precision 9, 13 and
174 17 constructed in this paper are simple rational numbers, which is a positive advantage over the rules
175 constructed by the predecessors (other researchers) .

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